Letterplace ideals and non-commutative Gröbner bases

Roberto La Scala, Viktor Levandovskyy

Università di Bari, RWTH Aachen

9.12.2009, Aachen

Notations

- $X = \{x_0, x_1, \dots\}$ a finite or countable set
- $F\langle X\rangle$ the free associative algebra generated by X
- I a two-sided ideal of $F\langle X\rangle$

All associative algebras, generated by a finite or countable number of elements, can be presented as $F\langle X\rangle/I$.

Examples

- algebras of finite or countable dimension
- (quantized) universal enveloping algebras of Lie algebras of finite or countable dimension
- relatively free algebras defined by the polynomial identities satified by an associative algebra
- etc.

Notations

- $X = \{x_0, x_1, \dots\}$ a finite or countable set
- $F\langle X\rangle$ the free associative algebra generated by X
- I a two-sided ideal of $F\langle X\rangle$

All associative algebras, generated by a finite or countable number of elements, can be presented as $F\langle X\rangle / I$.

Examples

- algebras of finite or countable dimension
- (quantized) universal enveloping algebras of Lie algebras of finite or countable dimension
- relatively free algebras defined by the polynomial identities satified by an associative algebra
- etc.

Trivial

- surjective homomorphism $F\langle X\rangle \to F[X]$
- 1-to-1 correspondence between all ideals $J \subset F[X]$ and a class of two-sided ideals $I \subset F\langle X\rangle$
- the ideals I contain all the commutators $[x_i, x_j] = x_i x_j - x_j x_i$

Problem

- Is there a 1-to-1 correspondence between all two-sided ideals $I \subset F\langle X\rangle$ and a class of ideals $J \subset F[Y]$ for some Y?
- Is there a "good" correspondence given between generating sets?
- in particular, between their Gröbner bases?

We propose a solution for the case I is an homogeneous ideal.

Trivial

- surjective homomorphism $F\langle X\rangle \to F[X]$
- 1-to-1 correspondence between all ideals $J \subset F[X]$ and a class of two-sided ideals $I \subset F\langle X\rangle$
- the ideals I contain all the commutators $[x_i, x_j] = x_i x_j - x_j x_i$

Problem

- Is there a 1-to-1 correspondence between all two-sided ideals $I \subset F\langle X\rangle$ and a class of ideals $J \subset F[Y]$ for some Y?
- Is there a "good" correspondence given between generating sets?
- in particular, between their Gröbner bases?

We propose a solution for the case I is an homogeneous ideal.

Notations

- $P = \mathbb{N} = \{0, 1, \ldots\}$ the set of "places" (X the set of "letters")
- $(x_i|j) = (x_i, j)$ element of the product set $X \times P$
- $F[X|P]$ the polynomial ring in the (commutative) variables $(x_i|j)$
- $\langle X \rangle$ the set of words, $[X|P]$ the set of letterplace monomials

Multi-gradings

- $F\langle X \rangle_\mu :=$ space generated by the words with multidegree μ
- $F[X|P]_{\mu,\nu} :=$ space generated by the monomials with multidegree μ for the letters and ν for the places.

Example

If $m = (x_2|0)(x_0|0)(x_4|2)(x_2|4) \in F[X|P]$,
then $\mu(m) = (1, 0, 2, 0, 1)$ and $\nu(m) = (2, 0, 1, 0, 1)$.

Notations

- $P = \mathbb{N} = \{0, 1, \ldots\}$ the set of "places" (X the set of "letters")
- $(x_i|j) = (x_i, j)$ element of the product set $X \times P$
- $F[X|P]$ the polynomial ring in the (commutative) variables $(x_i|j)$
- $\langle X \rangle$ the set of words, $[X|P]$ the set of letterplace monomials

Multi-gradings

- $F\langle X \rangle_\mu :=$ space generated by the words with multidegree μ
- $F[X|P]_{\mu,\nu} :=$ space generated by the monomials with multidegree μ for the letters and ν for the places.

Example

If $m = (x_2|0)(x_0|0)(x_4|2)(x_2|4) \in F[X|P]$,
then $\mu(m) = (1, 0, 2, 0, 1)$ and $\nu(m) = (2, 0, 1, 0, 1)$.

Known ideas

Equivalent representations

- Put $V = \bigoplus_{n\in\mathbb{N}} F[X|P]_{*,1^n}$ ($*$ = any, $1^n = (1, 1, \ldots, 1)$).
- If $m = \#X$, the groups GL_m and S_n act resp. from left and right over the spaces $F\langle X\rangle_n$ and V_n.
- One has the bijection (Feynmann, Rota)

$$\iota : F\langle X\rangle \to V \quad w = x_{i_1}\cdots x_{i_n} \mapsto (x_{i_1}|0)\cdots(x_{i_n}|n-1).$$

- The restriction $\iota_n : F\langle X\rangle_n \to V_n$ is a module isomorphism.

Clearly $\iota : F\langle X\rangle \to V \subset F[X|P]$ is not a ring homomorphism.

New Ideas

Act by shift

The monoid $\mathbb{N}$ has a faithful action over the graded algebra $F[X|P]$ (total degree). For each variable $(x_i|j)$ and $s \in \mathbb{N}$ we put

$$s \cdot (x_i|j) := (x_i|s + j)$$

In other words, one has a monomorphism $\mathbb{N} \to \mathrm{End}(F[X|P])$.

