Introduction to Singularities and D-module Theory

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Riemann-Hilbert Correspondence

$$\begin{array}{ccc}\{\text{integrable connections}\} & \longleftrightarrow & \{\text{Local systems}\} \\ \big\downarrow\hookrightarrow & & \\ \{D\text{-modules}\} & & \end{array}$$

To understand the inclusion we need:

- $\mathrm{Char}(M)$
- $\dim(M) \longrightarrow$ Holonomy

The Weyl Algebra

Let $W := \mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ be the Weyl algebra.

$$x_i x_j = x_j x_i, \qquad \partial_i \partial_j = \partial_j \partial_i, \qquad \partial_i x_j = x_j \partial_i + \delta_{ij}$$

- $W = \mathbb{C}\left\langle \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}\}_{\alpha,\beta \in \mathbb{N}^n} \right\rangle$.
- Noetherian domain.
- Simple: does not have proper two-sided ideals.
- Every left ideal can be generated by just two operators.
- There exists one-generator maximal ideal.

The Bernstein Filtration

- $P = \sum_{\alpha,\beta} f_{\alpha\beta} x^\alpha \partial^\beta \in W$
- $\deg(P) := \max_{\alpha,\beta}\{|\alpha| + |\beta| : f_{\alpha\beta} \neq 0\}$
- $B_k = \{P \in W \mid \deg(P) \leq k\}$
- $\mathrm{gr}^B(W) := \bigoplus_{k\geq 0} B_k / B_{k-1}$

Theorem

$$\mathrm{gr}^B(W) \;\cong\; \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n] : \quad x_i \mapsto x_i, \quad \partial_i \mapsto \xi_i.$$

Explanation: $B_2/B_1 \;\ni\; \partial_i x_i = x_i \partial_i + 1 \;=\; x_i \partial_i \quad (1 \in B_1)$

Good Filtrations

Definition

Let M a left W-module. A family $\Gamma = \{\Gamma_i\}_{i\geq 0}$ of $\mathbb{C}$-vector subspaces of M is a filtration of M it satisties

1. $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq M$,
2. $\bigcup_{i\geq 0} \Gamma_i = M$, $\qquad \longrightarrow \qquad \mathrm{gr}^{\Gamma}(M) := \bigoplus_{\geq 0} \Gamma_i/\Gamma_{i-1}$
3. $B_i\Gamma_j \subseteq \Gamma_{i+j}$,
4. Γ_i is $\mathbb{C}$-vector space of finite dimension,
5. $\mathrm{gr}^{\Gamma}(M)$ is a finitely generated $\mathrm{gr}^{B}(W)$-module.

Characteristic Varieties

- Let $W := \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ be the Weyl algebra.
- Consider M a finitely generated left W-module.
- Take a good filtration Γ.

Then $\mathrm{gr}^\Gamma(M)$ is a finitely generated $\mathrm{gr}^B(W)$-module

Definition-Proposition

The following variety does not depend on the good filtration and is called the characteristic variety of M.

$$\mathrm{Char}(M) := V(\mathrm{Ann}_{\mathrm{gr}^F(W)}(\mathrm{gr}^\Gamma M)) \subseteq \mathbb{C}^{2n}$$

Characteristic Varieties

Here it is a list with the main properties of $\text{Char}(M)$.

- $\text{Char}(M) = \text{Char}(M/N) \cup \text{Char}(N)$.
- $\text{Char}(M)$ is involutive and homogeneous.
- $n \leq \dim \text{Char}(M) \leq 2n, \ \forall M \neq 0$.

Remark

There exists the notion of Hilbert polynomial of a D-module and its degree coincides with dimension of $\text{Char}(M)$.

Holonomic Modules

$$\boxed{\dim(M) = n}$$

- Submodules and quotients of holonomic modules are holonomic.
- Finite sums of holonomic modules are holonomic.
- Holonomic modules are cyclic.
- Inverse image preserves holonomic modules.
- The same holds for direct images.

Example

- $S := \operatorname{gr}^B W \cong \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$.
- Let $\mathbb{C}[\mathbf{x}]$ be the polynomial ring.
- Take Γ the classical filtration by the degree of a polynomial.
- As $\mathbb{C}$-vector spaces $\operatorname{gr}^\Gamma \mathbb{C}[\mathbf{x}] \cong \mathbb{C}[\mathbf{x}]$.

$$\begin{aligned} \frac{B_1}{B_0} \times \frac{\Gamma_d}{\Gamma_{d-1}} &\longrightarrow \frac{\Gamma_{d+1}}{\Gamma_d} \\ (\partial_i + B_0, f + \Gamma_{d-1}) &\longmapsto \frac{\partial f}{\partial x_i} + \Gamma_d = 0 \end{aligned}$$

- $S \times M \to M, \quad x_i \cdot f = x_i f, \quad \xi_i \cdot f = 0$.
- $\operatorname{Ann}_S(\operatorname{gr}^\Gamma \mathbb{C}[\mathbf{x}]) = S\langle \xi_1, \dots, \xi_n \rangle$.
- $\operatorname{Char}(\operatorname{gr}^\Gamma \mathbb{C}[\mathbf{x}]) = V(\xi_1, \dots, \xi_n) =: T_X^* X$.

