Codes and invariant theory II.

Gabriele Nebe

Lehrstuhl D für Mathematik

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Higher genus complete weight enumerators.

Definition

Let $c^{(i)} := (c_1^{(i)}, \dots, c_N^{(i)}) \in V^N$, $i = 1, \dots, m$, be m not necessarily distinct codewords. For $v := (v_1, \dots, v_m) \in V^m$, let

$$a_v(c^{(1)}, \dots, c^{(m)}) := \left|\{j \in \{1, \dots, N\} \mid c_j^{(i)} = v_i \text{ for all } i \in \{1, \dots, m\}\}\right|.$$

The genus-m complete weight enumerator of C is

$$\mathrm{cwe}_m(C) := \sum_{(c^{(1)}, \dots, c^{(m)}) \in C^m} \prod_{v \in V^m} x_v^{a_v(c^{(1)}, \dots, c^{(m)})} \in \mathbb{C}[x_v : v \in V^m].$$

$$\begin{array}{cccccc}
c_1^{(1)} & c_2^{(1)} & \dots & c_j^{(1)} & \dots & c_N^{(1)} \\
c_1^{(2)} & c_2^{(2)} & \dots & c_j^{(2)} & \dots & c_N^{(2)} \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
c_1^{(m)} & c_2^{(m)} & \dots & c_j^{(m)} & \dots & c_N^{(m)} \\
 & & & \uparrow & & \\
 & & & v \in V^m & &
\end{array}$$

Examples.

$C = i_2 = \{(0,0),(1,1)\}$, then $\mathrm{cwe}_2(C) = x_{00}^2 + x_{11}^2 + x_{01}^2 + x_{10}^2$.

$$C = e_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

$$\begin{aligned} \mathrm{cwe}_2(e_8) &= x_{00}^8 + x_{01}^8 + x_{10}^8 + x_{11}^8 + 168 x_{00}^2 x_{01}^2 x_{10}^2 x_{11}^2 + \\ & 14(x_{00}^4 x_{01}^4 + x_{00}^4 x_{10}^4 + x_{00}^4 x_{11}^4 + x_{01}^4 x_{10}^4 + x_{01}^4 x_{11}^4 + x_{10}^4 x_{11}^4) \end{aligned}$$

Higher genus complete weight enumerators.

Remark.

For $C \leq V^N$ and $m \in \mathbb{N}$ let

$$C(m) := R^{m\times 1} \otimes C = \{(c^{(1)}, \ldots, c^{(m)})^{\mathrm{Tr}} \mid c^{(1)}, \ldots, c^{(m)} \in C\} \leq (V^m)^N$$

Then $\mathrm{cwe}_m(C) = \mathrm{cwe}(C(m))$.

If C is a self-dual isotropic code of Type $T = (R, V, \beta, \Phi)$, then $C(m)$ is a self-dual isotropic code of Type

$$T^m = (R^{m\times m}, V^m, \beta^{(m)}, \Phi^{(m)})$$

and hence $\mathrm{cwe}_m(C)$ is invariant under $\mathcal{C}_m(T) := \mathcal{C}(T^m)$ the genus-m Clifford-Weil group.

Main theorem implies.

$$\mathrm{Inv}(\mathcal{C}_m(T)) = \langle \mathrm{cwe}_m(C) : C \text{ of Type } T \rangle.$$

(if R is a direct product of matrix rings over chain rings).

Higher genus Clifford-Weil groups.

