Generalized Weyl algebras and their global dimension

V. V. Bavula

1 Generalized Weyl algebras

Definition of the generalized Weyl algebras. Let $D$ be a ring, $\sigma = (\sigma_1, \ldots, \sigma_n)$ a set of commuting automorphisms of $D$, $(\sigma_i \sigma_j = \sigma_j \sigma_i)$, $a = (a_1, \ldots, a_n)$ a set of elements of the centre $Z(D)$ of $D$, such that $\sigma_i(a_j) = a_j$ for all $i \neq j$.

The generalized Weyl algebra $A = D(\sigma, a)$ (briefly GWA) of degree $n$ with a base ring $D$ is the ring generated by $D$ and the $2n$ indeterminates $X_1^+, \ldots, X_n^+, X_1^-, \ldots, X_n^-$ subject to the defining relations:

\begin{align*}
X_i^- X_i^+ &= a_i, & X_i^+ X_i^- &= \sigma_i(a_i), \\
X_i^\pm \alpha &= \sigma_i^\pm(\alpha) X_i^\pm, & \forall \alpha \in D, \\
[X_i^-, X_j^+] &= [X_i^+, X_j^+] = [X_i^+, X_j^-] = 0, & \forall i \neq j,
\end{align*}

where $[x, y] = xy - yx$. We say that $a$ and $\sigma$ are the sets of defining elements and automorphisms of $A$ respectively. We use also the following notation:

\begin{align*}
X_i := X_i^+ \text{ and } Y_i := X_i^-.
\end{align*}

$\mathbb{Z}$-grading on GWA. For an vector $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ we put

\begin{align*}
v_k = v_{k_1}(1) \cdots v_{k_n}(n)
\end{align*}

where for $1 \leq i \leq n$ and $m \geq 0$:

\begin{align*}
v_{\pm m}(i) = (X_i^\pm)^m, & \quad v_0(i) = 1.
\end{align*}

In the case $n = 1$, we write $v_m$ for $v_m(1)$. It follows from the definition of the GWA that

\begin{align*}
A = \bigoplus_{k \in \mathbb{Z}^n} A_k
\end{align*}

is a $\mathbb{Z}^n$-graded algebra ($A_k A_e \subseteq A_{k+e}$, for all $k, e \in \mathbb{Z}^n$), where $A_k = Dv_k = v_k D$. 

1
The category of generalized Weyl algebras is closed under the tensor product (over a base field) of algebras:
\[ A \otimes A' = D \otimes D'(\sigma \cup \sigma', a \cup a'). \]

Noetherian property.

**Proposition 1.1** Let \( A \) be a generalized Weyl algebra with base ring \( D \). Then
1. if \( D \) is left (right) Noetherian, then \( A \) is left (right) Noetherian;
2. if \( D \) is a domain and \( a_i \neq 0 \), for all \( i = 1, \ldots, n \), then \( A \) is a domain.

The Weyl algebra \( A_n \). Define the \( n \)-th Weyl algebra, \( A_n = A_n(K) \), over a field (a ring) \( K \) to be the associative \( K \)-algebra with identity generated by the \( 2n \) indeterminates \( X_1, \ldots, X_n, \partial_1, \ldots, \partial_n \), subject to the relations:
\[ [X_i, X_j] = [\partial_i, \partial_j] = [\partial_i, X_j] = 0 \text{ for } i \neq j, \ [\partial_i, X_i] = 1 \text{ for all } i. \]
The Weyl algebra \( A_n \) is the generalized Weyl algebra \( A = D(\sigma; a) \) of degree \( n \) where
\[ D = K[H_1, \ldots, H_n] \]
is a polynomial ring in \( n \) variables with coefficients in \( K \). The sets of defining elements and automorphisms of \( A \) are
\[ \{a_i = H_i \mid 1 \leq i \leq n\} \text{ and } \{\sigma_i \mid \sigma_i(H_j) = H_j - \delta_{ij}\}, \]
respectively, where \( \delta_{ij} \) is the Kronecker delta. Moreover, the map
\[ A_n \to A, \ X_i \mapsto X_i^+, \ \partial_i \mapsto X_i^-, \ \partial_i X_i \mapsto H_i, \ i = 1, \ldots, n, \]
is an algebra isomorphism.