Decompose by shift

- If $m = (x_{i_1}|j_1) \cdots (x_{i_n}|j_n) \in [X|P]$ we define the **shift of** m to be the integer $\mathrm{sh}(m) = \min\{j_1, \ldots, j_n\}$.
- $F[X|P]^{(s)}$ space, gen. by the monomials with shift $s \in \mathbb{N}$.
- One has

$$F[X|P] = \bigoplus_{s \in \mathbb{N}} F[X|P]^{(s)} \quad s \cdot F[X|P]^{(t)} = F[X|P]^{(s+t)}$$

New Ideas

Act by shift

The monoid $\mathbb{N}$ has a faithful action over the graded algebra $F[X|P]$ (total degree). For each variable $(x_i|j)$ and $s \in \mathbb{N}$ we put

$$s \cdot (x_i|j) := (x_i|s+j)$$

In other words, one has a monomorphism $\mathbb{N} \to \mathrm{End}(F[X|P])$.

Decompose by shift

- If $m = (x_{i_1}|j_1) \cdots (x_{i_n}|j_n) \in [X|P]$ we define the **shift of** m to be the integer $\mathrm{sh}(m) = \min\{j_1, \ldots, j_n\}$.
- $F[X|P]^{(s)}$ space, gen. by the monomials with shift $s \in \mathbb{N}$.
- One has

$$F[X|P] = \bigoplus_{s \in \mathbb{N}} F[X|P]^{(s)} \quad s \cdot F[X|P]^{(t)} = F[X|P]^{(s+t)}$$

Example

If $m = (x_2|2)(x_1|2)(x_2|4)$ then $\mathrm{sh}(m) = 2$.
Moreover $3 \cdot m = (x_2|5)(x_1|5)(x_2|7)$ and $\mathrm{sh}(3 \cdot m) = 3 + sh(m) = 5$.

Definition

An ideal $J \subset F[X|P]$ is called

- **place-multigraded**, if $J = \sum_\nu J_{*,\nu}$ with $J_{*,\nu} = J \cap F[X|P]_{*,\nu}$
- **shift-decomposable**, if $J = \sum_s J^{(s)}$ with $J^{(s)} = J \cap F[X|P]^{(s)}$.

A place-multigraded ideal is also graded and shift-decomposable.

Definition

A shift-decomposable ideal $J \subset F[X|P]$ is called **shift-invariant** if $s \cdot J^{(t)} = J^{(s+t)}$ for all $s, t \in \mathbb{N}$.

Example

If $m = (x_2|2)(x_1|2)(x_2|4)$ then $\mathrm{sh}(m) = 2$.
Moreover $3 \cdot m = (x_2|5)(x_1|5)(x_2|7)$ and $\mathrm{sh}(3 \cdot m) = 3 + sh(m) = 5$.

Definition

An ideal $J \subset F[X|P]$ is called

- **place-multigraded**, if $J = \sum_\nu J_{*,\nu}$ with $J_{*,\nu} = J \cap F[X|P]_{*,\nu}$
- **shift-decomposable**, if $J = \sum_s J^{(s)}$ with $J^{(s)} = J \cap F[X|P]^{(s)}$.

A place-multigraded ideal is also graded and shift-decomposable.

Definition

A shift-decomposable ideal $J \subset F[X|P]$ is called **shift-invariant** if $s \cdot J^{(t)} = J^{(s+t)}$ for all $s, t \in \mathbb{N}$.

Proposition

Let J be an ideal of $F[X|P]$. We put $I = \iota^{-1}(J \cap V) \subset F\langle X \rangle$.

- *If J is shift-invariant, then I is a left ideal.*
- *If J is place-multigraded, then I is graded right ideal.*

Proof.

Assume J is shift-invariant and let $f \in I, w \in \langle X \rangle$. Denote $g = \iota(f) \in J \cap V$ and $m = \iota(w)$. If $\deg(w) = s$ we have clearly $\iota(wf) = m(s \cdot g) \in J \cap V$ and hence $wf \in I$.

Suppose now that J is place-multigraded and hence graded. Since V is a graded subspace, it follows that $J \cap V = \sum_d (J_d \cap V)$ and hence, setting $I_d = \iota^{-1}(J_d \cap V)$ we obtain $I = \sum_d I_d$. Let $f \in I_d$ that is $\iota(f) = g \in J_d \cap V$. For all $w \in \langle X \rangle$ we have that $\iota(fw) = g(d \cdot m) \in J \cap V$ that is $fw \in I$. $\square$

Proposition

Let J be an ideal of $F[X|P]$. We put $I = \iota^{-1}(J \cap V) \subset F\langle X \rangle$.

- *If J is shift-invariant, then I is a left ideal.*
- *If J is place-multigraded, then I is graded right ideal.*

Proof.

Assume J is shift-invariant and let $f \in I, w \in \langle X \rangle$. Denote $g = \iota(f) \in J \cap V$ and $m = \iota(w)$. If $\deg(w) = s$ we have clearly $\iota(wf) = m(s \cdot g) \in J \cap V$ and hence $wf \in I$.

Suppose now that J is place-multigraded and hence graded. Since V is a graded subspace, it follows that $J \cap V = \sum_d (J_d \cap V)$ and hence, setting $I_d = \iota^{-1}(J_d \cap V)$ we obtain $I = \sum_d I_d$. Let $f \in I_d$ that is $\iota(f) = g \in J_d \cap V$. For all $w \in \langle X \rangle$ we have that $\iota(fw) = g(d \cdot m) \in J \cap V$ that is $fw \in I$. □

Proposition

Let I be a left ideal of $F\langle X\rangle$ and put $I' = \iota(I)$. Define J the ideal of $F[X|P]$ generated by $\bigcup_{s\in\mathbb{N}} s\cdot I'$. Then J is a shift-invariant ideal. Moreover, if I is graded then J is place-multigraded.

