Algebraic systems theory and computer algebraic methods for some classes of linear control systems

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Algebraic systems theory has been greatly advanced in the last 15 years. One the one hand, this is due to the behavioral approach to systems and control theory, which was introduced by J. C. Willems [17] in the 1980s, and which has proven to be particularly fruitful for algebraic approaches, as it studies solution sets rather than the representing equations, and it does not divide the system variables into differently treated classes a priori. On the other hand, already in the 1960s, B. Malgrange [10], V. Palamodov [12], and others started to study systems of linear partial differential equations using algebraic tools such as module theory and homological methods. They founded what is now commonly referred to as the algebraic analysis approach. In 1990, a seminal paper by U. Oberst [11] established a link between the two approaches, leading to a deeper understanding of both. This stimulated the lively research activity in the area of multidimensional systems, and contributed to algebraic and behavioral systems theory in general.

The aim of this paper is to give a tutorial introduction to algebraic systems theory, focussing in particular on linear

- multidimensional shift-invariant systems (PDE with constant coefficients);
- one-dimensional time-varying systems (ODE with variable coefficients in the field of rational or meromorphic functions);
- one-dimensional parameter-dependent systems (ODE whose coefficients are polynomial or rational functions of several parameters).

Moreover, we describe a recent implementation of related algorithms in the SINGULAR [5] library `control.lib` [20].

1 Abstract linear systems

Let $\mathcal{D}$ be a ring (with 1, not necessarily commutative), and let $\mathcal{A}$ be a left $\mathcal{D}$-module. For a given matrix $R \in \mathcal{D}^{g \times q}$, we consider the abstract linear system

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\}.$$

The letter $\mathcal{B}$ has been chosen in allusion to be behavioral approach. One should think of $\mathcal{A}$ as a set of signals, and of $\mathcal{D}$ as a ring of (differential) operators acting on them. To say that $\mathcal{A}$ carries a $\mathcal{D}$-module structure amounts more or less to the requirement that one can apply any "operator" $d \in \mathcal{D}$ to any "signal" $a \in \mathcal{A}$ to obtain a new signal $da \in \mathcal{A}$. In the same way, the expression Rw from above becomes a well-defined element of $\mathcal{A}^g$.

An important observation by Malgrange [10] says that $\mathcal{B}$ is an Abelian group (with respect to addition) that is isomorphic to the group of $\mathcal{D}$-homomorphisms from $\mathcal{M} := \mathcal{D}^{1 \times q} / \mathcal{D}^{1 \times g} R$ to $\mathcal{A}$, that is,

$$\mathcal{B} \cong \operatorname{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{A}).$$

The result itself is not hard to prove, but its importance lies in the fact that it draws attention to the algebraic object $\mathcal{M}$, called the system module, and to the contravariant functor $\operatorname{Hom}_{\mathcal{D}}(\cdot, \mathcal{A})$ which transforms left $\mathcal{D}$-modules into Abelian groups.

One says that the $\mathcal{D}$-module $\mathcal{A}$ is an injective cogenerator [8] if the functor $F := \operatorname{Hom}_{\mathcal{D}}(\cdot, \mathcal{A})$ preserves and reflects exactness, that is, a sequence of left $\mathcal{D}$-modules and $\mathcal{D}$-homomorphisms

$$\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{P}$$

is exact (that is, $\operatorname{im}(f) = \ker(g)$) if and only if the sequence of Abelian groups and group homomorphisms

$$F\mathcal{M} \xleftarrow{Ff} F\mathcal{N} \xleftarrow{Fg} F\mathcal{P}$$

is exact (that is, $\operatorname{im}(Fg) = \ker(Ff)$), where $(Ff)(\varphi) = \varphi \circ f$ for $\varphi \in F\mathcal{N} = \operatorname{Hom}_{\mathcal{D}}(\mathcal{N}, \mathcal{A})$, and Fg is defined analogously.