**Examples of generalized Weyl algebras of degree 1.** Let \( A = D(\sigma; a) \) be a generalized Weyl algebra of degree 1, \( a \in Z(D) \), \( \sigma \in \text{Aut}(D) \).

The ring \( A \) is generated by \( D, X = X_1^+ \) and \( Y = X_1^- \) subject to the defining relations:
\[ X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \ \forall \alpha \in D, \ YX = a \text{ and } XY = \sigma(a). \]
The algebra
\[ A = \bigoplus_{n \in \mathbb{Z}} A_n \]
is \( \mathbb{Z} \)-graded, where \( A_n = Dv_n \),
\[
v_n = X^n \ (n > 0), \quad v_n = Y^{-n} \ (n < 0), \quad v_0 = 1.
\]

It follows from the above relations that
\[
v_nv_m = (n, m)v_{n+m} = v_{n+m} < n, m >
\]
for some \((n, m) \in D\). If \( n > 0 \) and \( m > 0 \) then
\[
n \geq m : (n, -m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a), \quad (n, m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),
\]
\[
n \leq m : (n, -m) = \sigma^n(a) \cdots \sigma(a), \quad (n, m) = \sigma^{-n+1}(a) \cdots a,
\]
in other cases \((n, m) = 1\).

Let \( A_{(i)} = D_i(\sigma_i, a_i) \ (i = 1, ..., n) \) be a GWA of degree 1 over a field \( K \) and assume that each \( \sigma_i \) is a \( K \)-automorphism, then their tensor product
\[
\otimes^n_i A_{(i)} = (\otimes^n_i D_i)((\sigma_i), (a_i))
\]
is a GWA of degree \( n \) over \( K \).

This construction allows us to build a great deal of examples of generalized Weyl algebras of degree \( n \). For example, the \( n \)-th Weyl algebra \( A_n(K) \) can be written in this way as
\[
A_n(K) = A_1 \otimes \cdots \otimes A_1,
\]
\( n \) times.

**Example 1.** \( A = K[H][\sigma, a] \), where \( \sigma \) is an arbitrary automorphism of \( K[H] \), i.e.
\[
\sigma(H) = \lambda H + \mu, \quad \lambda \neq 0, \quad \mu \in K,
\]
\( a \in K[H] \).

When \( \sigma(H) = H - 1 \) and \( a = H \) we get the 1st Weyl algebra:
\[
A_1 \simeq K[H](\sigma, a = H).
\]

Let \( F_m = A_1^G \) be the fixed ring where \( G \) is the cyclic group of order \( m \), acting on the Weyl algebra \( A_1 \) as follows:
\[
\omega : A_1 \rightarrow A_1, \quad \partial \mapsto \omega \partial, \quad X \mapsto \omega^{-1}X,
\]
where \( \omega \) is a primitive \( m' \)th root of unity. Then
\[
F_m = K < \partial^m, \partial X, X^m > \simeq K[H](\sigma, a = m^m H(H - 1/m) \cdots (H - (m - 1)/m),
\]
\( m \in \mathbb{Z} \).
\[ \partial^m \leftrightarrow Y, \ X^m \leftrightarrow X, \ \partial X/m \leftrightarrow H, \]

is a GWA of degree 1, where \( \sigma(H) = H - 1, \ char \ K = 0. \)

\textit{Case}, \( \mu = 0. \) Let

\[ \Lambda = K < X, Y | XY = \lambda YX >, \]

the \textbf{quantum plane}, then

\[ \Lambda \simeq K[H](\sigma, a = H), \ \sigma(H) = \lambda H. \]