Example

If $f = 2x_2x_3x_1 - 3x_3x_1x_3 \in I$, all the following polynomials belong to J:

$$\begin{aligned}
\iota(f) &= 2(x_2|1)(x_3|2)(x_1|3) - 3(x_3|1)(x_1|2)(x_3|3)\\
1\cdot\iota(f) &= 2(x_2|2)(x_3|3)(x_1|4) - 3(x_3|2)(x_1|3)(x_3|4)\\
2\cdot\iota(f) &= 2(x_2|3)(x_3|4)(x_1|5) - 3(x_3|3)(x_1|4)(x_3|5)\\
&\text{etc}
\end{aligned}$$

Proposition

Let I be a left ideal of $F\langle X\rangle$ and put $I' = \iota(I)$. Define J the ideal of $F[X|P]$ generated by $\bigcup_{s\in\mathbb{N}} s \cdot I'$. Then J is a shift-invariant ideal. Moreover, if I is graded then J is place-multigraded.

Example

If $f = 2x_2x_3x_1 - 3x_3x_1x_3 \in I$, all the following polynomials belong to J:

$$
\begin{array}{rcl}
\iota(f) & = & 2(x_2|1)(x_3|2)(x_1|3) - 3(x_3|1)(x_1|2)(x_3|3) \\
1 \cdot \iota(f) & = & 2(x_2|2)(x_3|3)(x_1|4) - 3(x_3|2)(x_1|3)(x_3|4) \\
2 \cdot \iota(f) & = & 2(x_2|3)(x_3|4)(x_1|5) - 3(x_3|3)(x_1|4)(x_3|5) \\
\text{etc} & &
\end{array}
$$

Definition

- Let $I \subset F\langle X\rangle$ be a graded two-sided ideal. Denote $\tilde{\iota}(I)$ the shift-invariant ideal $J \subset F[X|P]$ generated by $\bigcup_{s\in\mathbb{N}} s\cdot\iota(I)$ and call J the **letterplace analogue** of the ideal I.
- Let $J \subset F[X|P]$ be a shift-invariant place-multigraded ideal. Denote $\tilde{\iota}^{-1}(J)$ the graded two-sided ideal $I = \iota^{-1}(J \cap V) \subset F\langle X\rangle$.

Theorem

The following inclusions hold:

- $\tilde{\iota}^{-1}(\tilde{\iota}(I)) = I,\ \ \tilde{\iota}(\tilde{\iota}^{-1}(J)) \subseteq J,$
- $\tilde{\iota}(\tilde{\iota}^{-1}(J)) = J$ *if and only if* J *is generated by* $\bigcup_{s\in\mathbb{N}} s\cdot(J\cap V)$.

Definition

- Let $I \subset F\langle X\rangle$ be a graded two-sided ideal. Denote $\tilde{\iota}(I)$ the shift-invariant ideal $J \subset F[X|P]$ generated by $\bigcup_{s\in\mathbb{N}} s\cdot\iota(I)$ and call J the **letterplace analogue** of the ideal I.
- Let $J \subset F[X|P]$ be a shift-invariant place-multigraded ideal. Denote $\tilde{\iota}^{-1}(J)$ the graded two-sided ideal $I = \iota^{-1}(J\cap V) \subset F\langle X\rangle$.

Theorem

The following inclusions hold:

- $\tilde{\iota}^{-1}(\tilde{\iota}(I)) = I, \ \tilde{\iota}(\tilde{\iota}^{-1}(J)) \subseteq J,$
- $\tilde{\iota}(\tilde{\iota}^{-1}(J)) = J$ *if and only if* J *is generated by* $\bigcup_{s\in\mathbb{N}} s\cdot(J\cap V)$.

Definition

A graded ideal J of $F[X|P]$ is called a **letterplace ideal**, if J is generated by $\bigcup_{s\in\mathbb{N}} s\cdot(J\cap V)$.
In this case J is shift-invariant and place-multigraded.

We obtain finally

Corollary

The map $\iota : F\langle X\rangle \to V$ induces a 1-to-1 correspondence $\tilde{\iota}$ between graded two-sided ideals I of the free associative algebra $F\langle X\rangle$ and letterplace ideals J of the polynomial ring $F[X|P]$.

Definition

A graded ideal J of $F[X|P]$ is called a **letterplace ideal**, if J is generated by $\bigcup_{s\in\mathbb{N}} s\cdot(J\cap V)$.
In this case J is shift-invariant and place-multigraded.

We obtain finally

Corollary

The map $\iota : F\langle X\rangle \to V$ induces a 1-to-1 correspondence $\tilde{\iota}$ between graded two-sided ideals I of the free associative algebra $F\langle X\rangle$ and letterplace ideals J of the polynomial ring $F[X|P]$.

How generating sets behave under the ideal correspondence $\tilde{\iota}$?

Definition

Let J be a letterplace ideal of $F[X|P]$ and $H \subset F[X|P]$. We say that H is a **letterplace basis** of J if $H \subset J \cap V$, H homogeneous and $\bigcup_{s \in \mathbb{N}} s \cdot H$ is a generating set of the ideal J.

