What is a block?

**Definition**

Let $A$ be a ring. Then a block $b$ in $A$ is a central primitive idempotent. ($b^2 = b \in Z(A)$ not a sum)

Also the indecomposable ring $B := bA$ is called a block.

Then (under certain finiteness assumption)

$$A = b_1 A \oplus \ldots \oplus b_n A = B_1 \oplus \ldots \oplus B_n$$

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**Blocks of finite group rings.**

- $G$ finite group
- $R$ local integral domain
- $K = \frac{R}{\pi R}$ of char. 0
- $R/\pi R = k$ field of characteristic $p$
- $K, k$ big enough.

$K^G = K_{n_1 x n_1} + \ldots + K_{n_s x n_s}$, blocks $\xi_1, \ldots, \xi_s$

\[ RG = B_1 + \ldots + B_n, \text{ blocks } b_1, \ldots, b_n \]

\[ kG = \bar{B}_1 + \ldots + \bar{B}_n, \text{ blocks } \bar{b}_1, \ldots, \bar{b}_n \]

$b$ block in $RG$ is sum $b = \sum_{t \in T(b)} \epsilon_t$. 
Central idempotents

- $\Delta_t : G \to GL_{n_t}(K)$ $(t = 1, \ldots, s)$ the irreducible representations
- $\chi_t(g) := \text{trace}(\Delta_t(g))$, so $n_t = \chi_t(1)$, irreducible character.

Blocks of $RG$.

$$\epsilon_t = \frac{\chi_t(1)}{|G|} \sum_{g \in G} \chi_t(g) g^{-1} \in KG.$$  

The blocks of $RG$ are unique sums of the $\epsilon_t$:

$$b = \sum_{t \in T(b)} \epsilon_t = \sum_{t \in T(b)} \left( \sum_{g \in G} \frac{\chi_t(1) \chi_t(g)}{|G|} \right) g^{-1} \in RG$$

if and only if $\sum_{t \in T(b)} \chi_t(1) \chi_t(g) \in |G|R$ for all $g \in G$.

The character $\chi_t$ belongs to the block $b$ if $t \in T(b)$, so $b\epsilon_t = \epsilon_t$. 
\[ G = S_4, \ p = 3, \ R = \mathbb{Z}_3, \ K = \mathbb{Q}_3, \ k = \mathbb{F}_3, \ |G| = 2^33. \]

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\[ \begin{align*}
\bullet \ b_1 & = \epsilon_4 = \frac{3}{24}(3(\cdot)^+ + (\cdot)^+ - (\cdot)(\cdot)^+ - (\cdot\cdot\cdot)^+) \in RG \\
\bullet \ b_2 & = \epsilon_5 = \frac{3}{24}(3(\cdot)^+ - (\cdot)^+ - (\cdot)(\cdot)^+ + (\cdot\cdot\cdot)^+) \in RG \\
\bullet \ b & = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1}{4}((\cdot)^+ + (\cdot)(\cdot)^+) = \frac{1}{4}\sum_{g \in V_4} g \in RG \\
\mathbb{Z}_3S_4 & = B \oplus B_1 \oplus B_2 = \mathbb{Z}_3S_3 \oplus \mathbb{Z}_3^{3 \times 3} \oplus \mathbb{Z}_3^{3 \times 3} \\
\end{align*} \]
The defect group of a block

Definition.

Let \( b = \sum_{g \in G} a_g g \in RG \) be a block. Then

\[
D(b) = \max \{ \text{Syl}_p(C_G(g)) \mid a_g \mod \pi \neq 0 \}
\]

is called the defect group of \( b \) (unique up to conjugacy).

\( G = S_4 \) above, then \( D(b_4) = D(b_5) = 1, D(b) = C_3 \)

Numerical properties

\( |G| = p^a m, \ |D(b)| = p^{d(b)}, \ d := d(b) \) is the defect of \( b \).

- \( d = 0 \iff |T(b)| = 1. \) Then \( B = R^{n \times n} \) is a matrix ring.
- \( \gcd\{\chi_t(1) \mid t \in T(b)\} = p^{a-d} y \) with \( p \nmid y \).
- If \( \chi_t(1) = p^{a-d+h(t)} y' \) with \( p \nmid y' \) then \( h(t) \) is the height of \( \chi_t \).
- The principal block \( b \) is the one containing the trivial character. Its defect groups are the Sylow-p-subgroups of \( G \).
Brauer’s first main theorem

Let \( B = bRG \) be a block and \( D := D(b) \) its defect group. Let \( N := N_G(D) \) be the normalizer in \( G \) of \( D \).

The Brauer correspondence

The blocks of \( RG \) with defect group \( D \) are in bijection with the blocks of \( RN \) with defect group \( D \).

The Donovan conjecture

If \( k \) is algebraically closed, then there are only finitely many Morita equivalence classes of blocks with a given defect group.

Donovan’s conjecture for special cases

Donovan’s conjecture holds for blocks with cyclic defect groups. It also holds for tame blocks. Scopes proved the Donovan conjecture for symmetric groups. This allows to deduce it for the alternating groups.
The group structure of defect groups

**Cyclic defect group**

Let $b$ be a block of $RG$ such that $D(b)$ is cyclic. Then $b$ is a Brauer tree algebra and its structure is well understood. Brauer, Dade, Green, Thompson, Plesken

There is a similar (slightly weaker) result for tame blocks those with dihedral, semidihedral or quaternion defect group. Erdmann

**Abelian defect group**

- **Brauer’s height zero conjecture:**
  All characters in $b$ have height 0, if and only if $D(b)$ is abelian.

- **Broué’s perfect isometry conjecture:**
  If $D(b)$ is abelian then $b$ and its Brauer correspondent are perfectly isometric.

See [http://www.maths.bris.ac.uk/~majcr/adgc/adgc.html](http://www.maths.bris.ac.uk/~majcr/adgc/adgc.html) for results on this conjecture.
Exercises.

Blocks from character tables.

- Compute the blocks of $A_4$ for $p = 3$.
- Compute the blocks of $S_5$ for $p = 2, 3, 5$.

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