The injective cogenerator property is a very powerful tool for systems theory, because it enables us to translate any statement on abstract linear systems that can be formulated in terms of kernels and images, into an equivalent statement on $\mathcal{D}$-modules. In many cases, the relevant rings $\mathcal{D}$ are variants of polynomial rings, which can be efficiently manipulated using modern computer algebra systems. Using the correspondence outlined above, one can then re-interpret the results of these computations using the language of systems theory.

2 Multidimensional systems

Let $\mathcal{D} = \mathbb{K}[\partial_1, \ldots, \partial_n]$ denote the ring of linear partial differential operators with constant (real or complex) coefficients, and let $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{K})$. Here, $\mathcal{D}$ is commutative, and it is known that $\mathcal{A}$ is an injective cogenerator [11].

A linear system as defined in the previous section is then the smooth solution set of a homogeneous system of linear constant-coefficient PDE, which is also called a multidimensional (linear, shift-invariant) system. The following two properties are fundamental in multidimensional systems theory:

- $\mathcal{B}$ is autonomous if it has no free variables (inputs), or equivalently, if there exists no $0 \neq w \in \mathcal{B}$ with compact support [13].

- $\mathcal{B}$ is controllable if it is parametrizable, i.e., it has an image representation $\mathcal{B} = \{M\ell \mid \ell \in \mathcal{A}^l\}$ for some $M \in \mathcal{D}^{q \times l}$. Equivalently, for all $w_1, w_2 \in \mathcal{B}$ and for all open sets $U_1, U_2 \subset \mathbb{R}^n$ with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$, there exists $w \in \mathcal{B}$ such that [13]
$$w(x) = \begin{cases} w_1(x) & \text{if } x \in U_1 \\ w_2(x) & \text{if } x \in U_2. \end{cases}$$

We have the following characterizations of autonomy and controllability in terms of the system module $\mathcal{M} = \mathcal{D}^{1 \times q}/\mathcal{D}^{1 \times g}R$ [15, 16]:

- $\mathcal{B}$ is autonomous if and only if $\mathcal{M}$ is torsion, that is, any representation matrix R of $\mathcal{B}$ has full column rank.

- $\mathcal{B}$ is controllable if and only if $\mathcal{M}$ is torsion-free, that is, any representation matrix R of $\mathcal{B}$ is a left syzygy matrix, i.e., the rows of R generate the left kernel $\{z \in \mathcal{D}^{1 \times q} \mid zM = 0\}$ of some $M \in \mathcal{D}^{q \times l}$.

3 One-dimensional (1d) time-varying systems

Let $\mathcal{D} = K[\frac{d}{dt}]$ denote the ring of linear ordinary differential operators with coefficients in the field K of rational (meromorphic) functions. This ring is not commutative, but it is a simple left and right principal ideal domain [2, 4]. For every $R \in \mathcal{D}^{g \times q}$, there exist unimodular matrices U, V such that URV is diagonal (a non-commutative analogue of the Smith form, which is due to Jacobson [7]). To cope with singularities of the coefficients, one has to work with $\mathcal{A} = \mathcal{C}_{ae}^\infty(\mathbb{R}, \mathbb{K})$, which is the set of all functions that are smooth except for a finite (discrete) number of points; alternative approaches can be found in [3, 6, 14]. Then $\mathcal{A}$ is again an injective cogenerator [18, 19], and we have the following characterizations

of autonomy and controllability which are generalizations of characterizations that are well-known in the constant-coefficient (i.e., time-invariant) case:

- $\mathcal{B}$ is autonomous (i.e., has no inputs) if and only if it can be represented by a square matrix of full rank (since $\mathcal{D}$ can be embedded into a skew field of fractions, the notion of rank makes sense in the usual way).

- $\mathcal{B}$ is controllable (i.e., has an image representation) if and only if it can be represented by a right invertible representation matrix (indeed, any full-row-rank representation of a controllable system must be right invertible).