Proposition

Let I be a graded two-sided ideal of $F\langle X \rangle$ and put $J = \tilde{\iota}(I)$. Moreover, let $G \subset I$, G homogeneous and define $H = \iota(G) \subset J \cap V$. Then G is a generating set of I as two-sided ideal if and only if H is a letterplace basis of J.

How generating sets behave under the ideal correspondence $\tilde{\iota}$?

Definition

Let J be a letterplace ideal of $F[X|P]$ and $H \subset F[X|P]$. We say that H is a **letterplace basis** of J if $H \subset J \cap V$, H homogeneous and $\bigcup_{s \in \mathbb{N}} s \cdot H$ is a generating set of the ideal J.

Proposition

Let I be a graded two-sided ideal of $F\langle X \rangle$ and put $J = \tilde{\iota}(I)$. Moreover, let $G \subset I$, G homogeneous and define $H = \iota(G) \subset J \cap V$. Then G is a generating set of I as two-sided ideal if and only if H is a letterplace basis of J.

Now, we enter the realm of Gröbner bases.

- $A = F\langle X \rangle$ or $F[Y]$ ($Y = X \times P$).
- M is the monoid of all monomials of A.

A **term-ordering** of A is a total order on M which is a multiplicatively compatible well-ordering. Precisely one has:

(i) either $u \prec v$ or $v \prec u$, for any $u, v \in M, u \neq v$;

(ii) if $u \prec v$ then $wu \prec wv$ and $uw \prec vw$, for all $u, v, w \in M$;

(iii) every non-empty subset of M has a minimal element.

Remark

Even if the number of variables of the polynomial algebra A is infinite, there exist term-orderings. By Higman's lemma, any multiplicatively compatible total ordering on M, such that $1 \prec x_0 \prec x_1 \prec \ldots$, is a term-ordering.

Now, we enter the realm of Gröbner bases.

- $A = F\langle X\rangle$ or $F[Y]$ ($Y = X \times P$).
- M is the monoid of all monomials of A.

A **term-ordering** of A is a total order on M which is a multiplicatively compatible well-ordering. Precisely one has:

(i) either $u \prec v$ or $v \prec u$, for any $u, v \in M, u \neq v$;

(ii) if $u \prec v$ then $wu \prec wv$ and $uw \prec vw$, for all $u, v, w \in M$;

(iii) every non-empty subset of M has a minimal element.

Remark

Even if the number of variables of the polynomial algebra A is infinite, there exist term-orderings. By Higman's lemma, any multiplicatively compatible total ordering on M, such that $1 \prec x_0 \prec x_1 \prec \ldots$*, is a term-ordering.*

Notations

- $\mathrm{lm}(f)$ the leading (greatest) monomial of $f \in F\langle X\rangle, f \neq 0$
- $\mathrm{lm}(G) = \{\mathrm{lm}(g) \mid g \in G, g \neq 0\}$ with $G \subset F\langle X\rangle$
- $\mathrm{LM}(G)$ the two-sided ideal generated by $\mathrm{lm}(G)$

Definition

Let I be a two-sided ideal of $F\langle X\rangle$ and $G \subset I$. If $\mathrm{lm}(G)$ is a basis of $\mathrm{LM}(I)$ then G is called a **Gröbner basis** of I. In other words, for all $f \in I, f \neq 0$ there are $w_1, w_2 \in \langle X\rangle, g \in G \setminus \{0\}$ such that $\mathrm{lm}(f) = w_1 \mathrm{lm}(g) w_2$.

Notations

- $\mathrm{lm}(f)$ the leading (greatest) monomial of $f \in F\langle X\rangle, f \neq 0$
- $\mathrm{lm}(G) = \{\mathrm{lm}(g) \,|\, g \in G, g \neq 0\}$ with $G \subset F\langle X\rangle$
- $\mathrm{LM}(G)$ the two-sided ideal generated by $\mathrm{lm}(G)$

Definition

Let I be a two-sided ideal of $F\langle X\rangle$ and $G \subset I$. If $\mathrm{lm}(G)$ is a basis of $\mathrm{LM}(I)$ then G is called a **Gröbner basis** of I. In other words, for all $f \in I, f \neq 0$ there are $w_1, w_2 \in \langle X\rangle, g \in G \setminus \{0\}$ such that $\mathrm{lm}(f) = w_1 \mathrm{lm}(g) w_2$.

In a similar way, the notion of Gröbner basis is defined for an ideal of the commutative polynomial ring $F[Y]$.

Definition

Let $G \subset F[Y], f \in F[Y]$. By definition f has a **Gröbner representation** with respect to G if $f = 0$ or there are $f_i \in F[Y], g_i \in G$ such that $f = \sum_{i=1}^{n} f_i g_i$, with $f_i g_i = 0$ or $\mathrm{lm}(f) \succeq \mathrm{lm}(f_i)\mathrm{lm}(g_i)$ otherwise.

Proposition (Buchberger's criterion)

Let G be a basis of an ideal $J \subset F[Y]$. Then G is a Gröbner basis of J if and only if for all $f, g \in G \setminus \{0\}, f \neq g$ the S-polynomial $S(f,g)$ has a Gröbner representation with respect to G.

This criterion implies a "critical pair & completion" algorithm, transforming a generating set G_0 into a Gröbner basis G.

In a similar way, the notion of Gröbner basis is defined for an ideal of the commutative polynomial ring $F[Y]$.

