Bernstein-Sato Polynomial of Affine Algebraic Varieties

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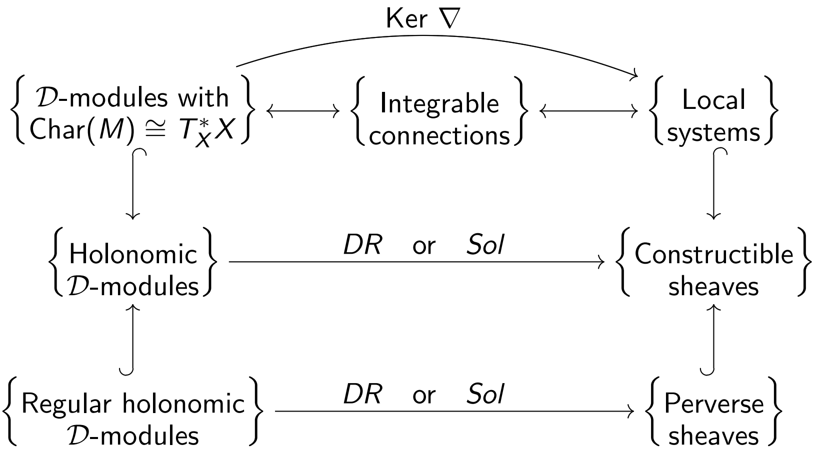
Riemann-Hilbert Correspondence

Kashiwara-Mebkhout (1980)



Stratification associated with $\mathbb{C}[\mathbf{x}, \frac{1}{f}]$

Theorem (Kashiwara)

$\mathbb{C}[\mathbf{x}, \frac{1}{f}]$ is a regular holonomic *D*-module.

- Walther (2004) gave an algorithm for computing such stratification.
- There is another stratification of $\mathbb{C}^n$ associated with local *b*-functions. $\longrightarrow$ Primary ideal decomposition is needed, Oaku (1997).

$$P(s)f^{s+1} = b_f(s)f^s$$

Joint work with . . .

- Viktor Levandovskyy
- Daniel Andres

Introduction and notations

Basic notations

- $\mathbb{C}$ the field of the complex numbers.
- $\mathbb{C}[s]$ the ring of polynomials in one variable over $\mathbb{C}$.
- $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ the ring of polynomials in n variables.
- $D_n = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ the ring of $\mathbb{C}$-linear differential operators on R_n, the n-th Weyl algebra:

$$\partial_i x_i = x_i \partial_i + 1$$

- $D_n[s] = \mathbb{C}[s] \otimes_{\mathbb{C}} D_n$.

The $D_n[s]$-module $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \bullet f^s$ (Bernstein, 1972)

- Let $f \in \mathbb{C}[\mathbf{x}]$ be a non-zero polynomial.
- By $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}]$ we denote the ring of rational functions of the form
$$\frac{g(\mathbf{x}, s)}{f^r}$$
where $g(\mathbf{x}, s) \in \mathbb{C}[\mathbf{x}, s] = \mathbb{C}[x_1, \ldots, x_n, s]$.
- We denote by $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \bullet f^s$ the free $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}]$-module of rank one generated by the symbol f^s.
- $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \bullet f^s$ has a natural structure of left $D_n[s]$-module.

$$\partial_i \bullet f^s = s \frac{\partial f}{\partial x_i} \frac{1}{f} \bullet f^s \quad \in \quad \mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \bullet f^s$$

The global b-function

Theorem (Bernstein, 1972)

For every polynomial $f \in \mathbb{C}[\mathbf{x}]$ there exists a non-zero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P(s) \in D_n[s]$ such that

$$P(s) f \bullet f^s = b(s) \bullet f^s \quad \in \quad \mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \bullet f^s.$$

Definition (Bernstein & Sato, 1972)

The set of all possible polynomials $b(s)$ satisfying the above equation is an ideal of $\mathbb{C}[s]$. The monic generator of this ideal is denoted by $b_f(s)$ and called the Bernstein-Sato polynomial of f.

The local b-function

Now assume that

- $f \in \mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ is a convergent power series.
- $\mathcal{D}_n$ is the ring of differential operators with coefficients in $\mathcal{O}$.

