Lattices and spherical designs.

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Definition. (Delsarte, Goethals, Seidel 1977)
A finite nonempty set

\[ X \subset S^{n-1} = \{ x \in \mathbb{R}^n \mid (x, x) = 1 \} \]

is a **spherical t-design** if for all polynomials \( f \) of degree \( \leq t \)

\[
\frac{1}{|X|} \sum_{x \in X} f(x) = \int_{S^{n-1}} f(x) d\mu(x).
\]

Since \( f_{\alpha,k} : x \mapsto (x, \alpha)^k \) with \( \alpha \in \mathbb{R}^n \) generate the space of homogeneous polynomials of degree \( k \), this is equivalent to asking

\[
\sum_{x \in X} (\alpha, x)^k = |X| \int_{S^{n-1}} (\alpha, x)^k d\mu(x) = \begin{cases} 
0 & k \text{ odd} \\
 c_k |X| (\alpha, \alpha)^{k/2} & k \text{ even}
\end{cases}
\]

where \( c_k = \prod_{j=1}^{k/2} \frac{2j-1}{n+2j-2} \) for \( k = 1, \ldots, t \).
If \( X \subset S^{n-1} \) is a spherical \( t \)-design, then

\[
|X| \geq \binom{n-1+t/2}{t/2} + \binom{n-2+t/2}{t/2-1} \quad t \text{ even}
\]
\[
|X| \geq 2^{\binom{n-1+(t-1)/2}{(t-1)/2}} \quad t \text{ odd}
\]

If equality holds, then \( X \) is called a \textbf{tight} \( t \)-design.
Tight \( t \)-designs in \( S^{n-1} \) with \( n \geq 3 \) only exist for \( t \leq 5 \) or \( t = 7, 11 \). They are classified completely for \( t \in \{1, 2, 3, 11\} \) and for \( t = 4, 5, 7 \) up to dimension \( n = 104 \).

Examples:
\( t = 1 \): \(|X| = 2^{\binom{n-1}{0}} = 2, \ X = \{x, -x\}\)
\( t = 2 \): \(|X| = n + 1, \ \text{simplex}\).
\( t = 3 \): \(|X| = 2^{\binom{n}{1}} = 2n, \ X = \{\pm e_1, \ldots, \pm e_n\}\)
for ON-basis \((e_1, \ldots, e_n)\) (cross polytope)
\( t = 7 \): \(n = 8\) and \( X = \text{Min}(E_8), \ |X| = 240\).
\( t = 7 \): \(n = 23\) and \( X = \text{Min}(O_{23}), \ |X| = 4600\).
\( t = 11 \): \(n = 24\) and \( X = \text{Min}(\wedge_{24}), \ |X| = 196560\).
Definition. A lattice $L \subset \mathbb{R}^n$ is the set of all integral linear combinations of a basis

$$L = \langle b_1, \ldots, b_n \rangle_{\mathbb{Z}}.$$

The dual lattice is

$$L^* = \{ \alpha \in \mathbb{R}^n \mid (\alpha, x) \in \mathbb{Z} \text{ for all } x \in L \}.$$

The minimum of $L$ is

$$\min(L) = \min\{(x, x) \mid 0 \neq x \in L\}$$

and we denote by

$$\text{Min}(L) := \{ x \in L \mid (x, x) = \min(L) \}$$

the set of minimal vectors in $L$.

$L$ is called a t-design lattice, if $\text{Min}(L)$ forms a spherical t-design and generates $L$. 
Fact. Let $X = -X \subset S^{n-1}$ be antipodal. Then $X$ is a $2h$-design $\iff X$ is a $2h + 1$-design.

Proof. $f$ homogeneous of odd degree, then $f(-x) = -f(x)$ hence

$$\sum_{x \in X} f(x) = 0$$

for any antipodal set $X$.

