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Computer algebraic methods for the structural analysis of linear control systems

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We give an overview of the mathematical background of the SINGULAR control library.

1 Multidimensional behavioral systems

Let $\mathcal{D} = \mathbb{K}[\partial_1, \dots, \partial_n]$ denote the ring of linear partial differential operators with constant (real or complex) coefficients, and let $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{K})$. A multidimensional behavioral system is defined as the smooth solution set of a homogeneous system of linear constant-coefficient PDE, that is, $\mathcal{B} = \ker_\mathcal{A}(R) = \{w \in \mathcal{A}^q \mid Rw = 0\}$ for some $R \in \mathcal{D}^{g\times q}$.

The following two properties are fundamental in systems theory: $\mathcal{B}$ is autonomous if it has no free variables (inputs), or equivalently, if there exists no $0 \neq w \in \mathcal{B}$ with compact support [5]. $\mathcal{B}$ is controllable if it is parametrizable, i.e., it has an image representation $\mathcal{B} = \ker_\mathcal{A}(R) = \operatorname{im}_\mathcal{A}(M)$ for some $M \in \mathcal{D}^{q\times l}$. Equivalently, for all $w_1, w_2 \in \mathcal{B}$ and for all open sets $U_1, U_2 \subset \mathbb{R}^n$ with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$, there exists $w \in \mathcal{B}$ such that [5]

$$w(x) = \begin{cases} w_1(x) & \text{if } x \in U_1 \\ w_2(x) & \text{if } x \in U_2. \end{cases}$$

To a system $\mathcal{B} = \ker_\mathcal{A}(R)$, one associates the system module $\mathcal{M} = \mathcal{D}^{1\times q}/\mathcal{D}^{1\times g}R$ and the transposed system module $\mathcal{N} = \mathcal{D}^g/R\mathcal{D}^q$. We have the following characterizations of autonomy and controllability in algebraic terms: $\mathcal{B}$ is autonomous if and only if $\mathcal{M}$ is torsion, that is, any representation matrix R of $\mathcal{B}$ has full column rank. $\mathcal{B}$ is controllable if and only if $\mathcal{M}$ is torsion-free, that is, any representation matrix R of $\mathcal{B}$ is a left syzygy matrix, i.e., the rows of R generate the left kernel $\{z \in \mathcal{D}^{1\times q} \mid zM = 0\}$ of some $M \in \mathcal{D}^{q\times l}$. Both characterizations hold due to the injective cogenerator property of the $\mathcal{D}$-module $\mathcal{A}$ [4]. Pommaret and Quadrat [6, 7] introduced the following refinements of these concepts.

1.1 Autonomy degrees

$\mathcal{B}$ is autonomous if and only if $\operatorname{Ext}^0_\mathcal{D}(\mathcal{M}, \mathcal{D}) = 0$. One says that $\mathcal{B}$ has autonomy degree at least r if $\operatorname{Ext}^0_\mathcal{D}(\mathcal{M}, \mathcal{D}) = \ldots = \operatorname{Ext}^r_\mathcal{D}(\mathcal{M}, \mathcal{D}) = 0$, or equivalently, if $\dim(\mathcal{M}) < n - r$. Clearly, $\mathcal{B}$ has autonomy degree at least 0 if and only if $\mathcal{B}$ autonomous, that is, all $w \in \mathcal{B}$ that vanish in a neighborhood of infinity must be identically zero.

A system has autonomy degree at least 1 if and only if $\mathcal{B}$ is overdetermined, i.e., each $v \in \mathcal{C}^\infty(U_\infty, \mathbb{K})^q$ that satisfies the system law locally, where U_∞ is the complement of a bounded convex set, can be uniquely extended to $w \in \mathcal{B}$. The vanishing of $\operatorname{Ext}^1_\mathcal{D}(\mathcal{M}, \mathcal{D})$ guarantees the existence of such an extension, whereas $\operatorname{Ext}^0_\mathcal{D}(\mathcal{M}, \mathcal{D}) = 0$ yields its uniqueness [9].

A system has autonomy degree at least $n - 1$ if and only if $\mathcal{B}$ is finite dimensional over $\mathbb{K}$. Finally, autonomy degree at least n corresponds to $\mathcal{B} = 0$.

1.2 Controllability degrees

Instead of the system module $\mathcal{M}$, we investigate the transposed module $\mathcal{N}$. Note that $\mathcal{N}$ is not uniquely determined by $\mathcal{B}$, but $\operatorname{Ext}^i_\mathcal{D}(\mathcal{N}, \mathcal{D})$ for $i \geq 1$ is. Pommaret and Quadrat [6, 7] showed that $\mathcal{M}$ is torsion-free if and only if $\operatorname{Ext}^1_\mathcal{D}(\mathcal{N}, \mathcal{D}) = 0$. One says that $\mathcal{B}$ has controllability degree at least r if $\operatorname{Ext}^1_\mathcal{D}(\mathcal{N}, \mathcal{D}) = \ldots = \operatorname{Ext}^r_\mathcal{D}(\mathcal{N}, \mathcal{D}) = 0$. Clearly, controllability degree at least 1 corresponds to controllability of $\mathcal{B}$. Controllability degree at least n means that $\mathcal{B}$ is flat, i.e., $\mathcal{M}$ is projective, or equivalently, free, according to the Quillen-Suslin theorem. A discussion of flat systems in the context of multidimensional behavioral systems can be found in [9].