$\mathcal{C}_2(\mathrm{I})$.

$$R = \mathbb{F}_2^{2\times 2}, R^* = \mathrm{GL}_2(\mathbb{F}_2) = \langle a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ b := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \rangle$$

$V = \mathbb{F}_2^2 = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$, symmetric idempotent $e = \mathrm{diag}(1,0)$

$$\mathcal{C}_2(\mathrm{I}) \quad = \langle m_a = \begin{pmatrix} 1000 \\ 0010 \\ 0100 \\ 0001 \end{pmatrix}, \ m_b = \begin{pmatrix} 1000 \\ 0001 \\ 0100 \\ 0010 \end{pmatrix},$$

$$h_{e,e,e} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\,1\,0\,0 \\ 1\text{-}1\,0\,0 \\ 0\,0\,1\,1 \\ 0\,0\,1\text{-}1 \end{pmatrix}, \ d_{\varphi e} = \mathrm{diag}(1,-1,1,-1)\rangle$$

$$\mathcal{C}_2(\mathrm{II}) = \langle m_a, m_b, h_{e,e,e}, d_{\phi e} = \mathrm{diag}(1,i,1,i)\rangle.$$

Molien series of $\mathcal{C}_2(\mathrm{I})$.

$\mathcal{C}_2(\mathrm{I})$ has order 2304 and Molien series

$$\frac{1+t^{18}}{(1-t^2)(1-t^8)(1-t^{12})(1-t^{24})}$$

where the generators correspond to the degree 2 complete weight enumerators of the codes:

$$i_2, e_8, d_{12}^+, g_{24}, \text{ und } (d_{10}e_7f_1)^+$$

$W(F_4)$ is a subgroup of index 2 in $\mathcal{C}_2(\mathrm{I})$.

$$\mathrm{Inv}(\mathcal{C}_2(\mathrm{I})) = \mathbb{C}[i_2, e_8, d_{12}^+, g_{24}] \oplus (d_{10}e_7f_1)^+\mathbb{C}[i_2, e_8, d_{12}^+, g_{24}]$$

Molien series of $\mathcal{C}_2(\mathrm{II})$.

$$\mathcal{C}_2(\mathrm{II}) = \langle m_a, m_b, h_{e,e,e}, d_{\phi e} = \mathrm{diag}(1, i, 1, i) \rangle.$$

$\mathcal{C}_2(\mathrm{II})$ has order 92160 and Molien series

$$\frac{1 + t^{32}}{(1 - t^8)(1 - t^{24})^2(1 - t^{40})}$$

where the generators correspond to the degree 2 complete weight enumerators of the codes:

$$e_8, g_{24}, d_{24}^+, d_{40}^+, \text{ and } d_{32}^+$$

$\mathcal{C}_2(\mathrm{II})$ has a reflection subgroup of index 2, No. 31 on the Shephard-Todd list.

$$\mathrm{Inv}(\mathcal{C}_2(\mathrm{II})) = \mathbb{C}[e_8, g_{24}, d_{24}^+, d_{40}^+] \oplus d_{32}^+ \mathbb{C}[e_8, g_{24}, d_{24}^+, d_{40}^+]$$

Higher genus Clifford-Weil groups for Type I, II, III, IV.

$$\mathcal{C}_m(\mathrm{I}) = 2_+^{1+2m}.O_{2m}^+(\mathbb{F}_2)$$

$$\mathcal{C}_m(\mathrm{II}) = Z_8 \star 2^{1+2m}.\,\mathrm{Sp}_{2m}(\mathbb{F}_2)$$

$$\mathcal{C}_m(\mathrm{III}) = Z_4.\,\mathrm{Sp}_{2m}(\mathbb{F}_3)$$

$$\mathcal{C}_m(\mathrm{IV}) = Z_2.U_{2m}(\mathbb{F}_4)$$

Higher genus Clifford-Weil groups for the classical Types of codes over finite fields.

$$\mathcal{C}_m(T) = S.(\ker(\lambda) \times \ker(\lambda)).\mathcal{G}_m(T)$$

$$\lambda(\phi) : (x, y) \mapsto \phi(x+y) - \phi(x) - \phi(y)$$

R	J	ϵ	$\mathcal{G}_m(T)$
$\mathbb{F}_q \oplus \mathbb{F}_q$	$(r, s)^J = (s, r)$	1	$\mathrm{GL}_{2m}(\mathbb{F}_q)$
$\mathbb{F}_{q^2}$	$r^J = r^q$	1	$U_{2m}(\mathbb{F}_{q^2})$
$\mathbb{F}_q,\ q$ odd	$r^J = r$	1	$\mathrm{Sp}_{2m}(\mathbb{F}_q)$
$\mathbb{F}_q,\ q$ odd	$r^J = r$	-1	$O^+_{2m}(\mathbb{F}_q)$
$\mathbb{F}_q,\ q$ even	doubly even		$\mathrm{Sp}_{2m}(\mathbb{F}_q)$
$\mathbb{F}_q,\ q$ even	singly even		$O^+_{2m}(\mathbb{F}_q)$