Corollary. $L$ is $2h$-design lattice $\iff L$ is $2h + 1$-design lattice.
The 5-design lattices $L$ of dimension $n \leq 12$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$L$</th>
<th>$t$</th>
<th>$\vert \text{Min}(L) \vert$</th>
<th>$\text{min}(L) \text{min}(L^*)$</th>
<th>tight</th>
</tr>
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<tr>
<td>1</td>
<td>$\mathbb{Z}$</td>
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<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$A_2$</td>
<td>5</td>
<td>6</td>
<td>$2/3$</td>
<td>yes</td>
</tr>
<tr>
<td>4</td>
<td>$D_4$</td>
<td>5</td>
<td>24</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$E_6$</td>
<td>5</td>
<td>72</td>
<td>$8/3$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$E_6^*$</td>
<td>5</td>
<td>54</td>
<td>$8/3$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$E_7$</td>
<td>5</td>
<td>126</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$E_7^*$</td>
<td>5</td>
<td>56</td>
<td>3</td>
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</tr>
<tr>
<td>8</td>
<td>$E_8$</td>
<td>7</td>
<td>240</td>
<td>4</td>
<td>yes</td>
</tr>
<tr>
<td>10</td>
<td>$K_{10}$</td>
<td>5</td>
<td>270</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$K_{10}^*$</td>
<td>5</td>
<td>240</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$K_{12}$</td>
<td>5</td>
<td>756</td>
<td>$16/3$</td>
<td></td>
</tr>
</tbody>
</table>
7-design lattices $L$ of dimension $n \leq 24$
(List complete (for $n \neq 23$))

| $n$ | $L$  | $t$ | $|\text{Min}(L)|$ | $\text{min}(L) \text{ min}(L^*)$ | tight |
|-----|------|-----|------------------|----------------------------------|-------|
| 1   | $\mathbb{Z}$ | $\infty$ | 2               | 1                               |       |
| 8   | $E_8$ | 7   | 240             | 4                               | yes   |
| 16  | $\Lambda_{16}$ | 7   | 4320            | 8                               |       |
| 23  | $O_{23}$ | 7   | 4600            | 9                               | yes   |
| 23  | $\Lambda_{23}$ | 7   | 93150           | 12                              |       |
| 24  | $\Lambda_{24}$ | 11  | 196560          | 16                              | yes   |

No $t$-design lattices are known for $t \geq 12$. 
Theorem. 5-design lattices are local maxima for the density function.

The theory of designs provides tools to classify $t$-design lattices of small dimension and hence a method to find certain nice dense lattices.

Note. Local maxima for the density function are similar to rational lattices. In particular $t$-design lattices are rational if $t \geq 4$. 
The Hermite function $\gamma$ is a positive function on the space of similarity classes of $n$-dimensional lattices defined by

$$\gamma(L) = \frac{\min(L)}{\det(L)^{1/n}}$$

with $\det(L) = \text{vol}(L)^2$ the determinant of a Gram matrix of $L$.

$\gamma(L)$ is proportional to the density of the sphere packing associated with $L$.

$\gamma$ has only finitely many local maxima which may be characterized as those lattices $L$ that are perfect and eutactic (Voronoi, Korkine, Zolotareff ~ 1900).
A lattice $L$ is **eutactic**, if there are $\lambda_x > 0$ ($x \in \text{Min}(L)$) such that

$$(\alpha, \alpha) = \sum_{x \in \text{Min}(L)} \lambda_x (x, \alpha)^2 \text{ for all } \alpha \in \mathbb{R}^n.$$ 

2-design lattices are eutactic with $\lambda_x = (c_2 |\text{Min}(L)|)^{-1}$ for all $x \in \text{Min}(L)$. 

$L$ is **perfect**, if the orthogonal projections

$$p_x : \alpha \mapsto (x, \alpha)x$$

along $x \in \text{Min}(L)$ span the space of all symmetric endomorphisms of $\mathbb{R}^n$. 

4-design lattices are perfect. 
(proof quite similar to above).
Bounds on the minimum of $t$-design lattices.

The **Bergé-Martinet-invariant** of a lattice $L$ is

$$
\gamma'(L)^2 := \gamma(L)\gamma(L^*) = \min(L)\min(L^*).
$$

**Theorem.** If $L \subset \mathbb{R}^n$ is a 4-design lattice, then $\gamma'(L)^2 \geq (n+2)/3$.
If equality holds then $(\alpha, x) \in \{0, \pm 1\}$ for all $\alpha \in \text{Min}(L^*)$ and $x \in \text{Min}(L)$.

**Proof.** $D_4(\alpha) - D_2(\alpha) = \sum_{x \in \text{Min}(L)} (x, \alpha)^2((x, \alpha)^2 - 1) = \frac{|\text{Min}(L)|}{n} \frac{(x, x)(\alpha, \alpha)^3}{n + 2} - 1$.