Definition

Let $G \subset F[Y], f \in F[Y]$. By definition f has a **Gröbner representation** with respect to G if $f = 0$ or there are $f_i \in F[Y], g_i \in G$ such that $f = \sum_{i=1}^{n} f_i g_i$, with $f_i g_i = 0$ or $\mathrm{lm}(f) \succeq \mathrm{lm}(f_i)\mathrm{lm}(g_i)$ otherwise.

Proposition (Buchberger's criterion)

Let G be a basis of an ideal $J \subset F[Y]$. Then G is a Gröbner basis of J if and only if for all $f, g \in G \setminus \{0\}, f \neq g$ the S-polynomial $S(f, g)$ has a Gröbner representation with respect to G.

This criterion implies a ”critical pair & completion” algorithm, transforming a generating set G_0 into a Gröbner basis G.

- If Y is infinite, then the ring $F[Y]$ is not noetherian. Hence, it is not guaranteed that G_0, G are finite sets, that is the procedure to terminate in a finite number of steps.
- If G_0 is a finite basis of the ideal J, then its Gröbner basis G is contained in $F[Y']$, where Y' is the set of variables occuring in G_0. Therefore G is also finite by noetherianity of $F[Y']$.
- Assume J is graded and has a finite number of generators of degree $\leq d$. Then, the number of elements in the Gröbner basis of J of degree $\leq d$ is finite and the truncated algorithm terminates up to degree d.

- When a monoid S acts (by algebra endomorphisms) over a polynomial ring, one has the notion of **Gröbner S-basis.**
- In a paper by Drensky-LS. (JSC, 2006), such notion has been introduced and applied to the ideal $I \subset F\langle X\rangle$ defining the universal enveloping algebra of the free 2-nilpotent Lie algebra.
- The ideal I is generated by all commutators $[x_i, x_j, x_k]$ and hence it is stable under the action of all endomorphisms $x_i \to x_j$.

We introduced here this notion for the specific action of $\mathbb{N}$ over $F[X|P]$.

Definition

Let J be an ideal of $F[X|P]$ and $H \subset J$. Then H is said a **shift-basis** (resp. a **Gröbner shift-basis**) of J if $\bigcup_{s\in\mathbb{N}} s \cdot H$ is a basis (resp. a Gröbner basis) of J (then $\mathbb{N} \cdot J = J$).

If J is a letterplace ideal, then any letterplace basis of J is a shift-basis but not generally a Gröbner shift-basis of J.

- When a monoid S acts (by algebra endomorphisms) over a polynomial ring, one has the notion of **Gröbner S-basis.**
- In a paper by Drensky-LS. (JSC, 2006), such notion has been introduced and applied to the ideal $I \subset F\langle X\rangle$ defining the universal enveloping algebra of the free 2-nilpotent Lie algebra.
- The ideal I is generated by all commutators $[x_i, x_j, x_k]$ and hence it is stable under the action of all endomorphisms $x_i \to x_j$.

We introduced here this notion for the specific action of $\mathbb{N}$ over $F[X|P]$.

Definition

Let J be an ideal of $F[X|P]$ and $H \subset J$. Then H is said a **shift-basis** (resp. a **Gröbner shift-basis**) of J if $\bigcup_{s\in\mathbb{N}} s \cdot H$ is a basis (resp. a Gröbner basis) of J (then $\mathbb{N} \cdot J = J$).

If J is a letterplace ideal, then any letterplace basis of J is a shift-basis but not generally a Gröbner shift-basis of J.

Problem

- **Question**: is it possible to “reduce by symmetry” the Buchberger algorithm with respect to the action by shifting?
- **Answer**: YES, if the term-ordering is compatible with such action.

Definition

A term-ordering on $F[X|P]$ is called **shift-invariant**, when $u \prec v$ if and only if $s \cdot u \prec s \cdot v$ for any $u, v \in [X|P]$ and $s \in \mathbb{N}$. In this case, one has that $\mathrm{lm}(s \cdot f) = s \cdot \mathrm{lm}(f)$ for all $f \in F[X|P] \setminus \{0\}$ and $s \in \mathbb{N}$.

It is clear that many of the usual term-orderings are shift-invariant. From now on we assume $F[X|P]$ endowed with a shift-invariant term-ordering.

Problem

- **Question**: is it possible to "reduce by symmetry" the Buchberger algorithm with respect to the action by shifting?
- **Answer**: YES, if the term-ordering is compatible with such action.

Definition

A term-ordering on $F[X|P]$ is called **shift-invariant**, when $u \prec v$ if and only if $s \cdot u \prec s \cdot v$ for any $u, v \in [X|P]$ and $s \in \mathbb{N}$. In this case, one has that $\mathrm{lm}(s \cdot f) = s \cdot \mathrm{lm}(f)$ for all $f \in F[X|P] \setminus \{0\}$ and $s \in \mathbb{N}$.

It is clear that many of the usual term-orderings are shift-invariant. From now on we assume $F[X|P]$ endowed with a shift-invariant term-ordering.

One proves immediately:

Proposition

Let $J \subset F[X|P]$ be an ideal and $H \subset J$. We have that H is a Gröbner shift-basis of J if and only if $\mathrm{lm}(H)$ is a shift-basis of $\mathrm{LM}(J)$.

Lemma

Let $f_1, f_2 \in F[X|P] \setminus \{0\}, f_1 \neq f_2$. Then $S(s \cdot f_1, s \cdot f_2) = s \cdot S(f_1, f_2)$.