Hecke operators in coding theory.

Motivation.

Determine linear relations between $\mathrm{cwe}_m(C)$ for
$C \in M_N(T) = \{C \leq V^N \mid C$ of Type $T\}$.

- $M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^+]$ and these two codes have the same genus 1 and 2 weight enumerator, but $\mathrm{cwe}_3(e_8 \perp e_8)$ and $\mathrm{cwe}_3(d_{16}^+)$ are linearly independent.
- $h(M_{24}(\mathrm{II})) = 9$ and only the genus 6 weight enumerators are linearly independent, there is one relation for the genus 5 weight enumerators.
- $h(M_{32}(\mathrm{II})) = 85$ and here the genus 10 weight enumerators are linearly independent, whereas there is a unique relation for the genus 9 weight enumerators.

Linear relations between weight enumerators.

Three different approaches:

- Determine all the codes and their weight enumerators.
  If $\dim(C) = n = N/2$ there are $\prod_{i=0}^{d-1}(2^n - 2^i)/(2^d - 2^i)$ subspaces of dimension d in C.
  $N = 32, d = 10$ yields more than 10^{18} subspaces.
- Use Molien's theorem:
  $\mathrm{Inv}_N(\mathcal{C}_m(\mathrm{II})) = \langle \mathrm{cwe}_m(C) \mid C \in M_N(\mathrm{II}) \rangle$
  and if $a_N := \dim(\mathrm{Inv}_N(\mathcal{C}_m(\mathrm{II})))$ then

$$\sum_{N=0}^{\infty} a_N t^N = \frac{1}{|\mathcal{C}_m(\mathrm{II})|} \sum_{g \in \mathcal{C}_m(\mathrm{II})} (\det(1 - tg))^{-1}$$

  Problem: $\mathcal{C}_{10}(\mathrm{II}) \leq \mathrm{GL}_{1024}(\mathbb{C})$ has order $> 10^{69}$.
  with the use of normal subgroup structure, we know the Molien series of these Clifford-Weil groups for $m \leq 4$.
- Use Hecke operators.

Kneser-Hecke operators.

- Fix a Type $T = (\mathbb{F}_q, \mathbb{F}_q, \beta, \Phi)$ of self-dual codes over a finite **field** with q elements.

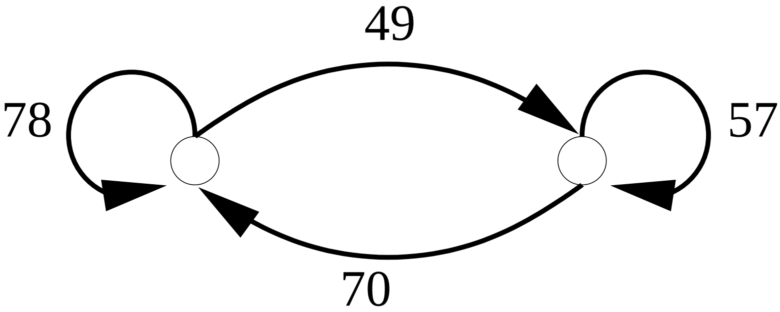
  $$M_N(T) = \{C \leq \mathbb{F}_q^N \mid C \text{ of Type } T\} = [C_1] \,\dot\cup\, \ldots \,\dot\cup\, [C_h]$$

  where $[C]$ denotes the permutation equivalence class of the code C.
- $$\mathcal{V} = \mathbb{C}[C_1] \oplus \ldots \oplus \mathbb{C}[C_h] \cong \mathbb{C}^h$$
- Then $n := \frac{N}{2} = \dim(C)$ for all $C \in M_N(T)$.
- $C, D \in M_N(T)$ are called neighbours, if $\dim(C) - \dim(C \cap D) = 1$, $C \sim D$.
- $$K_N(T) \in \mathrm{End}(\mathcal{V}),\ K_N(T) : [C] \mapsto \sum_{D \in M_N(T), D \sim C} [D].$$