For $\alpha \in L^*$ this is nonnegative since $(x, \alpha) \in \mathbb{Z}$ for $x \in \text{Min}(L) \subset L$.
Choosing $\alpha \in \text{Min}(L^*)$ we find

$$(\alpha, \alpha)(x, x) = \min(L)\min(L^*) = \gamma'(L)^2 \geq (n + 2)/3$$

and "=" $\Leftrightarrow D_4(\alpha) - D_2(\alpha) = 0 \Leftrightarrow (\alpha, x) \in \{0, \pm 1\}\forall x \in \text{Min}(L)$.
General method: \( L \) a \( t \)-design lattice, \( t = 2h \), \( \alpha \in \text{Min}(L^*) \). Then

\[
\sum_{x \in \text{Min}(L)} \prod_{j=0}^{h-1} ((x, \alpha)^2 - j^2) = \frac{\left| \text{Min}(L) \right| \gamma'(L)^2}{n} P_{n,t}(\gamma'(L)^2) \geq 0
\]

where \( P_{n,t}(z) \) is a polynomial of degree \( h - 1 = t/2 - 1 \) in \( z = \gamma'(L)^2 \).

For small \( t \), the polynomials \( P_{n,t} \) are as follows:

\[
\begin{align*}
P_{n,2}(z) &= 1 \\
P_{n,4}(z) &= \frac{3}{n+2}z - 1 \\
P_{n,6}(z) &= \frac{3 \cdot 5}{(n+2)(n+4)}z^2 - 5\frac{3}{n+2}z + 4 \\
P_{n,8}(z) &= \frac{3 \cdot 5 \cdot 7}{(n+2)(n+4)(n+6)}z^3 - 14\frac{3 \cdot 5}{(n+2)(n+4)}z^2 + 49\frac{3}{n+2}z - 36
\end{align*}
\]
Remark. Let $L$ be a 6-design lattice of dimension $n > 1$. Then
\[ \gamma'(L)^2 > \frac{n+2}{3}, \]

Proof. If $\gamma'(L)^2 = \frac{n+2}{3}$ then $(\alpha, x) \in \{0, \pm 1\}$ for all $\alpha \in \text{Min}(L^*)$, $x \in \text{Min}(L)$. Hence $\frac{n+2}{3}$ is also a zero of $P_{n,6}(t)$ which implies that $5(n + 2) = 3(n + 4)$ whence $n = 1$. 
For an 8-design lattice $L \leq \mathbb{R}^n$, we have $\gamma'(L)^2 \geq b(n)$ where $b(n)$ is the real root of $P_{n,8}(z)$.

For a 12-design lattice $L \leq \mathbb{R}^n$, we have $\gamma'(L)^2 \geq c(n)$ where $c(n)$ is the real root of $P_{n,12}(z)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>26</th>
<th>32</th>
<th>36</th>
<th>40</th>
<th>48</th>
<th>50</th>
<th>66</th>
<th>72</th>
<th>80</th>
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<tbody>
<tr>
<td>$\frac{n+2}{3}$</td>
<td>9.33</td>
<td>11.33</td>
<td>12.66</td>
<td>14</td>
<td>16.66</td>
<td>17.33</td>
<td>22.66</td>
<td>24.66</td>
<td>27.33</td>
</tr>
<tr>
<td>$b(n)$</td>
<td>16</td>
<td>20.66</td>
<td>24</td>
<td>27.49</td>
<td>34.38</td>
<td>36</td>
<td>48</td>
<td>52.31</td>
<td>58.01</td>
</tr>
<tr>
<td>$c(n)$</td>
<td>17.88</td>
<td>23.35</td>
<td>27.24</td>
<td>31.16</td>
<td>38.54</td>
<td>40.29</td>
<td>53.64</td>
<td>58.53</td>
<td>64.99</td>
</tr>
</tbody>
</table>
The **Hermite constant** \( \gamma_n := \max\{\gamma(L) \mid L \leq \mathbb{R}^n\} \) satisfies

\[
\frac{1}{2\pi e} \leq \frac{\gamma_n}{n} \leq \frac{1.744}{2\pi e}
\]

The best bound for \( \gamma'(L)^2 \) currently available is

\[
\gamma'(L)^2 \leq \gamma_n^2 \sim n^2.
\]

If \( n \) tends to infinity then the real roots of \( P_{n,t} \) are approximately \( n \), yielding

\[
n \lesssim \gamma'(L)^2 \lesssim n^2
\]

for a \( t \)-design lattice \( L \leq \mathbb{R}^n \) which does not give a contradiction to the possible existence of such lattices for arbitrarily big \( t \).
Towards a classification of \( t \)-design lattices.

