Codes and invariant theory II.

Gabriele Nebe

Lehrstuhl D für Mathematik

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Higher genus complete weight enumerators.

Definition

Let \( c^{(i)} := (c_1^{(i)}, \ldots, c_N^{(i)}) \in V^N \), \( i = 1, \ldots, m \), be \( m \) not necessarily distinct codewords. For \( v := (v_1, \ldots, v_m) \in V^m \), let

\[
a_v(c^{(1)}, \ldots, c^{(m)}) := |\{ j \in \{1, \ldots, N\} \mid c_j^{(i)} = v_i \text{ for all } i \in \{1, \ldots, m\}\}|.
\]

The \textit{genus-} \( m \) \textit{complete weight enumerator} of \( C \) is

\[
cwe_m(C) := \sum_{(c^{(1)}, \ldots, c^{(m)}) \in C^m} \prod_{v \in V^m} x_v^{a_v(c^{(1)}, \ldots, c^{(m)})} \in \mathbb{C}[x_v : v \in V^m].
\]
Examples.

\[ C = i_2 = \{(0, 0), (1, 1)\}, \text{ then } \text{cwe}_2(C) = x_{00}^2 + x_{11}^2 + x_{01}^2 + x_{10}^2. \]

\[ C = e_8 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}. \]

\[ \text{cwe}_2(e_8) = x_{00}^8 + x_{01}^8 + x_{10}^8 + x_{11}^8 + 168x_{00}^2x_{01}^2x_{10}^2x_{11}^2 + 14(x_{00}^4x_{01}^4 + x_{00}^4x_{10}^4 + x_{00}^4x_{11}^4 + x_{01}^4x_{10}^4 + x_{01}^4x_{11}^4 + x_{10}^4x_{11}^4) \]
Higher genus complete weight enumerators.

Remark.

For $C \leq V^N$ and $m \in \mathbb{N}$ let

$C(m) := R^{m \times 1} \otimes C = \{(c^{(1)}, \ldots, c^{(m)})^T | c^{(1)}, \ldots, c^{(m)} \in C\} \leq (V^m)^N$

Then \(\text{cwe}_m(C) = \text{cwe}(C(m))\).

If $C$ is a self-dual isotropic code of Type $T = (R, V, \beta, \Phi)$, then $C(m)$ is a self-dual isotropic code of Type

$$T^m = (R^{m \times m}, V^m, \beta^{(m)}, \Phi^{(m)})$$

and hence $\text{cwe}_m(C)$ is invariant under $\mathcal{C}_m(T) := \mathcal{C}(T^m)$ the genus-m Clifford-Weil group.

Main theorem implies.

$$\text{Inv}(\mathcal{C}_m(T)) = \langle \text{cwe}_m(C) : C \text{ of Type } T \rangle.$$ (if $R$ is a direct product of matrix rings over chain rings).
Higher genus Clifford-Weil groups.

$C_2(I)$. 

$R = \mathbb{F}_2^{2 \times 2}, R^* = \text{GL}_2(\mathbb{F}_2) = \langle a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, b := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \rangle$

$V = \mathbb{F}_2^2 = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \}, \text{symmetric idempotent } e = \text{diag}(1, 0)$

$C_2(I) = \langle m_a = \begin{pmatrix} 1000 \\ 0010 \\ 0100 \\ 0001 \end{pmatrix}, m_b = \begin{pmatrix} 1000 \\ 0001 \\ 0100 \\ 0010 \end{pmatrix},

h_{e,e,e} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \ d_{\varphi e} = \text{diag}(1, -1, 1, -1) \rangle$

$C_2(II) = \langle m_a, m_b, h_{e,e,e}, d_{\varphi e} = \text{diag}(1, i, 1, i) \rangle$. 
Molien series of $C_2(I)$.