It follows that we can "reduce by symmetry" the Buchberger's criterion with respect to the shift action.

Proposition

Let H be a shift-basis of an ideal $J \subset F[X|P]$. Then H is a Gröbner shift-basis of J if and only if for all $f, g \in H \setminus \{0\}, s \in \mathbb{N}, f \neq s \cdot g$ the S-polynomial $S(f, s \cdot g)$ has a Gröbner representation with respect to $\bigcup_{t \in \mathbb{N}} t \cdot H$.

One proves immediately:

Proposition

Let $J \subset F[X|P]$ be an ideal and $H \subset J$. We have that H is a Gröbner shift-basis of J if and only if $\mathrm{lm}(H)$ is a shift-basis of $\mathrm{LM}(J)$.

Lemma

Let $f_1, f_2 \in F[X|P] \setminus \{0\}, f_1 \neq f_2$. Then $S(s \cdot f_1, s \cdot f_2) = s \cdot S(f_1, f_2)$.

It follows that we can "reduce by symmetry" the Buchberger's criterion with respect to the shift action.

Proposition

Let H be a shift-basis of an ideal $J \subset F[X|P]$. Then H is a Gröbner shift-basis of J if and only if for all $f, g \in H \setminus \{0\}, s \in \mathbb{N}, f \neq s \cdot g$ the S-polynomial $S(f, s \cdot g)$ has a Gröbner representation with respect to $\bigcup_{t \in \mathbb{N}} t \cdot H$.

Proof.

We have to prove now that $G = \bigcup_s s \cdot H$ is a Gröbner basis of J, that is for any $f, g \in H \setminus \{0\}, s, t \in \mathbb{N}, s \cdot f \neq t \cdot g$ the S-polynomial $S(s \cdot f, t \cdot g)$ has a Gröbner representation with respect to G. Assume $s \leq t$ and put $u = t - s$. By the previous lemma we have $S(s \cdot f, t \cdot g) = S(s \cdot f, s \cdot (u \cdot g)) = s \cdot S(f, u \cdot g)$. By hypothesis, the S-polynomial $S = S(f, u \cdot g)$ is zero or $S = \sum_i f_i g_i$, where $f_i \in F[X|P], g_i \in G$ and $\mathrm{lm}(S) \succeq \mathrm{lm}(f_i)\mathrm{lm}(g_i)$ for all i such that $f_i g_i \neq 0$. By acting with the shift s (algebra endomorphism), it is clear that $s \cdot S$ has also a Gröbner representation with respect to G. $\square$

By the above proposition, we obtain the correctness of the following Buchberger algorithm "reduced by symmetry".

Algorithm SGBasis

Input: H_0 a shift-basis of an ideal $J \subset F[X|P]$.
Output: H a Gröbner shift-basis of J.

$H := H_0 \setminus \{0\}$;
$P := \{(f, s \cdot g) \mid f, g \in H, s \in \mathbb{N}, f \neq s \cdot g, \mathrm{gcd}(\mathrm{lm}(f), \mathrm{lm}(s \cdot g)) \neq 1\}$;
while $P \neq \emptyset$ do
  choose $(f, s \cdot g) \in P$;
  $P := P \setminus \{(f, s \cdot g)\}$;
  $h := \textsc{Reduce}(S(f, s \cdot g), \bigcup_t t \cdot H)$;
  if $h \neq 0$ then
    $P := P \cup \{(h, s \cdot g) \mid g \in H, s \in \mathbb{N}, \mathrm{gcd}(\mathrm{lm}(h), \mathrm{lm}(s \cdot g)) \neq 1\}$;
    $P := P \cup \{(g, s \cdot h) \mid g \in H, s \in \mathbb{N}, \mathrm{gcd}(\mathrm{lm}(g), \mathrm{lm}(s \cdot h),) \neq 1\}$;
    $P := P \cup \{(h, s \cdot h) \mid s \in \mathbb{N}, \mathrm{gcd}(\mathrm{lm}(h), \mathrm{lm}(s \cdot h)) \neq 1\}$;
    $H := H \cup \{h\}$;
return H.

We want now to understand what happens when we apply this algorithm to letterplace ideals.

Notations

- If $\nu = (\nu_k)$ is a multidegree, denote $\sqrt{\nu} = (\eta_k)$ the multidegree defined as $\eta_k = 1$ if $\nu_k > 0$ and $\eta_k = 0$ otherwise.
- Define then

$$V' = \bigoplus_{\sqrt{\nu}=1^n,\, n\in\mathbb{N}} F[X|P]_{*,\nu}$$

Example

$(x_2|0)(x_0|1)(x_4|1)(x_2|2) \in V'$, but $(x_2|0)(x_0|1)(x_4|3)(x_2|4) \notin V'$.

Proposition

Let $J \subset F[X|P]$ be a letterplace ideal. There exists a Gröbner shift-basis of J contained in $\bigcup_\nu J_{*,\nu} \cap V'$.

We want now to understand what happens when we apply this algorithm to letterplace ideals.

Notations

- If $\nu = (\nu_k)$ is a multidegree, denote $\sqrt{\nu} = (\eta_k)$ the multidegree defined as $\eta_k = 1$ if $\nu_k > 0$ and $\eta_k = 0$ otherwise.
- Define then
$$V' = \bigoplus_{\sqrt{\nu}=1^n,\, n\in\mathbb{N}} F[X|P]_{*,\nu}$$

Example

$(x_2|0)(x_0|1)(x_4|1)(x_2|2) \in V'$, but $(x_2|0)(x_0|1)(x_4|3)(x_2|4) \notin V'$.