Let \( L \) be a \( t \)-design lattice with \( t = 2h \) even. For \( \alpha \in L^* \) and \( i \in \mathbb{N} \) put

\[
N_i(\alpha) := \{ x \in \text{Min}(L) \mid (x, \alpha) = i \}.
\]

If \( N_i(\alpha) = \emptyset \) for all \( i > h \) then

\[
|N_h(\alpha)| = \frac{1}{h \cdot (t - 1)!} \sum_{x \in \text{Min}(L)} \prod_{j=0}^{h-1} ((x, \alpha)^2 - j^2) = \]

\[
\frac{|\text{Min}(L)| (\alpha, \alpha)(x, x)}{nh \cdot (t - 1)!} P_{n,t}((\alpha, \alpha)(x, x))
\]

and there are similar expressions for \( |N_i(\alpha)| \) for \( 0 \leq i \leq h \).
Theorem. If \((\alpha, x) \in \{0, \pm 1, \ldots, \pm h\}\) for all \(x \in \text{Min}(L)\) then

\[
\sum_{x \in N_h(\alpha)} x = \frac{|N_h(\alpha)|h}{(\alpha, \alpha)}.
\]

Proof. For \(\gamma \in \mathbb{R}^n\)

\[
\sum_{x \in \text{Min}(L)} (x, \gamma)(x, \alpha) \prod_{j=1}^{h-1} ((x, \alpha)^2 - j^2) = c \sum_{x \in N_h(\alpha)} (x, \gamma) = c'(\alpha, \gamma).
\]

This implies that \(\sum_{x \in N_h(\alpha)} x = \frac{c'}{c} \alpha\) where one gets the constant by taking the scalar product with \(\alpha\).
Let $X \subset S^{n-1}$ be a spherical $t$-design then for all $k \leq t$ and all $\alpha \in \mathbb{R}^n$

$$(D_k)(\alpha) : \sum_{x \in X} (\alpha, x)^k = \begin{cases} 0 & k \text{ odd} \\ c_k |X| (\alpha, \alpha)^{k/2} & k \text{ even} \end{cases}$$

where $c_k = \prod_{j=1}^{k/2} \frac{2j-1}{n+2j-2}$.

Substituting $\alpha = \xi_1 \alpha_1 + \xi_2 \alpha_2$ in $(D_k)$ and comparing coefficients
we find that for all $i, j \in \mathbb{N}$ with $i + j \leq t$ there is a polynomial $p_{i,j}$
such that for all $\alpha, \beta \in \mathbb{R}^n$

$$(D_{ij})(\alpha, \beta) : \sum_{x \in X} (x, \alpha)^i (x, \beta)^j = p_{i,j}((\alpha, \alpha), (\beta, \beta), (\alpha, \beta))$$
Theorem. Let $L$ be a $t$-design lattice with $t = 2h$ even and let $\alpha \in L^*$ such that $(\alpha, x) \in \{0, \pm 1, \ldots, \pm (h - d)\}$ for all $x \in \text{Min}(L)$. Then the projection of $N_{h-d}(\alpha)$ onto $\alpha^\perp$ is a $2d + 1$-design.

Proof. (idea) For $j \in \{0, \ldots d\}$ consider

$$\sum_{x \in N_{h-d}(\alpha)} (x, \beta)^{2j} = c \sum_{x \in \text{Min}(L)} \prod_{i=0}^{h-d-1} ((x, \alpha)^2 - i^2)(x, \beta)^{2j}$$

which is a linear combination of the $p_{2\ell,2j}$ with $\ell + j \leq h$.

Corollary. Let $L \subset \mathbb{R}^n$ be a 6-design lattice with $\gamma'(L)^2 = 8$ scaled such that $\min(L) = 2$, $\min(L^*) = 4$. Then $n = 16$ and for all $\alpha \in \text{Min}(L^*)$

$N_2(\alpha) = \{x_i, \alpha - x_i \mid 1 \leq i \leq 15\}$ and $\langle N_2(\alpha), \alpha \rangle \cong D_{16}$. 
Proof. $\alpha \in \text{Min}(L^*)$, $x \in \text{Min}(L) \Rightarrow (\alpha, x) \in \{0, \pm 1, \pm 2\}$. Hence $P_{n,6}((\alpha, \alpha)(x,x)) = P_{n,6}(8) = 0$ which yields $n = 8$ or $n = 16$. Since $\gamma_8 = 2$ the only possibility is that $n = 16$.