  Kneser-Hecke operator.
- $K_N(T)$ is the adjacency matrix of the neighbouring graph.

Examples for Kneser-Hecke operators.

Example. $M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^+]$



$$K_{16}(\mathrm{II}) = \begin{pmatrix} 78 & 49 \\ 70 & 57 \end{pmatrix}$$

The Hilbert space $\mathcal{V}$.

$\mathcal{V}$ has a Hermitian positive definite inner product defined by

$$\langle [C_i], [C_j] \rangle := |\operatorname{Aut}(C_i)| \delta_{ij}.$$

Theorem. (N. 2006)

The Kneser-Hecke operator K is a self-adjoint linear operator.

$$\langle v, Kw \rangle = \langle Kv, w \rangle \text{ for all } v, w \in \mathcal{V}.$$

Example.

$\frac{7}{10} = \frac{|\operatorname{Aut}(e_8 \perp e_8)|}{|\operatorname{Aut}(d_{16}^+)|}$ hence

$$\operatorname{diag}(7, 10) K_{16}(\mathrm{II})^{\mathrm{Tr}} = K_{16}(\mathrm{II}) \operatorname{diag}(7, 10).$$

The filtration of $\mathcal{V}$.

$$\mathrm{cwe}_m : \mathcal{V} \to \mathbb{C}[X], \sum_{i=1}^{h} a_i[C_i] \mapsto \sum_{i=1}^{h} a_i \, \mathrm{cwe}_m(C_i)$$

is a linear mapping with kernel

$$\mathcal{V}_m := \ker(\mathrm{cwe}_m).$$

Then

$$\mathcal{V} =: \mathcal{V}_{-1} \geq \mathcal{V}_0 \geq \mathcal{V}_1 \geq \ldots \geq \mathcal{V}_n = \{0\}.$$

is a filtration of $\mathcal{V}$ yielding the orthogonal decomposition

$$\mathcal{V} = \bigoplus_{m=0}^{n} \mathcal{Y}_m \text{ where } \mathcal{Y}_m = \mathcal{V}_{m-1} \cap \mathcal{V}_m^{\perp}.$$

$$\mathcal{V}_0 = \{\sum_{i=1}^{h} a_i[C_i] \mid \sum a_i = 0\}$$

and

$$\mathcal{V}_0^{\perp} = \mathcal{Y}_0 = \langle \sum_{i=1}^{h} \frac{1}{|\mathrm{Aut}(C_i)|}[C_i] \rangle.$$

Eigenvalues of Kneser-Hecke operator.

Theorem. (N. 2006)

The space $\mathcal{Y}_m = \mathcal{Y}_m(N)$ is the $K_N(T)$-eigenspace to the eigenvalue $\nu_N^{(m)}(T)$ with $\nu_N^{(m)}(T) > \nu_N^{(m+1)}(T)$ for all m.

Type	$\nu_N^{(m)}(T)$
q_{I}^E	$(q^{n-m} - q - q^m + 1)/(q-1)$
q_{II}^E	$(q^{n-m-1} - q^m)/(q-1)$
q^E	$(q^{n-m} - q^m)/(q-1)$
q_1^E	$(q^{n-m-1} - q^m)/(q-1)$
q^H	$(q^{n-m+1/2} - q^m - q^{1/2} + 1)/(q-1)$
q_1^H	$(q^{n-m-1/2} - q^m - q^{1/2} + 1)/(q-1)$

Corollary.

