Algorithms and Strategies for Computing Minimal Free Resolutions

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The methods for computing (minimal) free resolutions of ideals and modules over polynomial rings, can be divided in two classes.

(1) Hilbert's approach

Algorithms that are based on the **Hilbert's syzygy computation** for graded modules. If $f_1, \ldots, f_n$ are the homogeneous generators of a module and $(g_1, \ldots, g_n)$ is the "unknown" homogeneous syzygy of some degree, by imposing that the form $\sum_i g_i f_i$ is equal to zero, we get a large linear system in the unknown coefficients of the polynomials g_i.

Usually, the Hilbert's approach is considered unfeasible since the size of such linear systems have a fast growth with respect to the number of variables and the degree of syzygies.

In the paper [3] we faced such problem and proved that, at least for the computation of the **linear strand (linear syzygies)**, the use of the Koszul syzygies together with some "sparse linear algebra" techniques improves much these computations.

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(2) Gröbner bases methods

Algorithms that use the computation of a **Gröbner basis** $G = \{f_i\}$ to generate syzygies. If $S(f_h, f_k) = t_h f_h - t_k f_k$ is a S-polynomial and $t_h f_h - t_k f_k = \sum_i g_i f_i$ is the expression obtained by keeping track of the reduction of $S(f_h, f_k)$ to zero (**Buchberger criterion**), then one obtains the syzygy $(g_1, \ldots, g_h - t_h, \ldots, g_k + t_k, \ldots, g_n)$. The collection of these syzygies gives a basis for $\mathrm{syz}(G)$.

The algorithms of the class (2) can be further divided into two subclasses.

(2a) Step-by-step minimalization

This approach is essentially defined only in the graded case and it leads to a **direct computation of a minimal resolution** module by module. It consists in computing **iteratively** a Gröbner basis G of a module M_k in the resolution, in order to obtain a basis G' for $\mathrm{syz}(G)$ and then to transform G into a **minimal basis** H of M_k and G' into a basis of $M_{k+1} = \mathrm{syz}(H)$.

This method is quite good, but it has the drawback that Gröbner basis computations has to be restarted from scratch at each step. Moreover, the "Gröbner structure" of the resolution is lost.

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(2b) Schreyer approach

Algorithms that are based on an idea due to F. Schreyer which consists in defining **term-orderings induced by maps.** In this way, the syzygies obtained by the computation of a Gröbner basis G are themselves a Gröbner basis of the syzyzy module $\mathrm{syz}(G)$.
This method leads to the notion of **Schreyer resolution** that is a free resolution where any basis in the complex is a Gröbner one. This is a generalization to free resolutions of the notion of Gröbner basis of a module.

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Orderings induced by maps

Let $M = F_0/M_0$ be a module over a ring $R = K[x_1, \ldots, x_n]/J$ and let

$$\Phi : \cdots \longrightarrow F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

be a sequence of R-homomorphisms between free modules. If R, M are graded, then we assume that also the homomorphisms are such. If τ is a sequence of term-orderings of the free modules F_i, we say that τ is an **ordering induced by** Φ if

$$s \cdot \mathrm{lt}(\varphi_i(e_h)) < t \cdot \mathrm{lt}(\varphi_i(e_k)) \Rightarrow s \cdot e_h < t \cdot e_k$$

for all terms s, t and e_h, e_k elements of the canonical basis $\mathcal{E}_i$ of F_i.

We denote by $\mathrm{init}(\Phi)$ the sequence of homomorphisms:

$$\mathrm{init}(\Phi) : \cdots \longrightarrow F_l \xrightarrow{\xi_l} F_{l-1} \xrightarrow{\xi_{l-1}} \cdots \xrightarrow{\xi_2} F_1 \xrightarrow{\xi_1} F_0$$

such that $\xi_i(e_k) = \mathrm{lt}(\varphi_i(e_k))$, for all elements $e_k \in \mathcal{E}_i$. Then, the elements $\xi_i(e_k)$ are all monomials.

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Schreyer resolution

A Schreyer resolution of an R-module M is by definition an exact sequence:

$$\Phi : \cdots \longrightarrow F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

together with a **term-ordering** τ **induced by** Φ such that $\varphi_i(\mathcal{E}_i)$ is a **minimal Gröbner basis** of $\ker(\varphi_{i-1})$.

The notion of initial module can be generalized to the following one.