For $x \in N_2(\alpha)$ let $\overline{x} := x - \frac{2}{(\alpha, \alpha)} \alpha \in \alpha^\perp$. Then for all $x, y \in N_2(\alpha)$ we get

$$(\overline{x}, y) = (x, y) - 1 = \begin{cases} 1 & x = y \\ \leq 0 & x \neq y \end{cases}$$

since $x$ and $y$ are minimal vectors of a lattice. Hence $N_2(\alpha)$ is a set of vectors of length 1 in an $n - 1$-dimensional space such that distinct vectors have non-positive inner products. Therefore $|N_2(\alpha)| \leq 2(n - 1)$. Since $N_2(\alpha)$ is a 3-design, we find that $|N_2(\alpha)| \geq 2(n - 1)$ and hence equality holds and $\overline{N_2(\alpha)}$ is a cross polytope $(\overline{x}, \overline{y}) = 0$ if $\overline{x} \neq \pm \overline{y}$.
This gives the Gramm matrix of $N_2(\alpha)$.

Gramm matrix for $(x_1, \ldots, x_{15}, \alpha)$

$$
\begin{pmatrix}
2 & 1 & \ldots & 1 & 2 \\
1 & 2 & \ldots & 1 & 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & 2 & 2 \\
2 & 2 & \ldots & 2 & 4 \\
\end{pmatrix}
$$
Theorem. The 16-dimensional 6-design-lattices are similar to the Barnes-Wall lattice.

Proof follows from

Lemma. Let $L$ be a 6-design lattice of dimension 16. Then $\gamma'(L)^2 = 8$.

together with the Corollary above:

Corollary. Let $L \subset \mathbb{R}^n$ be a 6-design lattice with $\gamma'(L)^2 = 8$ Then $n = 16$ and for all $\alpha \in \text{Min}(L^*)$

$$L \supset \langle N_2(\alpha), \alpha \rangle \cong D_{16}.$$
General strategy: If $L$ is a $2h$-design lattice and $\alpha \in \text{Min}(L^*)$ then for $1 \leq j \leq h$

$$\sum_{x \in \text{Min}(L)} (x, \alpha)^{2j} = \gamma'(L)^{2j} |\text{Min}(L)| \prod_{k=1}^{j} \frac{2k-1}{n-2k+2} \in 2\mathbb{Z}$$

$$\frac{1}{(2h-1)!} \sum_{x \in \text{Min}(L)} \prod_{j=0}^{h-1} ((x, \alpha)^2 - j^2) = \frac{|\text{Min}(L)|\gamma'(L)^2}{n} P_{n,t}(\gamma'(L)^2) \in 2\mathbb{Z}$$

and

$$\frac{1}{h!^2} \sum_{x \in \text{Min}(L)} (x, \alpha)^2 \prod_{j=1}^{(h-1)/2} ((x, \alpha)^2 - j^2)^2 = |\text{Min}(L)|p_{n,h}(\gamma'(L)^2) \in 2\mathbb{Z}$$

are even non-negative integers.
Proof of Lemma. Put \( r := \gamma'(L)^2 \). Then
\( r \in \mathbb{Q} \) and \( r \leq \gamma_1^2 \leq 9.163 \).
Bounds on kissing numbers yield \( s := |\text{Min}(L)| \leq 8160 \).
For \( \alpha \in \text{Min}(L^*) \) the sum
\[
\frac{1}{12} \sum_{x \in \text{Min}(L)} (\alpha, x)^2((\alpha, x)^2 - 1) = \frac{sr}{12 \cdot 16} \left( \frac{r}{6} - 1 \right)
\]
is a positive integer \( \leq s \frac{\gamma_1^2}{12 \cdot 16} \left( \frac{\gamma_1^2}{6} - 1 \right) \leq 0.0252s \).
Going through all possibilities for
\[
(s, a) \in [1632, 8160] \times [1, 0.0252s]
\]
using the fact that \( r \) is a positive rational solution of
\[
\frac{sr}{12n} \left( \frac{3r}{n + 2} - 1 \right) - a = 0
\]
satisfying integrality conditions above (with \( h = 3 \)) the only possibility is \( r = 8 \).