The neighbouring graph is connected.

Proof. The maximal eigenvalue ν_0 of the adjacency matrix is simple with eigenspace $\mathcal{Y}_0$.

Example: Type II, length 16.

$M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^{+}]$
$(2^{8-m-1} - 2^m : m = 0, 1, 2, 3) = (127, 62, 28, 8)$

$$K_{16}(\mathrm{II}) = \begin{pmatrix} 78 & 49 \\ 70 & 57 \end{pmatrix}$$

has eigenvalues 127 and 8 with eigenvectors $(7, 10)$ and $(1, -1)$.
Hence

$$\mathcal{Y}_0 = \langle 7[e_8 \perp e_8] + 10[d_{16}^{+}] \rangle$$

$$\mathcal{Y}_1 = \mathcal{Y}_2 = 0$$

$$\mathcal{Y}_3 = \langle [e_8 \perp e_8] - [d_{16}^{+}] \rangle.$$

Example: Type II, length 24.

$M_{24}(\mathrm{II}) = [e_8^3] \cup [e_8 d_{16}] \cup [e_7^2 d_{10}] \cup [d_8^3] \cup [d_{24}] \cup [d_{12}^2] \cup [d_6^4] \cup [d_4^6] \cup [g_{24}]$
$K_{24}(\mathrm{II}) =$

$$\begin{pmatrix} 213 & 147 & 344 & 343 & 0 & 0 & 0 & 0 & 0 \\ 70 & 192 & 896 & 490 & 7 & 392 & 0 & 0 & 0 \\ 10 & 14 & 504 & 490 & 0 & 49 & 980 & 0 & 0 \\ 1 & 3 & 192 & 447 & 0 & 36 & 1152 & 216 & 0 \\ 0 & 990 & 0 & 0 & 133 & 924 & 0 & 0 & 0 \\ 0 & 60 & 480 & 900 & 1 & 206 & 400 & 0 & 0 \\ 0 & 0 & 72 & 216 & 0 & 3 & 1108 & 648 & 0 \\ 0 & 0 & 0 & 45 & 0 & 0 & 720 & 1218 & 64 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1771 & 276 \end{pmatrix}$$

m	0	1	2	3	4	5	6
ν_m	2047	1022	508	248	112	32	-32
$\dim(\mathcal{Y}_m)$	1	1	1	2	2	1	1

$\langle 99[e_8^3] - 297[e_8 d_{16}] - 3465[d_8^3] + 7[d_{24}] + 924[d_{12}^2]$
$+4928[d_6^4] - 2772[d_4^6] + 576[g_{24}]\rangle = \ker(\mathrm{cwe}_5) = \mathcal{V}_5$

The Dimension of $\mathcal{Y}_m(N)$ for doubly-even binary self-dual codes.

N, m	0	1	2	3	4	5	6	7	8	9	≥ 10
8	1										
16	1	0	0	1							
24	1	1	1	2	2	1	1				
32	1	1	2	5	10	15	21	18	8	3	1

The Molien series of $\mathcal{C}_m(\mathrm{II})$ is

$$1 + t^8 + a(m)t^{16} + b(m)t^{24} + c(m)t^{32} + \dots$$

where

m	1	2	3	4	5	6	7	8	9	≥ 10
a	1	1	2	2	2	2	2	2	2	2
b	2	3	5	7	8	9	9	9	9	9
c	2	4	9	19	34	55	73	81	84	85

$\dim(\mathcal{Y}_m(N))$ for binary self-dual codes.

N, m	0	1	2	3	4	5	6	7	8	9	10	11
2	1											
4	1											
6	1											
8	1	1										
10	1	1										
12	1	1	1									
14	1	1	1	1								
16	1	2	1	2	1							
18	1	2	2	2	2							
20	1	2	3	4	4	2						
22	1	2	3	6	7	4	2					
24	1	3	5	9	15	13	7	2				
26	1	3	6	12	23	29	20	8	1			
28	1	3	7	18	40	67	75	39	10	1		
30	1	3	8	23	65	142	228	189	61	10	1	
32	1	4	10	33	111	341	825	1176	651	127	15	1

Application to Molien series.