Proposition

Let $J \subset F[X|P]$ be a letterplace ideal. There exists a Gröbner shift-basis of J contained in $\bigcup_\nu J_{*,\nu} \cap V'$.

Definition

Let J be a letterplace ideal of $F[X|P]$ and $H \subset J$. We say that H is a **Gröbner letterplace basis** of J if $H \subset \bigcup_\nu J_{*,\nu} \cap V'$ and H is a Gröbner shift-basis of J.

From such a basis we want to obtain a Gröbner basis of the graded two-sided ideal $I = \tilde{\iota}^{-1}(J)$.

Definition

Fix the term-orderings $<$ on $F\langle X \rangle$ and $\prec$ on $F[X|P]$. They are called **compatible with** ι, when $v < w$ holds if and only if $\iota(v) \prec \iota(w)$ for any $v, w \in \langle X \rangle$. In this case, it follows that $\mathrm{lm}(\iota(f)) = \iota(\mathrm{lm}(f))$ for all $f \in F\langle X \rangle \setminus \{0\}$.

Definition

Let J be a letterplace ideal of $F[X|P]$ and $H \subset J$. We say that H is a **Gröbner letterplace basis** of J if $H \subset \bigcup_\nu J_{*,\nu} \cap V'$ and H is a Gröbner shift-basis of J.

From such a basis we want to obtain a Gröbner basis of the graded two-sided ideal $I = \tilde{\iota}^{-1}(J)$.

Definition

Fix the term-orderings $<$ on $F\langle X\rangle$ and $\prec$ on $F[X|P]$. They are called **compatible with** ι, when $v < w$ holds if and only if $\iota(v) \prec \iota(w)$ for any $v, w \in \langle X\rangle$. In this case, it follows that $\mathrm{lm}(\iota(f)) = \iota(\mathrm{lm}(f))$ for all $f \in F\langle X\rangle \setminus \{0\}$.

Assume: $F\langle X\rangle, F[X|P]$ are endowed with term-orderings compatible with ι and the one of $F[X|P]$ is shift-invariant.

Proposition

Let $I \subset F\langle X\rangle$ be a graded two-sided ideal and put $J = \tilde{\iota}(I)$. Moreover, let H be a Gröbner letterplace basis of J and put $G = \iota^{-1}(H \cap V)$. Then G is a Gröbner basis of I as two-sided ideal.

Remark

- *Let G_0 be a homogeneous basis of I. Then, the computation of a homogeneous Gröbner basis G of I can be done by applying the algorithm* SGBASIS *to $H_0 = \iota(G_0)$.*
- *If $H = \text{SGBASIS}(H_0)$ then $G = \iota^{-1}(H \cap V)$, and hence one is interested to compute only the elements of $H \cap V$.*
- *We prove: all such elements are obtained from S-polynomials $S(f, s \cdot g)$ where f, g are already elements of V.*

Assume: $F\langle X\rangle, F[X|P]$ are endowed with term-orderings compatible with ι and the one of $F[X|P]$ is shift-invariant.

Proposition

Let $I \subset F\langle X\rangle$ be a graded two-sided ideal and put $J = \tilde{\iota}(I)$. Moreover, let H be a Gröbner letterplace basis of J and put $G = \iota^{-1}(H \cap V)$. Then G is a Gröbner basis of I as two-sided ideal.

Remark

- *Let G_0 be a homogeneous basis of I. Then, the computation of a homogeneous Gröbner basis G of I can be done by applying the algorithm* SGBasis *to $H_0 = \iota(G_0)$.*
- *If $H = \text{SGBasis}(H_0)$ then $G = \iota^{-1}(H \cap V)$, and hence one is interested to compute only the elements of $H \cap V$.*
- *We prove: all such elements are obtained from S-polynomials $S(f, s \cdot g)$ where f, g are already elements of V.*

Algorithm NCGBasis

$H := \iota(G_0 \setminus \{0\});$
$P := \{(f, s\cdot g) \mid f, g \in H, s \in \mathbb{N}, f \neq s\cdot g, \gcd(\mathrm{lm}(f), \mathrm{lm}(s\cdot g)) \neq 1,$
$\quad \mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(s\cdot g)) \in V\};$
while $P \neq \emptyset$ do
  choose $(f, s\cdot g) \in P;$
  $P := P \setminus \{(f, s\cdot g)\};$
  $h := \textsc{Reduce}(S(f, s\cdot g), \bigcup_t t\cdot H);$
  if $h \neq 0$ then
    $P := P \cup \{(h, s\cdot g) \mid g \in H, s \in \mathbb{N}, \gcd(\mathrm{lm}(h), \mathrm{lm}(s\cdot g)) \neq 1,$
    $\quad \mathrm{lcm}(\mathrm{lm}(h), \mathrm{lm}(s\cdot g)) \in V\};$
    $P := P \cup \{(g, s\cdot h) \mid g \in H, s \in \mathbb{N}, \gcd(\mathrm{lm}(g), \mathrm{lm}(s\cdot h)) \neq 1,$
    $\quad \mathrm{lcm}(\mathrm{lm}(s\cdot h), \mathrm{lm}(g)) \in V\};$
    $P := P \cup \{(h, s\cdot h) \mid s \in \mathbb{N}, \gcd(\mathrm{lm}(h), \mathrm{lm}(s\cdot h)) \neq 1,$
    $\quad \mathrm{lcm}(\mathrm{lm}(h), \mathrm{lm}(s\cdot h)) \in V\};$
    $H := H \cup \{h\};$
$G := \iota^{-1}(H);$
return G.