Let

\[
A(S^2_\lambda) = K < X, Y, H | XH = \lambda HX, \ YH = \lambda^{-1}HY, \\
\lambda^{1/2}YX = -(c - H)(d + H), \\
\lambda^{-1/2}XY = -(c - \lambda H)(d + \lambda H) >
\]

be the \textbf{algebra of functions on the quantum 2-dimensional sphere}, then

\[ A(S^2_\lambda) \simeq K[H](\sigma, a = -\lambda^{-1/2}(c - H)(d + H)), \ \sigma(H) = \lambda H. \]

The \textbf{quantum Weyl} algebra

\[ A_1(q) = < x, \partial | \partial x - qx\partial = 1 > \]

of degree 1 over \( K \) \( (q \neq 0 \in K) \) is the GWA

\[ A_1(q) = K[H](\sigma, a = H) \]

of degree 1 where \( \sigma(H) = q^{-1}(H - 1). \)

\textbf{Example 2.}

\[ A = D(\sigma, a), \] \text{ where } \[ D = K[H, (H - \mu/(1 - \lambda))^{-1}], \sigma(H) = \lambda H + \mu, \lambda \neq 0, 1, \mu \in K, \] \( a \neq 0 \in D. \) In particular, when \( \mu = 0 \) we have

\[ A = K[H, H^{-1}](\sigma, a), \ \sigma(H) = \lambda H. \]

\textbf{Example 3.} Consider the \( K \)-algebra \( \Lambda(b) \), deformation of \textit{Usl}(2) which is generated by \( X, Y, Z \) subject to the relations:

\[ [H, X] = X, \ [H, Y] = -Y, \ [X, Y] = b \neq 0 \in K[H]. \]

Then

\[ \Lambda(b) \cong K[H, C](\sigma, a = C - \alpha) \]
where
\[ \sigma : K[H, C] \to K[H, C], \quad H \to H - 1, \quad C \to C, \]
and \( \alpha \in K[H] \) is a solution of the equation
\[ \alpha - \sigma(\alpha) = b. \]

For \( b = 2H \),
\[ \Lambda(b) = Usl(2). \]
If \( K \) is a field of characteristic zero, then the center of \( \Lambda(b) \) is \( K[C] \). For any \( \lambda \in K \) the factor algebra
\[ \Lambda(b, \lambda) := \Lambda(b)/\Lambda(b)(C - \lambda) \]
is isomorphic to the GWA from Example 1 with the defining element \( \lambda - \alpha \). Let
\[ U(\lambda) := Usl(2)/Usl(2)(C - \lambda) \simeq K[H](\sigma, \lambda - H(H + 1)) \]
be the infinite dimensional primitive factor of \( Usl(2) \).

**Example 4.** The quantum Heisenberg algebra:
\[ \mathcal{H}_q = K < X, Y, H \mid XH = q^2HX, YH = q^{-2}HY, XY - q^{-2}YX = q^{-1}H >, q \in K, q^4 \neq 1 \]
is isomorphic to the GWA of degree 1:
\[ \mathcal{H}_q \simeq \mathbb{k}[H, C](\sigma; a = \rho^{-1}C - \mu H = q^2(C - H/q(1 - q^4)), \sigma(H) = q^2H, \sigma(C) = q^{-2}C. \]
The element \( \Omega = HC \) belongs to the centre of \( \mathcal{H}_q \). For each \( \lambda \neq 0 \in K \) the factor algebra
\[ \mathcal{H}_q(\lambda) := \mathcal{H}_q/(\Omega - \lambda) \]
is the GWA of degree 1:
\[ \mathcal{H}_q(\lambda) \simeq K[H, H^{-1}](\sigma; a = q^2(\lambda H^{-1} - H/q(1 - q^4)), \sigma(H) = q^3H. \]

**The ambiskew polynomial rings** \( E \) **are GWAs.**
Let \( D \) be an ring, \( \sigma \in \text{Aut}(D) \). Suppose that elements \( b \) and \( \rho \) belong to the centre of \( D \), moreover, \( \rho \) is invertible and \( \sigma \)-stable, i.e. \( \sigma(\rho) = \rho \). Then the ambiskew polynomial ring
\[ E = D < \sigma; b, \rho > \]
is obtained by adjoining to \( D \) two symbols \( X \) and \( Y \) subject to the relations:
\[ X\alpha = \sigma(\alpha)X, \quad Y\alpha = \sigma^{-1}(\alpha)Y, \quad \forall \alpha \in D; \quad XY - \rho YX = b. \]
If $D = K[H]$ is the polynomial ring, $\rho = 1$, $b = 2H$, and $\sigma(H) = H - 1$, we get the universal enveloping algebra $\mathfrak{u}sl(2)$.