$C_2(I)$ has order 2304 and Molien series

$$1 + t^{18}$$

$$\frac{1}{(1 - t^2)(1 - t^8)(1 - t^{12})(1 - t^{24})}$$

where the generators correspond to the degree 2 complete weight enumerators of the codes:

$$i_2, e_8, d_{12}^+, g_{24}, \text{und } (d_{10}e_7f_1)^+$$

$W(F_4)$ is a subgroup of index 2 in $C_2(I)$.

$$\text{Inv}(C_2(I)) = \mathbb{C}[i_2, e_8, d_{12}^+, g_{24}] \oplus (d_{10}e_7f_1)^+\mathbb{C}[i_2, e_8, d_{12}^+, g_{24}]$$
Molien series of $\mathbb{C}_2(\text{II})$.

$$\mathbb{C}_2(\text{II}) = \langle m_a, m_b, h_{e,e,e}, d_{\phi e} = \text{diag}(1, i, 1, i) \rangle.$$ $\mathbb{C}_2(\text{II})$ has order 92160 and Molien series

$$1 + t^{32} \over (1 - t^8)(1 - t^{24})^2(1 - t^{40})$$

where the generators correspond to the degree 2 complete weight enumerators of the codes:

$$e_8, g_{24}, d_{24}^+, d_{40}^+, \text{ and } d_{32}^+$$

$\mathbb{C}_2(\text{II})$ has a reflection subgroup of index 2, No. 31 on the Shephard-Todd list.

$$\text{Inv}(\mathbb{C}_2(\text{II})) = \mathbb{C}[e_8, g_{24}, d_{24}^+, d_{40}^+] \oplus d_{32}^+ \mathbb{C}[e_8, g_{24}, d_{24}^+, d_{40}^+]$$
Higher genus Clifford-Weil groups for Type I, II, III, IV.

\[ C_m(I) = 2^{1+2m} \cdot O_{2m}^+(\mathbb{F}_2) \]

\[ C_m(II) = Z_8 \ast 2^{1+2m} \cdot \text{Sp}_{2m}(\mathbb{F}_2) \]

\[ C_m(III) = Z_4 \cdot \text{Sp}_{2m}(\mathbb{F}_3) \]

\[ C_m(IV) = Z_2 \cdot U_{2m}(\mathbb{F}_4) \]
Higher genus Clifford-Weil groups for the classical Types of codes over finite fields.

\[ \mathcal{C}_m(T) = S.(\ker(\lambda) \times \ker(\lambda)).\mathcal{G}_m(T) \]

\[ \lambda(\phi) : (x, y) \mapsto \phi(x + y) - \phi(x) - \phi(y) \]

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<thead>
<tr>
<th>( R )</th>
<th>( J )</th>
<th>( \epsilon )</th>
<th>( \mathcal{G}_m(T) )</th>
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<td>((r, s)^J = (s, r))</td>
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<td>( \text{GL}_{2m}(\mathbb{F}_q) )</td>
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<td>( \text{U}<em>{2m}(\mathbb{F}</em>{q^2}) )</td>
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<td>( r^J = r )</td>
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<td>( \text{Sp}_{2m}(\mathbb{F}_q) )</td>
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<tr>
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<td>( \text{O}^{+}_{2m}(\mathbb{F}_q) )</td>
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Motivation.

Determine linear relations between $\text{cwe}_m(C')$ for $C \in M_N(T) = \{C \leq V^N \mid C \text{ of Type } T\}$.

- $M_{16}(\text{II}) = [e_8 \perp e_8] \cup [d_{16}^+]$ and these two codes have the same genus 1 and 2 weight enumerator, but $\text{cwe}_3(e_8 \perp e_8)$ and $\text{cwe}_3(d_{16}^+)$ are linearly independent.

- $h(M_{24}(\text{II})) = 9$ and only the genus 6 weight enumerators are linearly independent, there is one relation for the genus 5 weight enumerators.

- $h(M_{32}(\text{II})) = 85$ and here the genus 10 weight enumerators are linearly independent, whereas there is a unique relation for the genus 9 weight enumerators.
Linear relations between weight enumerators.

Three different approaches:

- Determine all the codes and their weight enumerators.
  If $\dim(C') = n = N/2$ there are $\prod_{i=0}^{d-1} (2^n - 2^i)/(2^d - 2^i)$ subspaces of dimension $d$ in $C$.
  $N = 32, d = 10$ yields more than $10^{18}$ subspaces.

- Use Molien’s theorem:
  $\text{Inv}_N(C_m(\II)) = \langle \text{cwe}_m(C) \mid C \in M_N(\II) \rangle$
  and if $a_N := \dim(\text{Inv}_N(C_m(\II)))$ then
  $$\sum_{N=0}^{\infty} a_N t^N = \frac{1}{|C_m(\II)|} \sum_{g \in C_m(\II)} (\det(1 - tg))^{-1}$$

  Problem: $C_{10}(\II) \leq \text{GL}_{1024}(\mathbb{C})$ has order $> 10^{69}$.
  with the use of normal subgroup structure, we know the Molien series of these Clifford-Weil groups for $m \leq 4$.