Schreyer frame

A Schreyer frame of a module M is a sequence of homomorphisms:

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where $\xi_i(\mathcal{E}_i)$ is a **minimal basis of the initial module** $\mathrm{init}(\ker(\xi_{i-1})) = (\mathrm{lt}(f) \mid f \in \ker(\xi_{i-1}))$. Hence, the elements $\xi_i(e_k)$ are all monomials.

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The following propositions show that the notion of Schreyer resolution is essentially a generalization to free resolutions of the notion of (minimal) Gröbner basis of a module.

Proposition

Let F be a free module endowed with a term-ordering and $G \subset F$ a subset. Then G is **minimal Gröbner basis** *if and only if $\mathrm{init}(G) = \{\mathrm{lt}(f) \mid f \in G\}$ is a* **minimal basis** *of $\mathrm{init}(M)$, where M is the module generated by G.*

Proposition

Let Φ be a complex endowed with an induced ordering. Then Φ is a **Schreyer resolution** *if and only if $\mathrm{init}(\Phi)$ is a* **Schreyer frame**.

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Let Φ be a complex endowed with an induced ordering. Then Φ is a **Schreyer resolution** *if and only if* $\mathrm{init}(\Phi)$ *is a* **Schreyer frame***.*

The last proposition is based on the following result.

Theorem (Schreyer)

Let $\varphi : F' \to F$ be the homomorphism $e_i \mapsto f_i$. Denote $\xi : F' \to F$ the map $e_i \mapsto \mathrm{lt}(f_i)$. Assume F' is endowed with a term-ordering induced by φ. Then, a Gröbner basis of $\ker(\xi)$ is given by the elements $\sigma_{hk} = t_h e_h - t_k e_k$ such that $\xi(\sigma_{hk}) = 0$ is the S-polynomial between $\mathrm{lt}(f_h), \mathrm{lt}(f_k)$. Moreover, one has that $\mathrm{init}(\ker(\varphi)) = \mathrm{init}(\ker(\xi))$.

This implies also that the Schreyer frame of $M = F_0/M_0$ can be constructed **at once** starting from a Gröbner basis of M_0.

Main problem

How to lift efficiently the frame to a Schreyer resolution and obtain a minimal resolution of M from this, in the graded case.

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Denote $\mathcal{B}_i = \xi_i(\mathcal{E}_i)$ and $\mathcal{C}_i = \varphi_i(\mathcal{E}_i)$ where $\mathcal{E}_i$ is the canonical basis of F_i. **By the Schreyer theorem**, an element of the frame $te_k \in \mathcal{B}_i$ can be lifted to a syzygy $g \in \mathcal{C}_i$ by associating to it an S-polynomial $S = S(f_h, f_k)$, where $f_h, f_k \in \mathcal{C}_{i-1}$, and by reducing S with respect to $\mathcal{C}_{i-1}$. Then, the syzygy g is just the trace of such reduction.

The classical strategy

This approach consists in constructing the Gröbner bases $\mathcal{C}_i$ of the Schreyer resolution from the bases $\mathcal{B}_i$ of the frame iteratively, that is **one computes $\mathcal{C}_1$ first, then $\mathcal{C}_2$ and so on.** This implies that **the corresponding S-polynomial reductions all lead to zero.** In other words, there is a 1-1 correspondence between frame elements and syzygies of the resolution. To obtain a minimal resolution from this is complicated, because one has **no apriori knowledge of what elements in the bases $\mathcal{C}_i$ are minimal generators.**

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A step forward is to understand that this method is a **generalization to resolutions of the Buchberger's algorithm for computing Gröbner bases of modules.** Then, the classical approach to Schreyer resolutions consists essentially in a **special selection strategy** of the elements in the frame (pairs, S-polynomials), that is an **ordering** on the set $\mathcal{B} = \bigcup_i \mathcal{B}_i$ such that $\mathcal{B}_1 < \mathcal{B}_2 < \ldots$. Hence, **other selection strategies are also possible!**

General strategies

By a general strategy, the S-polynomial corresponding to an element of $\mathcal{B}_i$ may reduce eventually to an element $f \neq 0$. In this case, **by a single reduction one obtains two elements**, the remainder $f \in \mathcal{C}_{i-1}$ and the syzygy $g \in \mathcal{C}_i$ obtained by tracing the reduction. Clearly, the leading term $\mathrm{lt}(f)$ is equal to an element of $\mathcal{B}_{i-1}$ which becomes hence **useless and can be discarded.**
Moreover, it is clear that element f **is not a minimal syzygy** since it is obtained as a combination of previous elements in C_{i-1}. Then, f and g can be **immediately discarded** as elements of a minimal resolution.