The ring $E$ is the **iterated skew polynomial ring**

$$E = D[Y; \sigma^{-1}][X; \sigma, \partial]$$

where $\partial$ is the $\sigma$–derivation of $D[Y; \sigma^{-1}]$ such that $\partial D = 0$ and $\partial Y = b$ (here $\sigma$ is extended from $D$ to $D[Y; \sigma^{-1}]$ by the rule: $\sigma(Y) = \rho Y$).

Lemma 1.2 shows that the rings $E = D < \sigma; b, \rho >$ are generalized Weyl algebras of degree 1.

**Lemma 1.2** Each iterated skew polynomial ring $E$ is the generalized Weyl algebra of degree 1 with base polynomial ring $D[H]$ and defining automorphism $\sigma : \sigma(H) = \rho(H) + b$ ($\sigma$ acts on $D$ as before):

$$D < \sigma; b, \rho > \cong D[H](\sigma; a = H), \quad X \leftrightarrow X, \quad Y \leftrightarrow Y, \quad d \leftrightarrow d (\forall d \in D), \quad YX \leftrightarrow H.$$ 

An element $d$ of a ring $D$ is **normal** if $dD = Dd$.

**Lemma 1.3** The following are equivalent.

1. $C = \rho(YX + \alpha) = XY + \sigma(\alpha)$ is normal in $D < \sigma; b, \rho >$;
2. $\rho \alpha - \sigma(\alpha) = b$ for some $\alpha \in D$;
3. $D[H] = D[C_1]$ for some $C_1 \in D[H]$ such that $\sigma(C_1) = \gamma C_1$ where $\gamma \in D$ is invertible and $\sigma(\gamma) = \gamma$.

**Corollary 1.4** Let $E$ be as in Lemma 1.3. Then

$$D < \sigma; b, \rho > \cong D[C](\sigma, a = \rho^{-1}C - \alpha), \quad \sigma(C) = \rho C.$$ 

Putting $\rho = 1$ we obtain the following Lemma and Corollary.

**Lemma 1.5** The following are equivalent.

1. $C = YX + \alpha = XY + \sigma(\alpha)$ is central in $D < \sigma; b, \rho >$
2. $\alpha - \sigma(\alpha) = b$ for some $\alpha \in D$;
3. $D[H] = D[C_1]$ for some $C_1 \in D[H]$ such that $\sigma(C_1) = C_1$ (then $C_1 = \gamma C$ for some central invertible $\sigma$-stable element $\gamma$).

**Corollary 1.6** Suppose that Lemma 1.5 holds. Then

$$D < \sigma; b, \rho > \cong D[C](\sigma, a = C - \alpha), \quad \sigma(C) = C.$$
Further Examples. For \( q, h = q - q^{-1} \in K = \mathbb{C} \), the algebra \( U_q = U_q sl(2) \) is generated by \( X, Y, H_- , H_+ \) subject to the relations:

\[
H_+ H_- = H_- H_+ = 1, \quad X H_\pm = q^\pm 1 H_\pm X, \quad Y H_\pm = q^\pm 1 H_\pm Y, \quad [X,Y] = (H_+^2 - H_-^2)/h.
\]

It follows from the relations that:

\[
U_q \simeq K[C, H, H^{-1}] (\sigma, a = C + \{H^2/(q^2 - 1) - H^{-2}/(q^{-2} - 1)\}/2h)
\]

where \( \sigma(H) = qH, \quad \sigma(C) = C \).