- Use Hecke operators.
Kneser-Hecke operators.

- Fix a Type $T = (\mathbb{F}_q, \mathbb{F}_q, \beta, \Phi)$ of self-dual codes over a finite field with $q$ elements.

$$M_N(T) = \{ C \leq \mathbb{F}_q^N | C \text{ of Type } T \} = [C_1] \cup \ldots \cup [C_h]$$

where $[C]$ denotes the permutation equivalence class of the code $C$.

- $\mathcal{V} = \mathbb{C}[C_1] \oplus \ldots \oplus \mathbb{C}[C_h] \cong \mathbb{C}^h$

- Then $n := \frac{N}{2} = \dim(C)$ for all $C \in M_N(T)$.

- $C, D \in M_N(T)$ are called neighbours, if $\dim(C) - \dim(C \cap D) = 1$, $C \sim D$.

- $K_N(T) \in \text{End}(\mathcal{V})$, $K_N(T) : [C] \mapsto \sum_{D \in M_N(T), D \sim C} [D]$.

Kneser-Hecke operator.

- $K_N(T)$ is the adjacency matrix of the neighbouring graph.
Examples for Kneser-Hecke operators.

Example. \( M_{16}(\Pi) = [e_8 \perp e_8] \cup [d_{16}^+] \)

\[
K_{16}(\Pi) = \begin{pmatrix} 78 & 49 \\ 70 & 57 \end{pmatrix}
\]
The Hilbert space $\mathcal{V}$.

$\mathcal{V}$ has a Hermitian positive definite inner product defined by

$$\langle [C_i], [C_j] \rangle := |\text{Aut}(C_i)| \delta_{ij}.$$  

**Theorem. (N. 2006)**

The Kneser-Hecke operator $K$ is a self-adjoint linear operator.

$$\langle v, Kw \rangle = \langle Kv, w \rangle \text{ for all } v, w \in \mathcal{V}.$$  

**Example.**

$$\frac{7}{10} = \frac{|\text{Aut}(e_8 \perp e_8)|}{|\text{Aut}(d^+_16)|} \quad \text{hence}$$

$$\text{diag}(7, 10)K_{16}(\Pi)^{\text{Tr}} = K_{16}(\Pi) \text{ diag}(7, 10).$$
The filtration of $\mathcal{V}$.

$$cwe_m : \mathcal{V} \to \mathbb{C}[X], \sum_{i=1}^{h} a_i [C_i] \mapsto \sum_{i=1}^{h} a_i \ cwe_m(C_i)$$

is a linear mapping with kernel

$$\mathcal{V}_m := \ker(cwe_m).$$

Then

$$\mathcal{V} =: \mathcal{V}_{-1} \geq \mathcal{V}_0 \geq \mathcal{V}_1 \geq \ldots \geq \mathcal{V}_n = \{0\}.$$ is a filtration of $\mathcal{V}$ yielding the orthogonal decomposition

$$\mathcal{V} = \bigoplus_{m=0}^{n} \mathcal{Y}_m \text{ where } \mathcal{Y}_m = \mathcal{V}_{m-1} \cap \mathcal{V}_m^\perp.$$

$$\mathcal{V}_0 = \left\{ \sum_{i=1}^{h} a_i [C_i] \mid \sum_{i=1}^{h} a_i = 0 \right\}$$

and

$$\mathcal{V}_0^\perp = \mathcal{Y}_0 = \langle \sum_{i=1}^{h} \frac{1}{|\text{Aut}(C_i)|} [C_i] \rangle.$$
Eigenvalues of Kneser-Hecke operator.

**Theorem. (N. 2006)**

The space $\mathcal{Y}_m = \mathcal{Y}_m(N)$ is the $K_N(T)$-eigenspace to the eigenvalue $\nu_{N}^{(m)}(T)$ with $\nu_{N}^{(m)}(T) > \nu_{N}^{(m+1)}(T)$ for all $m$.