2 Strong controllability

$\mathcal{B}$ is strongly controllable if for all $v_\infty \in \mathcal{C}^\infty(U_\infty, \mathbb{K})^q$ and all $v_0 \in \mathcal{C}^\infty(U_0, \mathbb{K})^q$, where U_0 is bounded and convex and U_∞ is the complement of a bounded convex set with $\overline{U}_0 \cap \overline{U}_\infty = \emptyset$, each satisfying the system law locally, there exists $w \in \mathcal{B}$ that connects v_∞ and v_0. This property is equivalent to $\mathcal{B}$ being both controllable and extendable, that is, $\operatorname{Ext}^1_\mathcal{D}(\mathcal{N}, \mathcal{D}) = 0$ and $\operatorname{Ext}^1_\mathcal{D}(\mathcal{M}, \mathcal{D}) = 0$ [8].

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3 One-dimensional parameter-dependent systems

Let $\mathcal{D} = \mathbb{K}[p_1, \ldots, p_N][\frac{d}{dt}]$ and $R \in \mathcal{D}^{g \times q}$, where $p = (p_1, \ldots, p_N)$ is a vector of system parameters. Then R describes a family of ODE systems: for each choice of $p_0 \in \mathbb{K}^N$, we obtain $R|_{p=p_0} \in \mathbb{K}[\frac{d}{dt}]^{g \times q}$, and thus a one-dimensional ($n = 1$) behavioral system

$$\mathcal{B}|_{p=p_0} = \{w \in \mathcal{A}^q \mid R|_{p=p_0} w = 0\}.$$

First, suppose that R has full column rank. Then we call the system family $\mathcal{B}$ generically autonomous. Specific parameter constellations may cause a rank drop in R. This determines the parameter values p_0 in which the system $\mathcal{B}|_{p=p_0}$ loses autonomy.

However, even if the rank of the representation matrix is constant for all parameter values, special parameter constellations may destroy controllability. For this, assume that $R|_{p=p_0}$ has full row rank for all $p_0 \in \mathbb{K}^N$. Then $\mathcal{B}|_{p=p_0}$ is controllable if and only if $R|_{p=p_0}$ is right invertible over $\mathbb{K}[\frac{d}{dt}]$. We say that the system family $\mathcal{B}$ is generically controllable if R is right invertible over $\mathbb{K}(p_1, \ldots, p_N)[\frac{d}{dt}]$. This implies that $\mathcal{B}|_{p=p_0}$ is controllable for almost all $p_0 \in \mathbb{K}^N$. More precisely, $\mathcal{B}|_{p=p_0}$ is controllable for all p_0 outside the algebraic variety

$$V = \mathcal{V}(\mathrm{ann}(\mathcal{N}) \cap \mathbb{K}[p_1, \ldots, p_N]),$$

where $\mathcal{N} = \mathcal{D}^g / R\mathcal{D}^q$ is defined as above, and thus $\mathrm{ann}(\mathcal{N}) = \{d \in \mathcal{D} \mid \exists X \in \mathcal{D}^{q \times g} : RX = dI\}$. However, in view of applicability to large examples, one would like to avoid the computation of the annihilator ideal. A heuristic method for detecting critical parameter constellations consists in checking generic controllability over $\mathbb{K}(p_1, \ldots, p_N)[\frac{d}{dt}]$ and keeping track of all denominators and content extractions in the computations. The result may be conservative in the sense that it may yield more candidates for controllability-destroying parameter constellations than necessary. However, the same is true for the approach using the annihilator ideal, because the set of points in which the system actually loses controllability will usually be a proper subset of V. On the other hand, the heuristic method can also be applied to rationally (rather than polynomially) parameter-dependent system families, and it can be used in the multidimensional case.

4 Implementation

The calculation of autonomy and controllability degrees is implemented in the SINGULAR [2] library `control.lib`. It is included in the distribution of SINGULAR from version 3.0 on. The procedure `control` provides additional output such as parametrizations, flat outputs etc. The main aims of our implementation are computational efficiency and user-friendliness, in particular, by requiring minimal algebraic preknowledge, its target audience consists mainly of control theorists.

A comparable functionality is offered by the MAPLE package OREMODULES [1], however, since the computational cost is not a primary issue of OREMODULES, it is outperformed by `control.lib` in the vast majority of examples.

The SINGULAR control library also contains the procedure `genericity` realizing the heuristic method for detecting critical parameter constellations described above. For the upcoming extension to variable coefficients, we will use the non-commutative subsystem SINGULAR:PLURAL [3] of the system SINGULAR.

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