Theorem (Björk & Kashiwara, 1976)

For every $f \in \mathcal{O}$ there exists a non-zero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P(s) \in \mathcal{D}_n[s]$ such that

$$P(s)f \bullet f^s = b(s) \bullet f^s \quad \in \quad \mathcal{O}[s, \frac{1}{f}] \bullet f^s.$$

Definition

The monic polynomial in $\mathbb{C}[s]$ of lowest degree which satisfies the above equation is denoted by $b_{f,0}(s)$ and called the local b-function of f.

Why a functional equation ?

1. On zeta function associated with prehomogeneous vector spaces, (Sato, 1972).

2. The analytic continuation of generalized functions with respect to a parameter, (Bernstein, 1972).

$$f^s = \frac{1}{b_f(s)} P(s) f^{s+1}$$

SOME WELL-KNOWN PROPETIES OF THE b-FUNCTION

1. The b-function is always a multiple of $(s+1)$. The equality holds if and only f is smooth.

2. The set $\{e^{2\pi i\alpha} \mid b_{f,0}(\alpha) = 0\}$ coincides with the eigenvalues of the monodromy of the Milnor fibration. (Malgrange, 1975 and 1983).

3. Every root of $b_f(s)$ is rational. (Kashiwara, 1976).

4. The roots of the b-function are negative rational numbers of the real interval $(-n, 0)$. (Varchenko, 1980; Saito, 1994).

5. $b_f(s) = \text{lcm}_{p\in\mathbb{C}^n}(b_{f,p}(s))$ (Briançon-Maisonobe and Mebkhout-Narváez, 1991).

Algorithms for computing the b-function

1. Functional equation, $P(s) f \bullet f^s = b_f(s) \bullet f^s$.
2. By definition, $(\mathrm{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] = \langle b_f(s) \rangle$.
3. Now find a system of generator of the annihilator and proceed with the elimination.

Annihilator	Elimination
Oaku-Takayama (1997)	Noro (2002)
Briançon-Maisonobe (2002)	Andre-Levandovskyy-MM (2009)
Levandovskyy (2008)	

Computing preimages under G-algebras morphisms

Two more approaches

1. Initial parts (Saito-Sturmfels-Takayama, 1991).

$$\mathrm{in}_{(-w,w)}(I_f) \cap \mathbb{C}[s] = \langle b_f(s) \rangle$$

2. Isolated singularities.

- Brieskorn (1970), Gauss-Manin connexion for computing monodromy.
- Malgrange (1975), minimal polynomial of the residue of the connexion with respect to a saturated lattice.
- Nachen (1990), implementation in Maple.
- Schulze (2004), implementation in Singular.

Partial solution

Theorem (the `checkRoot` algorithm)

Let m_α be the multiplicity of α as a root of $b_f(-s)$ and let us consider the ideal $I = \mathrm{Ann}_{D_n[s]}(f^s) + D_n[s]\langle f \rangle$. The following conditions are equivalent:

1. $m_\alpha > i$.
2. $\left(I : (s+\alpha)^i \right) + D_n[s]\langle s+\alpha \rangle \neq D_n[s]$.
3. $\left(I : (s+\alpha)^i \right)|_{s=-\alpha} \neq D_n$.

Applications

1. Computation of the *b*-functions via upper bounds.
    - Embedded resolutions.
    - Topologically equivalent singularities.
    - A'Campo's formula.
    - Spectral numbers.
2. Intergral roots of *b*-functions.
    - Logarithmic comparison problem.
    - Intersection homology D-module.
3. Stratification associated with local *b*-functions.
4. Bernstein-Sato polynomials for Varieties.
5. Narvaez's paper.

Stratification associated with local b-functions

Stratrat. associated with local b-functions

Theorem

- $\mathrm{Ann}_{D[s]}(f^s) + D[s]\langle f \rangle = D[s]\langle \{P_1(s), \ldots, P_k(s), f\} \rangle$
- $I_{\alpha,i} = \big(I : (s+\alpha)^i\big) + D[s]\langle s+\alpha \rangle$, $(i = 0, \ldots, m_\alpha - 1)$

$$m_\alpha(p) > i \iff p \in V(I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}])$$

1. $V_{\alpha,i} = V(I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}])$
2. $\emptyset =: V_{\alpha,m_\alpha} \subset V_{\alpha,m_\alpha-1} \subset \cdots \subset V_{\alpha,0} \subset V_{\alpha,-1} := \mathbb{C}^n$
3. $m_\alpha(p) = i \iff p \in V_{\alpha,i-1} \setminus V_{\alpha,i}$