<table>
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<tr>
<th>Type</th>
<th>$\nu_{N}^{(m)}(T)$</th>
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<tr>
<td>$q_{E}^1$</td>
<td>$(q^{n-m} - q - q^m + 1)/(q - 1)$</td>
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<tr>
<td>$q_{E}^2$</td>
<td>$(q^{n-m-1} - q^m)/(q - 1)$</td>
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<tr>
<td>$q_{E}^3$</td>
<td>$(q^{n-m} - q^m)/(q - 1)$</td>
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<tr>
<td>$q_{E}^4$</td>
<td>$(q^{n-m-1} - q^m)/(q - 1)$</td>
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<tr>
<td>$q_{H}^1$</td>
<td>$(q^{n-m+1/2} - q^m - q^{1/2} + 1)/(q - 1)$</td>
</tr>
<tr>
<td>$q_{H}^2$</td>
<td>$(q^{n-m-1/2} - q^m - q^{1/2} + 1)/(q - 1)$</td>
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**Corollary.**

The neighbouring graph is connected.

**Proof.** The maximal eigenvalue $\nu_0$ of the adjacency matrix is simple with eigenspace $\mathcal{Y}_0$. 
Example: Type II, length 16.

\[ M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^+] \]
\begin{equation}
(2^{8-m-1} - 2^m : m = 0, 1, 2, 3) = (127, 62, 28, 8)
\end{equation}

\[ K_{16}(\mathrm{II}) = \begin{pmatrix}
78 & 49 \\
70 & 57
\end{pmatrix} \]

has eigenvalues 127 and 8 with eigenvectors \((7, 10)\) and \((1, -1)\).

Hence
\begin{align*}
y_0 &= \langle 7[e_8 \perp e_8] + 10[d_{16}^+] \rangle \\
y_1 &= y_2 = 0 \\
y_3 &= \langle [e_8 \perp e_8] - [d_{16}^+] \rangle.
\end{align*}
Example: Type II, length 24.

\[ M_{24}(\Pi) = [e_8^3] \cup [e_8 d_{16}] \cup [e_7^2 d_{10}] \cup [d_8^3] \cup [d_{24}] \cup [d_{12}^2] \cup [d_6^4] \cup [g_{24}] \]

\[ K_{24}(\Pi) = \]

\[
\begin{pmatrix}
213 & 147 & 344 & 343 & 0 & 0 & 0 & 0 & 0 \\
70 & 192 & 896 & 490 & 7 & 392 & 0 & 0 & 0 \\
10 & 14 & 504 & 490 & 0 & 49 & 980 & 0 & 0 \\
1 & 3 & 192 & 447 & 0 & 36 & 1152 & 216 & 0 \\
0 & 990 & 0 & 0 & 133 & 924 & 0 & 0 & 0 \\
0 & 60 & 480 & 900 & 1 & 206 & 400 & 0 & 0 \\
0 & 0 & 72 & 216 & 0 & 3 & 1108 & 648 & 0 \\
0 & 0 & 0 & 45 & 0 & 0 & 720 & 1218 & 64 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1771 & 276
\end{pmatrix}
\]

<table>
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<th>( m )</th>
<th>( \nu_m )</th>
<th>( \dim(\mathcal{Y}_m) )</th>
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\[ \langle 99[e_8^3] - 297[e_8 d_{16}] - 3465[d_8^3] + 7[d_{24}] + 924[d_{12}^2] + 4928[d_6^4] - 2772[d_4^6] + 576[g_{24}] \rangle = \ker(cw_{e_5}) = \mathcal{V}_5 \]
The Dimension of $\mathcal{Y}_m(N)$ for doubly-even binary self-dual codes.

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<td>21</td>
<td>18</td>
<td>8</td>
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The Molien series of $\mathcal{C}_m(\Pi)$ is

$$1 + t^8 + a(m)t^{16} + b(m)t^{24} + c(m)t^{32} + \ldots$$

where

<table>
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<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
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\( \dim(y_m(N)) \) for binary self-dual codes.

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<td></td>
</tr>
<tr>
<td>32</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>33</td>
<td>111</td>
<td>341</td>
<td>825</td>
<td>1176</td>
<td>651</td>
<td>127</td>
<td>15</td>
<td>1</td>
</tr>
</tbody>
</table>
Application to Molien series.