**Woronowicz’s deformation** \( V \) is generated by \( V_0, V_-, V_+ \) subject to the relations:

\[
[V_0, V_+]_{s^2} \equiv s^2 V_0 V_+ - s^{-2} V_+ V_0 = V_+, \quad [V_-, V_0]_{s^2} = V_-, \quad [V_+, V_-]_{1/s} \equiv s^{-1} V_+ V_- - s V_- V_+ = V_0.
\]

The algebra \( V \) is the GWA:

\[
V \simeq K[u, v](\sigma, a = v), \quad V_\pm \leftrightarrow X_\pm, \quad V_0 \leftrightarrow u, \quad V_- V_+ \leftrightarrow v,
\]

where

\[
\sigma : u \rightarrow s^2(s^2 u - 1), \quad v \rightarrow s^2 v + s u,
\]

is the automorphism of the polynomial ring \( K[u, v] \) in two variables \( u \) and \( v \). If we put

\[
H = u + s^2/(1 - s^4), \quad Z = v + u/s(1 - s^2) + s^3/(1 - s^2)(1 - s^4),
\]

then

\[
\sigma(H) = s^4 H, \quad \sigma(Z) = s^2 Z
\]

and \( K[u, v] = K[H, Z] \). So,

\[
V \simeq K[H, Z](\sigma, a = Z + \alpha H + \beta), \quad V_\pm \leftrightarrow X_\pm, \quad V_0 \leftrightarrow H - s^2/(1 - s^4),
\]

where

\[
\alpha = -1/s(1 - s^2) \text{ and } \beta = s/(1 - s^4).
\]

**Witten’s first deformation** \( E \) is the algebra generated by \( E_0, E_-, E_+ \):

\[
[E_0, E_+]_p \equiv p E_0 E_+ - p^{-1} E_+ E_0 = E_+, \quad [E_-, E_0]_p = E_-, \quad [E_+, E_-] = E_0 - (p - 1/p) E_0^2,
\]

where \( p \neq 0, \pm 1, \pm i \in K \). The Casimir operator which commutes with all generators is:

\[
C = E_- E_+ + E_0 (E_0 + p)/p (p^2 + 1).
\]
Witten’s first deformation is the GWA:

\[ E \cong K[C, H](\sigma, a = C - H(H + 1)/(p + p^{-1})), \quad E_{\pm} \leftrightarrow X^\pm, \quad E_0 \leftrightarrow pH, \]

where

\[ \sigma : C \to C, \quad H \to p^2(H - 1). \]

The quantum group

\[ \mathcal{O}_{q^2}(so(k, 3)) = K[H] < \sigma; b = (q - q^{-1})H, \rho = 1 >, \quad \sigma(H) = q^2 H, \]

\[ q \in K, \text{ by Corollary 1.6 is isomorphic to the GWA of degree } 1: \]

\[ \mathcal{O}_{q^2}(so(k, 3)) = K[H, C](\sigma, a = C + H^2/(1 + q^2)), \quad \sigma(H) = q^2 H, \quad \sigma(C) = C. \]

## 2 Simplicity Criteria for GWAs

**Theorem 2.1** Let \( A = D(\sigma, a) \) be a GWA of degree 1. The algebra \( A \) is a simple algebra iff

1. the ring \( D \) has no proper \( \sigma \)-stable ideal,
2. no power of \( \sigma \) is an inner automorphism of \( D \),
3. For all \( n \geq 1 \), \( D = Da + D\sigma^n(a) \), and
4. \( a \) is not a zero divisor in \( D \).

**Theorem 2.2** Let \( A = D(\sigma, a) \) be a GWA of degree \( n \) and \( D \) is a domain. The algebra \( A \) is a simple algebra iff

1. the ring \( D \) has no proper \( \sigma \)-stable ideal,
2. the subgroup of the group of outer automorphisms \( \text{Aut}(R)/\text{Inn}(R) \) of \( R \) generated by the automorphisms \( \sigma_1, \ldots, \sigma_n \) is a free abelian group of rank \( n \), and
3. For all \( m \geq 1 \) and \( i = 1, \ldots, n \), \( D = Da_i + D\sigma_i^m(a_i) \).