Riemann-Hilbert Correspondence

$$\begin{array}{ccc} \{\text{Integrable connections}\} & \longleftrightarrow & \{\text{Local systems}\} \\ \updownarrow & & \\ \left\{ \begin{array}{c} D\text{-modules} + \\ \text{extra conditions ?} \end{array} \right\} & & \end{array}$$

To understand the inclusion we need:

- Char(M)
- dim(M) $\longrightarrow$ Holonomy

A Technical Remark

- This above theory can be devoloped for $\mathcal{D} = \mathcal{O}\langle \partial_1, \dots, \partial_n \rangle$.

$$\mathcal{O} = \mathbb{C}[\mathbf{x}],\ \mathbb{C}\{\mathbf{x}\},\ \mathbb{C}[[\mathbf{x}]],\ \mathcal{O}_X$$

- Main difference: take the filtration F given by the operators.

$$P = \sum_{\alpha} f_\alpha \partial^\alpha, \quad f_\alpha \in \mathcal{O}$$

$$\deg(P) = \max_{\alpha}\{|\alpha| \ : \ f_\alpha \neq 0\}$$

$$F_k = \{P \in \mathcal{D} \mid \deg(P) \leq k\}$$

- Sheaf theory: finitely generated $\longrightarrow$ coherent.

Riemann-Hilbert Correspondence

$$\left\{\begin{array}{c}\mathcal{D}\text{-modules with}\\ \mathrm{Char}(M) \cong T^*_X X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Integrable}\\ \text{connections}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Local}\\ \text{systems}\end{array}\right\}$$

$$\text{Ker } \nabla : \left\{\begin{array}{c}\mathcal{D}\text{-modules with}\\ \mathrm{Char}(M) \cong T^*_X X\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Local}\\ \text{systems}\end{array}\right\}$$

Lemma

Every $\mathbb{C}\{\mathbf{x}\}\langle\partial\rangle$-module finitely generated as a $\mathbb{C}\{\mathbf{x}\}$-module, is free over $\mathbb{C}\{\mathbf{x}\}$.

Riemann-Hilbert Correspondence

$$\left\{\begin{matrix}\mathcal{D}\text{-modules with} \\ \mathrm{Char}(M) \cong T_X^* X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Integrable} \\ \text{connections}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Local} \\ \text{systems}\end{matrix}\right\}$$

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$$\left\{\begin{matrix}\mathcal{D}\text{-modules with} \\ \mathrm{Char}(M) \cong T_X^* X\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{Holonomic} \\ \mathcal{D}\text{-modules}\end{matrix}\right\}$$

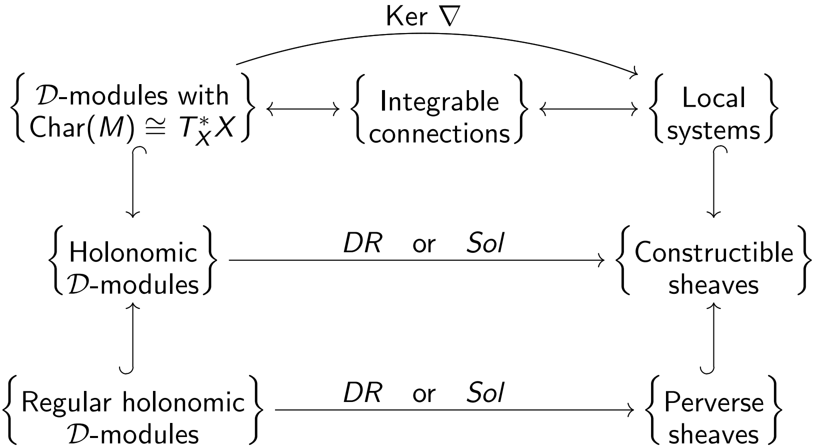
Constructible Sheaves

Definition

A sheaf $\mathcal{F}$ is constructible if the following holds.

1. There is a partition $\mathcal{P} = (X_j)_{j \in J}$ such that the restriction
$$\mathcal{F}|X_j$$
is a local system for all $j \in J$.
2. All its stalks $\mathcal{F}_x$ for $x \in X$ are finite type $\mathbb{C}$-vector spaces.

Riemann-Hilbert Correspondence



Stratification associated with $\mathbb{C}[\mathbf{x}, \frac{1}{f}]$

Theorem (Kashiwara)

$\mathbb{C}[\mathbf{x}, \frac{1}{f}]$ is a regular holonomic *D*-module.

- Walther gave an algorithm to compute such stratification, 2004.
- There is another stratification of $\mathbb{C}^n$ associated with local *b*-functions. $\longrightarrow$ Primary ideal decomposition is needed.

$$P(s) f^{s+1} = b_f(s) f^s$$

Danke für Ihre Aufmerksamkeit

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