The Molien series of $C_m(I)$ is

$$1 + t^2 + t^4 + t^6 + 2t^8 + 2t^{10} + \sum_{N=12}^{\infty} a_N(m)t^N$$

where

$$a_N(m) := \dim \langle \text{cwe}_m(C) : C = C^\perp \leq \mathbb{F}_2^N \rangle$$

is given in the following table:

<table>
<thead>
<tr>
<th>$m, N$</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>22</th>
<th>24</th>
<th>26</th>
<th>28</th>
<th>30</th>
<th>32</th>
</tr>
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<tbody>
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<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
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</tr>
<tr>
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<td>9</td>
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<td>19</td>
<td>33</td>
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<td>69</td>
<td>100</td>
<td>159</td>
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<tr>
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<td>4</td>
<td>7</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>55</td>
<td>103</td>
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<td>3152</td>
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<tr>
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<td>4</td>
<td>7</td>
<td>9</td>
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<td>55</td>
<td>103</td>
<td>261</td>
<td>731</td>
<td>3294</td>
</tr>
<tr>
<td>$\geq 11$</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>55</td>
<td>103</td>
<td>261</td>
<td>731</td>
<td>3295</td>
</tr>
</tbody>
</table>
A group theoretic interpretation of the Kneseer-Hecke operator.

In modular forms theory, Hecke operators are double cosets of the modular group. So I tried to find a similar interpretation for the Kneseer-Hecke operator.

Let \( T = (R, V, \beta, \Phi) \) be a Type. Then the invariant ring

\[
\text{Inv}(C_m(T)) = \langle \text{cwe}_m(C) \mid C \text{ of Type } T \rangle
\]

The finite Siegel \( \Phi \)-operator

\[
\Phi_m : \text{Inv}(C_m(T)) \to \text{Inv}(C_{m-1}(T)), \text{cwe}_m(C) \mapsto \text{cwe}_{m-1}(C)
\]

defines a surjective graded \( \mathbb{C} \)-algebra homomorphism between invariant rings of complex matrix groups of different degree.

\( \Phi \) is given by the variable substitution:

\[
x(v_1, \ldots, v_m) \mapsto \begin{cases} 
x(v_1, \ldots, v_{m-1}) & \text{if } v_m = 0 \\
0 & \text{else}
\end{cases}
\]
cwe_{m-1}(C) is obtained from cwe_m(C) by counting only those matrices

\[
\begin{array}{cccccccc}
c^{(1)}_1 & c^{(1)}_2 & \ldots & c^{(1)}_j & \ldots & c^{(1)}_N \\
c^{(2)}_1 & c^{(2)}_2 & \ldots & c^{(2)}_j & \ldots & c^{(2)}_N \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
c^{(m)}_1 & c^{(m)}_2 & \ldots & c^{(m)}_j & \ldots & c^{(m)}_N \\
\end{array}
\]

in which the last row is zero.

This is expressed by the variable substitution

\[
x(v_1, \ldots, v_m) \mapsto \begin{cases} 
x(v_1, \ldots, v_{m-1}) & \text{if } v_m = 0 \\
0 & \text{else}
\end{cases}
\]
A canonical right inverse of $\Phi$.

$$(p, q)_m := p\left(\frac{\partial}{\partial x}\right)(\bar{q}) \text{ for } p, q \in \mathbb{C}[x_v : v \in V^m]_N$$

defines a positive definite Hermitian product on the homogeneous component $\mathbb{C}[x_v : v \in V^m]_N$. The monomials of degree $N$ form an orthogonal basis and

$$(\prod_{v \in V^m} x_v^{n_v}, \prod_{v \in V^m} x_v^{n_v})_m = \prod_{v \in V^m} (n_v!).$$

Then $\Phi_m : \ker(\Phi_m)^\perp \rightarrow \text{Inv}(\mathbb{C}_{m-1}(T))$ is an isomorphism with inverse

$$\varphi_m : \text{Inv}(\mathbb{C}_{m-1}(T)) \rightarrow \text{Inv}(\mathbb{C}_m(T)), x(v_1,\ldots,v_{m-1}) \mapsto R(x(v_1,\ldots,v_{m-1},0))$$

where $R(p) = \frac{1}{|\mathbb{C}_m(T)|} \sum_{g \in \mathbb{C}_m(T)} p(gx)$ is the Reynolds operator (the orthogonal projection onto the invariant ring). Note that $R$ is not a ring homomorphism.
The orthogonal decomposition of the invariant space.