EXPERIMENTS

- $f = (x^2 + 9/4y^2 + z^2 - 1)^3 - x^2z^3 - 9/80y^2z^3 \in \mathbb{C}[x, y, z]$
- $b_f(s) = (s + 1)^2(s + 4/3)(s + 5/3)(s + 2/3)$
- $V_1 = V(x^2 + 9/4y^2 - 1, z)$
- $V_2 = V(x, y, z^2 - 1) \longrightarrow$ two points
- $V_3 = V(19x^2 + 1, 171y^2 - 80, z) \longrightarrow$ four points
- $V_3 \subset V_1, \quad V_1 \cap V_3 = \emptyset$
- $Sing(f) = V_1 \cup V_2$

$$\begin{array}{rclcccc} \alpha & = & -1, & \emptyset & \subset & V_1 \subset V(f) & \subset & \mathbb{C}^3; \\ \alpha & = & -4/3, & \emptyset & \subset & V_1 \cup V_2 & \subset & \mathbb{C}^3; \\ \alpha & = & -5/3, & \emptyset & \subset & V_2 \cup V_3 & \subset & \mathbb{C}^3; \\ \alpha & = & -2/3, & \emptyset & \subset & V_1 & \subset & \mathbb{C}^3. \end{array}$$

From this, one can easily find a stratification of $\mathbb{C}^3$ into constructible sets such that $b_{f,p}(s)$ is constant on each stratum.

$$\begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s+1 & p \in V(f) \setminus (V_1 \cup V_2), \\ (s+1)^2(s+4/3)(s+2/3) & p \in V_1 \setminus V_3, \\ (s+1)^2(s+4/3)(s+5/3)(s+2/3) & p \in V_3, \\ (s+1)(s+4/3)(s+5/3) & p \in V_2. \end{cases}$$

Bernstein-Sato Polynomial of Affine Algebraic Varieties

Thereom (Budur-Mustata-Saito, 2006)

For every r-tuple $f = (f_1, \ldots, f_r) \in \mathbb{C}[x]^r$ there exists a non-zero polynomial in one variable $b(s) \in \mathbb{C}[s]$ and r differential operators $P_1(S), \ldots, P_r(S) \in D_n\langle S\rangle$ such that

$$\sum_{k=1}^{r} P_k(S) f_k \bullet f^s = b(s_1 + \cdots + s_r) \bullet f^s$$

$$(\mathrm{Ann}_{D_n\langle S\rangle}(f^s) + \langle f_1, \ldots, f_r\rangle) \cap \mathbb{C}[s_1 + \cdots + s_r] = \langle b_f(s_1 + \ldots + s_r)\rangle$$

- Computation of $\mathrm{Ann}_{D_n\langle S\rangle}(f^s)$? (preimage approach)
- Difference: $D_n\langle S\rangle = D_n \otimes_{\mathbb{C}} \mathbb{C}\big\langle \{s_{ij}\} \ : \ [s_{ij}, s_{kl}] = \delta_{jk} s_{il} - \delta_{il} s_{kj} \big\rangle$

THEOREM (Levandovskyy, 2005)

Given two $\mathbb{C}$-algebras $A = \mathbb{C}\langle x_1, \ldots, x_n \mid x_j x_i = x_i x_j + d_{ij}\rangle$, $B = \mathbb{C}\langle y_1, \ldots, y_m \mid y_j y_i = y_i y_j + d'_{ij}\rangle$ and a morphism of $\mathbb{C}$-algebras $\phi : A \to B$, consider the $\mathbb{C}$-algebra

$$E := A \otimes_{\mathbb{C}}^{\phi} B$$

generated by A, B and new relations $\{y_j x_i = x_i y_j + [y_j, \phi(x_i)]\}$. Then for all $J \subseteq B$ left ideal one has

$$\phi^{-1}(J) = \Big(E \cdot \langle x_i - \phi(x_i)\rangle \, + \, E \cdot J\Big) \bigcap A.$$

REMARK. The above intersection can be computed using Gröbner bases if there exists an elimination monomial ordering for B in E such that $LM\big(y_j \phi(x_i) - \phi(x_i) y_j\big) < x_i y_j$.

How to compute the annihilator from the preimage theorem ?

THANK YOU VERY MUCH !!

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