4 1d parameter-dependent systems

Let $\mathcal{D} = \mathbb{K}[p_1, \ldots, p_N][\frac{d}{dt}]$ and $R \in \mathcal{D}^{g \times q}$, where $p = (p_1, \ldots, p_N)$ is a vector of system parameters. Then R describes a family of ODE systems: for each choice of $p_0 \in \mathbb{K}^N$, we obtain $R|_{p=p_0} \in \mathbb{K}[\frac{d}{dt}]^{g \times q}$, and thus a one-dimensional ($n = 1$) system

$$\mathcal{B}|_{p=p_0} = \{w \in \mathcal{A}^q \mid R|_{p=p_0} w = 0\}.$$

First, suppose that R has full column rank. Then we call the system family $\mathcal{B}$ generically autonomous. Specific parameter constellations may cause a rank drop in R. This determines the parameter values p_0 in which the system $\mathcal{B}|_{p=p_0}$ loses autonomy.

However, even if the rank of the representation matrix is constant for all parameter values, special parameter constellations may destroy controllability. For this, assume that $R|_{p=p_0}$ has full row rank for all $p_0 \in \mathbb{K}^N$. Then $\mathcal{B}|_{p=p_0}$ is controllable if and only if $R|_{p=p_0}$ is right invertible over $\mathbb{K}[\frac{d}{dt}]$. We say that the system family $\mathcal{B}$ is generically controllable if R is right invertible over $\mathbb{K}(p_1, \ldots, p_N)[\frac{d}{dt}]$. This implies that $\mathcal{B}|_{p=p_0}$ is controllable for almost all $p_0 \in \mathbb{K}^N$. More precisely, $\mathcal{B}|_{p=p_0}$ is controllable for all p_0 outside the algebraic variety

$$V = \mathcal{V}(\mathrm{ann}(\mathcal{N}) \cap \mathbb{K}[p_1, \ldots, p_N]),$$

where $\mathcal{N} := \mathcal{D}^g / R\mathcal{D}^q$, and thus $\mathrm{ann}(\mathcal{N}) = \{d \in \mathcal{D} \mid \exists X \in \mathcal{D}^{q \times g} : RX = dI\}$. However, in view of applicability to large examples, one would like to avoid the computation of the annihilator ideal. A heuristic method for detecting critical parameter constellations consists in checking generic controllability over $\mathbb{K}(p_1, \ldots, p_N)[\frac{d}{dt}]$, and keeping track of all denominators appearing in the computations. The result may be conservative in the sense that it may yield more candidates for controllability-destroying parameter constellations than necessary. However, the same is true for the approach using the annihilator ideal, because the set of points in which the system actually loses controllability will usually be a proper subset of V. On the other hand, the heuristic method can also be applied to rationally (rather than polynomially) parameter-dependent system families.

5 Implementation

Tests for autonomy and controllability of multidimensional systems are implemented in the SINGULAR [5] library `control.lib` [20], which is available with SINGULAR from version 3.0 on. A finer classification in terms of autonomy and controllability degrees [15, 16] is also provided, as well as additional output such as parametrizations, flat outputs etc. The SINGULAR control library also contains a procedure realizing the heuristic method for detecting critical parameter constellations described above. The main aims of our implementation are computational efficiency and user-friendliness, in particular, by requiring minimal algebraic preknowledge, its target audience consisting mainly of control theorists.

A broader functionality is offered by the MAPLE package OREMODULES [1], which focusses on non-commutative calculations, and which was the first implementation of algorithms for the computational solution of control-related problems in the algebraic analysis approach. It is expected that in the commutative case, where a direct comparison is possible, `control.lib` will eventually outperform OREMODULES with large examples, due to the high efficiency of SINGULAR with polynomial standard basis computations. For the upcoming extension of `control.lib` to variable coefficients, we will use the non-commutative computer algebra system PLURAL [9].

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