## 3 The global dimension of GWAs

Let \( A = D(\sigma, a) \) be a GWA. Denote by \( n = \text{lgd} D \) the left global dimension of the ring \( D \).

- **(Theorem)** Then \( \text{lgd} A = n, n + 1 \) or \( \infty \).

- **(Theorem)** If \( D \) is Noetherian and \( \text{lgd} A < \infty \). Then \( \text{gld} A = \max\{n, \mu\} \) where \( \mu = \sup \{\text{pd} M | \_A M \text{ is simple and finitely generated over } D\} \).
• (Corollary) If $D$ is semiprime Noetherian, $n < \infty$, $\operatorname{gl}A < \infty$, then $\operatorname{gld}A = n + 1$ if and only if there is a simple $A$-module $M$ which is finitely generated over $D$ and with $\operatorname{pd}_DM$.

• (Theorem) If $D$ is a commutative Noetherian domain of finite global dimension $n$, $a \neq 0$. Then $\operatorname{gld}A < \infty$ if and only if $\operatorname{pd}_DA/\mathfrak{a} = \infty$ for all prime ideals $\mathfrak{p}$ of $D$ which contain $a$.

• (Theorem) If $D$ is a commutative Noetherian ring, $n < \infty$, $a$ is regular, $\operatorname{gld}A < \infty$. Then $\operatorname{gld}A = n + 1$ if and only if either there is a semistable (i.e. $\sigma^i(\mathfrak{m}) = \mathfrak{m}$ for some $i \geq 1$) maximal ideal $\mathfrak{m}$ of $D$ of height $n$ or there are maximal ideals $\mathfrak{p}, \mathfrak{q}$ of $D$ of height $n$ such that $\sigma^i(\mathfrak{p}) = \mathfrak{q}$ for some $i \neq 0 \in \mathbb{Z}$ and $a \in \mathfrak{p}. \mathfrak{q}$.

In many cases the considered algebras are the generalized Weyl algebras $A = D(\sigma, a)$ of degree 1 where $D$ is a Dedekind ring $D$. By Theorem 3.1 we can compute their global homological dimension.

**Theorem 3.1** Let $A = D(\sigma, a)$ be the generalized Weyl algebra of degree 1 where $D$ is a Dedekind ring, $D(\sigma, a)$ is the product of the distinct maximal ideals of $D$. Then the global dimension of $A$ is

$$
\operatorname{gl}\dim A = \begin{cases} 
\infty, & \text{if } a = 0 \text{ or } n_i \geq 2 \text{ for some } i; \\
2, & \text{if } a \neq 0, n_1 = \ldots = n_s = 1, s \geq 1 \text{ or } a \text{ is invertible and there exists an integer } k \geq 1 \text{ such that either } \sigma^k(\mathfrak{p}_i) = \mathfrak{p}_j \text{ for some } i, j \\
& \text{or } \sigma^k(\mathfrak{q}) = \mathfrak{q} \text{ for some maximal ideal } \mathfrak{q} \text{ of } D; \\
1, & \text{otherwise.}
\end{cases}
$$

**Tensor homological minimal algebras.** Denote by $\mathcal{Alg}$ (respectively, $\mathcal{LN}$; $\mathcal{LFN}$; $\mathcal{SC})$ the class of all (respectively, left Noetherian; left Noetherian and finitely generated; somewhat commutative) algebras over a fixed field $K$. The tensor product $\otimes$ means $\otimes_K$ and a module means a left module. Denote by $d$ one of the following dimensions of rings: $\operatorname{wd}$, the weak dimension; $\operatorname{lgd}$, the left homological dimension; $\operatorname{kd}$, the Krull dimension (in the sense of Gabriel-Rentschler).