This yields an orthogonal decomposition of the space of degree \( N \) invariants of \( C_m(T) \)

\[
\text{Inv}_N(C_m(T)) = \ker(\Phi_m) \perp \varphi_m^{-1}(\text{Inv}_N(C_{m-1}(T))) = \\
\ker(\Phi_m) \perp \varphi_m^{-1}(\ker(\Phi_{m-1}) \perp \varphi_{m-1}^{-1}(\text{Inv}_N(C_{m-2})(T))) = \\
Y_m \perp Y_{m-1} \perp \ldots \perp Y_0
\]

such that for all \( 0 \leq k \leq m \) the mapping

\[
cwe_m : y_k \rightarrow Y_k.
\]

is an isomorphism of vector spaces.
The Kneser-Hecke operator $K_N(T)$ acts on $\text{Inv}_N(\mathcal{C}_m(T))$ as $\delta_m(K_N(T))$ having $Y_m \perp Y_{m-1} \perp \ldots \perp Y_0$ as the eigenspace decomposition.

$$\mathcal{C}_m(T) = S.(\ker(\lambda) \times \ker(\lambda)) \cdot \mathcal{G}_m(T)$$

Choose a suitable subgroup $\mathcal{U}_1$ of $\mathcal{E}_m(T)$ that corresponds to a 1-dimensional subspace of $(\ker(\lambda) \times \ker(\lambda))$ and let

$$p_1 := \frac{1}{q} \sum_{u \in \mathcal{U}_1} u \in \mathbb{C}^{q^m \times q^m}$$

be the orthogonal projection onto the fixed space of $\mathcal{U}_1$ and let

$$H_m(T) := \mathcal{C}_m(T)p_1\mathcal{C}_m(T) = \bigcup_{U \in X} p_U \mathcal{C}_m(T).$$

Then this double coset acts on $\text{Inv}_N(\mathcal{C}_m(T))$ via

$$\Delta_N(H_m(T)) : f \mapsto \frac{1}{|X|} \sum_{U \in X} f(xp_U)$$
Explicit description as sum of double cosets.

**Theorem. (N. 2006)**

\[(q - 1)\delta_m(K_N(T)) = q^{n-m-e}((q - 1)\Delta_N(H_m(T)) + \text{id}) - (q^m + a) \text{id}\]

where \(n = \frac{N}{2}\) and \(e, a\) are as follows:

<table>
<thead>
<tr>
<th></th>
<th>(T)</th>
<th>(q^E)</th>
<th>(q_{I}^{E})</th>
<th>(q_{I}^{E})</th>
<th>(q_{II}^{E})</th>
<th>(q_{I}^{H})</th>
<th>(q_{II}^{H})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a)</td>
<td>0</td>
<td>(q - 1)</td>
<td>0</td>
<td>0</td>
<td>(\sqrt{q} - 1)</td>
<td>(\sqrt{q} - 1)</td>
</tr>
<tr>
<td></td>
<td>(e)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1/2</td>
<td>(-1/2)</td>
</tr>
</tbody>
</table>
Conclusion.

- formal notion of Type $T = (R, V, \beta, \Phi)$.
- self-dual code $C$ of Type $T$.
- genus-$m$ complete weight enumerators
- the associated Clifford-Weil group $C_m(T)$, a finite complex matrix group of degree $|V|^m$ such that
  \[
  \text{Inv}_N(C_m(T)) = \langle \text{cwe}_m(C) \mid C = C^\perp \leq V^N \text{ of Type } T \rangle
  \]
- In particular the scalar subgroup $C_m(T) \cap \mathbb{C}^* \text{id}$ is cyclic of order
  \[
  \min \{ N \mid \text{there is a code } C \leq V^N \text{ of Type } T \}.
  \]
- $C_m(T)$ has a nice group theoretic structure.
- $\Phi_m : \text{Inv}(C_m(T)) \to \text{Inv}(C_{m-1}(T))$
- if $R$ is a field then:
  - As in modular forms theory, the invariant ring of $C_m(T)$ can be investigated using Hecke operators.
  - The Hecke algebra is generated by the incidence matrix of the Kneser neighbouring graph.
- Obtain linear relations between weight enumerators.