The Molien series of $\mathcal{C}_m(\mathrm{I})$ is

$$1 + t^2 + t^4 + t^6 + 2t^8 + 2t^{10} + \sum_{N=12}^{\infty} a_N(m) t^N$$

where

$$a_N(m) := \dim\langle \text{cwe}_m(C) : C = C^\perp \leq \mathbb{F}_2^N \rangle$$

is given in the following table:

m, N	12	14	16	18	20	22	24	26	28	30	32
2	3	3	4	5	6	6	9	10	11	12	15
3	3	4	6	7	10	12	18	22	29	35	48
4	3	4	7	9	14	19	33	45	69	100	159
5	3	4	7	9	16	23	46	74	136	242	500
6	3	4	7	9	16	25	53	94	211	470	1325
7	3	4	7	9	16	25	55	102	250	659	2501
8	3	4	7	9	16	25	55	103	260	720	3152
9	3	4	7	9	16	25	55	103	261	730	3279
10	3	4	7	9	16	25	55	103	261	731	3294
≥ 11	3	4	7	9	16	25	55	103	261	731	3295

A group theoretic interpretation of the Kneser-Hecke operator.

In modular forms theory, Hecke operators are double cosets of the modular group. So I tried to find a similar interpretation for the Kneser-Hecke operator.
Let $T = (R, V, \beta, \Phi)$ be a Type. Then the invariant ring $\mathrm{Inv}(\mathcal{C}_m(T)) = \langle \mathrm{cwe}_m(C) \mid C$ of Type $T\rangle$

The finite Siegel Φ-operator

$$\Phi_m : \mathrm{Inv}(\mathcal{C}_m(T)) \to \mathrm{Inv}(\mathcal{C}_{m-1}(T)), \mathrm{cwe}_m(C) \mapsto \mathrm{cwe}_{m-1}(C)$$

defines a surjective graded $\mathbb{C}$-algebra homomorphism between invariant rings of complex matrix groups of different degree.

Φ is given by the variable substitution:

$$x_{(v_1,\ldots,v_m)} \mapsto \begin{cases} x_{(v_1,\ldots,v_{m-1})} & \text{if } v_m = 0 \\ 0 & \text{else} \end{cases}$$

Explanation for formula for Φ

$\mathrm{cwe}_{m-1}(C)$ is obtained from $\mathrm{cwe}_m(C)$ by counting only those matrices

$$\begin{array}{cccccc}
c_1^{(1)} & c_2^{(1)} & \dots & c_j^{(1)} & \dots & c_N^{(1)} \\
c_1^{(2)} & c_2^{(2)} & \dots & c_j^{(2)} & \dots & c_N^{(2)} \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
c_1^{(m)} & c_2^{(m)} & \dots & c_j^{(m)} & \dots & c_N^{(m)} \\
 & & & \uparrow & & \\
 & & & v \in V^m & &
\end{array}$$

in which the last row is zero.

This is expressed by the variable substitution

$$x_{(v_1,\dots,v_m)} \mapsto \begin{cases} x_{(v_1,\dots,v_{m-1})} & \text{if } v_m = 0 \\ 0 & \text{else} \end{cases}$$

A canonical right inverse of Φ.

$$(p,q)_m := p(\frac{\partial}{\partial x})(\overline{q}) \text{ for } p,q \in \mathbb{C}[x_v : v \in V^m]_N$$

defines a positive definite Hermitian product on the homogeneous component $\mathbb{C}[x_v : v \in V^m]_N$.
The monomials of degree N form an orthogonal basis and

$$(\prod_{v \in V^m} x_v^{n_v}, \prod_{v \in V^m} x_v^{n_v})_m = \prod_{v \in V^m} (n_v!).$$