We give for a large class of (infinite dimensional non-commutative) algebras the answer to the question:

• when the dimension of the tensor product of algebras is the sum of dimensions of the multiples,

$$
d(\Lambda_1 \otimes \cdots \otimes \Lambda_n) = d\Lambda_1 + \cdots + d\Lambda_n. \quad (1.1)
$$

Usually, (1.1) is not true. For example, if $L_n$ is the division ring of the polynomial ring $P_n = K[X_1, \ldots, X_n]$ in $n$ variables, then

$$
\operatorname{lgd} L_n \otimes L_n = n \neq 0 = \operatorname{lgd} L_n + \operatorname{lgd} L_n.
$$
Therefore, we should put on the algebras \( \Lambda_i \) some restricted conditions for (1.1) to be satisfied. It is well known, that for all above dimensions \( d=wd, \ k d \) (respectively, \( l g d \))

\[
d(A \otimes B) \geq dA + dB, \text{ for all (respectively, left Noetherian) algebras } A, B. \tag{1.2}
\]

So it is natural to give the following

**Definition.** An algebra \( A \) is a tensor \( d \)-minimal with respect to some class of algebras \( \Omega \), if

\[
d(A \otimes B) = dA + dB, \text{ for each } B \in \Omega. \tag{1.3}
\]

In the case \( wd \) (\( l g d \)) we say that the algebra \( A \) is tensor weak minimal (tensor homological minimal), briefly, TWM (THM).

The polynomial ring \( P_n \) is tensor homological minimal with respect to \( Alg \), since for each \( B \in \Omega \)

\[
lgd(P_n \otimes B) = n + lgd B = lgd P_n + lgd B \text{ (Hilbert's syzygy theorem)}.
\]

**Lemma 3.2** Let \( \Lambda_i \) (\( i = 1, \ldots, n \)) be tensor \( d \)-minimal algebras with respect to \( \Omega \) such that \( \Lambda_i \otimes \Omega \subseteq \Omega \) for all \( i \). Then the tensor product \( \otimes_{i=1}^n \Lambda_i \) is a tensor \( d \)-minimal algebra with respect to \( \Omega \) and

\[
d(\otimes_{i=1}^n \Lambda_i \otimes B) = \sum_{i=1}^n d\Lambda_i + dB
\]

for each \( B \in \Omega \).

**Theorem 3.3** Let \( K \) be an algebraically closed uncountable field. Consider the following generalized Weyl algebras of degree 1 with non-zero defining element \( a \).

1. \( \text{char } K = 0 \),

\[
K[H](\sigma, a), \ \sigma(H) = H - \mu, \ \mu \neq 0 \in K
\]

(if \( \sigma(H) = H - 1 \) we have the Weyl algebra \( A_1 \simeq K[H](\sigma, a = H) \) and all prime quotients of \( USl(2) : U(\lambda) = USl(2)/(C - \lambda) \simeq K[H](\sigma, a = \lambda - H(H + 1)) \), where \( C \) is the Casimir element);

2. \( K[H,(H - \mu/(1 - \lambda))^{-1}](\sigma, a), \ \sigma(H) = \lambda H + \mu, \lambda \neq 0, 1 \in K \) is not a root of 1, \( \mu \in K \) (if \( \sigma(H) = \lambda H, \lambda \neq 0, 1 \), then \( K[H,H^{-1}](\sigma, a = H) \) is a localization of the quantum plane \( \Lambda = K < X,Y | XY = \lambda YX > \) at the multiplicatively closed set \( S = \{ (XY)^n, n = 0,1,\ldots \} \)).
Each tensor product $\Lambda = \bigotimes_{i=1}^{n} \Lambda_i$ of these algebras is a tensor homological minimal algebra with respect to $\mathcal{LFN}$, therefore

$$\operatorname{lgd}(\bigotimes_{i=1}^{n} \Lambda_i \otimes B) = \sum_{i=1}^{n} \operatorname{lgd} \Lambda_i + \operatorname{lgd} B,$$

for any left Noetherian finitely generated algebra $B$.

If $K$ is not necessarily uncountable, then $\Lambda$, where all algebras $\Lambda_i$ from 1, is a tensor homological minimal algebra with respect to $\mathcal{SC}$.

In particular, the left global dimension of the Weyl algebra is

$$\operatorname{lgd} (A_n) = \operatorname{lgd} (A_1 \otimes \cdots \otimes A_1) = n \cdot \operatorname{lgd} A_1 = n.$$

References