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Resolution

```
INPUT: C̄_1, a minimal Gröbner basis of M_0 ⊂ F_0.
C_i, H_i := ∅, ∀i
while B ≠ ∅ do
   m := min[B]; B := B \ {m}
   i := lev(m)
   if i = 1 then
      g := the element of C̄_1 s.t. lt(g) = m
      C_1 := C_1 ∪ {g}; H_1 := H_1 ∪ {g}
   else
      (f, g) := Reduce[m, C_{i-1}]
      C_i := C_i ∪ {g}
      if f ≠ 0 then
         C_{i-1} := C_{i-1} ∪ {f}
         B := B \ {lt(f)}      ## discard a useless S-poly!!
      else
         H_i := H_i ∪ {g}
return C_i, H_i, ∀i
```

In general, the sets $\mathcal{H}_i$ define a free resolution of M **contained** in the Schreyer one (up to some back-substitutions).

The proposed strategy

From a computational viewpoint, it is clear that efficient strategies are the ones that maximize the number of elements in the frame that lead to non-zero S-polynomial reductions. In [1, 2], for the graded case we proposed the following strategy

$$\deg(m) < \deg(n) \text{ or } \deg(m) = \deg(n) \text{ and } \mathrm{lev}(m) > \mathrm{lev}(n)$$
$$\text{implies that } m < n, \text{ for any } m, n \in \mathcal{B} = \bigcup_i \mathcal{B}_i.$$

In other words, **for the same (induced) degree one prefers to perform higher syzygies computations**. This is exactly the **opposite** of the classical strategy.

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Theorem

Let $M = F_0/M_0$ be a graded module. The proposed algorithm defines the following short exact sequence for the Schreyer resolution $\Phi(\{\mathcal{C}_i\})$ of M:

$$0 \to \Phi' \to \Phi \to \Phi'' \to 0$$

where Φ' is a **trivial complex** *defined by the syzygies associated to reductions of S-polynomials with* **remainder different from zero***, and $\Phi''(\{\mathcal{H}_i\})$ is a* **minimal free resolution** *of M obtained by the syzygies corresponding to reductions with* **remainder equal to zero***.*

Note that the Schreyer resolution and its frame is a **combinatorial structure** based on least common multiples (S-polynomials). The theorem explains **how a minimal resolution sits in it!**. If one can predict S-polynomial reductions in an astract setting, this is also a method to define minimal free resolutions.

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Summing up

1. **The proposed strategy is computationally optimal**: it computes the Schreyer resolution with the minimal number of S-polynomials to be reduced. In fact, this strategy **minimize** the number of S-polynomials with **zero remainder** (the associated syzygies form in fact a minimal resolution). In other words, the strategy **maximizes** the number of S-polynomials with **non-zero remainder** which are **the most efficient ones** since they contribute to the resolution with two elements.
2. **The Betti numbers of M are known without actually computing a minimal free resolution.** It is sufficient to count in any degree the number of S-polynomials that reduce to zero (the sets $\mathcal{H}_i$ defines a minimal resolution, but up to some back-substitutions that can be also avoided).

This means that the **classic strategy** for Schreyer algorithm is "**completely wrong**" since it is inefficient and **destroys the "natural way" to find a minimal free resolution inside a Schreyer one.**

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In collaboration with Mike Stillman these algorithms have been implemented in the kernel of the computer algebra system **Macaulay2**, and later, by other people, in many computer algebra systems (**Singular**, etc) that provides computation of free resolutions and related homological objects.

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3.	15.	1.5
4.	15.	2.0
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Clearly, **the speedup is as higher as the Schreyer resolution diverges from a minimal one.** Beeing combinatorial, usually the Schreyer resolution is quite redundant.

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What is next?

The proposed algorithms have been generalized to modules on **algebras of differential operators** (Weyl algebras) by O. Toshinori e N. Takayama [4]. My interest for algorithmic methods for free resolutions has recently started again because I'm interested in extending them to (bi)modules over any quotient $K\langle x_1, \ldots, x_n \rangle / J$ of the free associative algebra.

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