Remark

- *The termination of a procedure that computes non-commutative (homogeneous) Gröbner bases is not provided in general, even if the set of variables X and a basis G_0 of I are both finite ($F\langle X\rangle$ is not noetherian).*
- *From the viewpoint of our method, this corresponds to the fact that the set of commutative variables $X \times P$ is infinite, and the letterplace ideal $J = \tilde{\iota}(I)$ is generated by $\bigcup_{s\in\mathbb{N}} s \cdot \iota(G_0)$ which is also an infinite set.*

Proposition

Let $I \subset F\langle X\rangle$ be a graded two-sided ideal and $d > 0$ an integer. If I has a finite number of homogeneous generators of degree $\leq d$ then the algorithm NCGBASIS *computes in a finite number of steps all elements of degree $\leq d$ of a homogeneous Gröbner basis of I.*

Proof.

Consider the elements $f, g \in H \subset V$ at the current step. If both these polynomials have degree $\leq d$ then the condition $\gcd(\mathrm{lm}(h), \mathrm{lm}(s \cdot g)) \neq 1$ implies that $s \leq d - 1$. It follows that the computation actually runs over the variables set $X' \times \{0, \ldots, d-1\}$, where X' is the finite set of variables occurring in the generators of I of degree $\leq d$. By noetherianity of the ring $F[X' \times \{0, \ldots, d-1\}]$ we conclude that the truncated procedure, up to degree d, stops after a finite number of steps. □

This generalizes a well-known result about solvability of word problems for finitely presented homogeneous associative algebras.

Proposition

Let $I \subset F\langle X \rangle$ be a graded two-sided ideal and $d > 0$ an integer. If I has a finite number of homogeneous generators of degree $\leq d$ then the algorithm NCGBASIS *computes in a finite number of steps all elements of degree $\leq d$ of a homogeneous Gröbner basis of I.*

Proof.

Consider the elements $f, g \in H \subset V$ at the current step. If both these polynomials have degree $\leq d$ then the condition $\gcd(\mathrm{lm}(h), \mathrm{lm}(s \cdot g)) \neq 1$ implies that $s \leq d - 1$. It follows that the computation actually runs over the variables set $X' \times \{0, \ldots, d-1\}$, where X' is the finite set of variables occurring in the generators of I of degree $\leq d$. By noetherianity of the ring $F[X' \times \{0, \ldots, d-1\}]$ we conclude that the truncated procedure, up to degree d, stops after a finite number of steps. □

This generalizes a well-known result about solvability of word problems for finitely presented homogeneous associative algebras.

- We have developed an implementation of the letterplace algorithm in the computer algebra system

  Singular: www.singular.uni-kl.de.

- Even is the implementation is still experimental, the comparisons with the best implementations of non-commutative Gröbner bases (classic algorithm) are very encouraging. They show that, in addition to the interesting feature to be portable in any commutative computer algebra system, the proposed method is really feasible.

Example	BERG	GBNP	SING	#In	#Out
nilp3-6	**0:01**	0:07	**0:01**	192	110
nilp3-10	0:23	1:49	**0:03**	192	110
nilp4-6	1:22	1:12	**0:14**	2500	891
nilp4-7	**1:24**	7:32	1:40	2500	1238
nilp4s-8	**13:52**	1h:14:54	0:57†	1200	1415
nilp4s-9	**5h:50:26**	40h:23:19	1:32†	1200	1415
metab5-10	**0:20**	13:58††	0:22	360	76
metab5-11	27:23	14:42†	**1:11**	360	113
metab5s-10	**0:32**	1h:42:43††	0:34	45	76
metab5s-11	27:33	25:27†	**2:05**	45	113
tri4-7	0:48	18h†	**0:08**	12240	672
tri4s-7	0:40	3:37	**0:07**	3060	672
ufn3-6	0:31	1:43	**0:23**	125	1065
ufn3-8	**2:18**	9:33	2:20	125	1763
ufn3-10	**5:24**	20:37	3:25†	125	2446

Examples on Serre's relations

Example	BERG	GBNP	SING	#In	#Out
`ser-f4-15`	16:05	1h:25:48	**0:08**	9	43
`ser-e6-12`	0:49	5:39	**0:07**	20	76
`ser-e6-13`	2:36	14:52	**0:14**	20	79
`ser-ha11-10`	0:04	7:82	**0:01**	5	33
`ser-ha11-15`	1h:03:21	4h:06:00	**1:58**	5	112
`ser-eha112-12`	0:56	3:44	**0:37**	5	126
`ser-eha112-13`	1h:12:50	34:53	**4:08**	5	174

Open problems

- Extend the letterplace method to the computation of non-homogeneous ideals (in preparation).
- one-sided Gröbner bases over fin. pres. algebras
- one- and two-sided syzygies and resolutions over f.p.a.
- Hilbert functions and dimensions
- homological algebra
- For ideals that are invariants under the actions of (semi)groups, algebras, etc, to integrate the methods of representation theory to Gröbner bases techniques.
- Do there exist letterplace analogues of Lie ideals? Of Gröbner-Shirshov bases?