Then $\Phi_m : \ker(\Phi_m)^\perp \to \mathrm{Inv}(\mathcal{C}_{m-1}(T))$ is an isomorphism with inverse

$$\varphi_m : \mathrm{Inv}(\mathcal{C}_{m-1}(T)) \to \mathrm{Inv}(\mathcal{C}_m(T)), x_{(v_1,\ldots,v_{m-1})} \mapsto R(x_{(v_1,\ldots,v_{m-1},0)})$$

where $R(p) = \frac{1}{|\mathcal{C}_m(T)|} \sum_{g \in \mathcal{C}_m(T)} p(gx)$ is the Reynolds operator (the orthogonal projection onto the invariant ring).
Note that R is not a ring homomorphism.

The orthogonal decomposition of the invariant space.

This yields an orthogonal decomposition of the space of degree N invariants of $\mathcal{C}_m(T)$

$$\mathrm{Inv}_N(\mathcal{C}_m(T)) = \ker(\Phi_m) \perp \varphi_m^{-1}(\mathrm{Inv}_N(\mathcal{C}_{m-1}(T))) =$$

$$\ker(\Phi_m) \perp \varphi_m^{-1}(\ker(\Phi_{m-1}) \perp \varphi_{m-1}^{-1}(\mathrm{Inv}_N(\mathcal{C}_{m-2})(T))) =$$

$$Y_m \perp Y_{m-1} \perp \ldots \perp Y_0$$

such that for all $0 \leq k \leq m$ the mapping

$$\mathrm{cwe}_m : \mathcal{Y}_k \to Y_k.$$

is an isomorphism of vector spaces.

$$\begin{array}{rccccccccccc} \mathcal{V} = & \mathcal{Y}_n & \perp \ldots \perp & \mathcal{Y}_{m+1} & \perp & \mathcal{Y}_m & \perp & \mathcal{Y}_{m-1} & \perp \ldots \perp & \mathcal{Y}_0 \\ \mathrm{cwe}_m & \downarrow & \ldots & \downarrow & & \downarrow & & \downarrow & \ldots & \downarrow \\ \mathrm{Inv}_N(\mathcal{C}_m(T)) = & 0 & \perp \ldots \perp & 0 & \perp & Y_m & \perp & Y_{m-1} & \perp \ldots \perp & Y_0 \end{array}$$

$K_N(T)$ and double cosets.

The Kneser-Hecke operator $K_N(T)$ acts on $\mathrm{Inv}_N(\mathcal{C}_m(T))$ as $\delta_m(K_N(T))$ having $Y_m \perp Y_{m-1} \perp \ldots \perp Y_0$ as the eigenspace decomposition.

$$\mathcal{C}_m(T) = \underbrace{S.(\ker(\lambda) \times \ker(\lambda))}_{\mathcal{E}_m(T)} .\mathcal{G}_m(T)$$

Choose a suitable subgroup $\mathcal{U}_1$ of $\mathcal{E}_m(T)$ that corresponds to a 1-dimensional subspace of $(\ker(\lambda) \times \ker(\lambda))$ and let

$$p_1 := \frac{1}{q} \sum_{u \in \mathcal{U}_1} u \in \mathbb{C}^{q^m \times q^m}$$

be the orthogonal projection onto the fixed space of $\mathcal{U}_1$ and let

$$H_m(T) := \mathcal{C}_m(T) p_1 \mathcal{C}_m(T) = \dot{\bigcup}_{U \in X} p_U \mathcal{C}_m(T).$$

Then this double coset acts on $\mathrm{Inv}_N(\mathcal{C}_m(T))$ via

$$\Delta_N(H_m(T)) : f \mapsto \frac{1}{|X|} \sum_{U \in X} f(x p_U)$$

Explicit description as sum of double cosets.

Theorem.(N. 2006)

$$(q-1)\delta_m(K_N(T)) = q^{n-m-e}((q-1)\Delta_N(H_m(T)) + \mathrm{id}) - (q^m + a)\,\mathrm{id}$$

where $n = N/2$ and e, a are as follows:

T	q^E	q_{I}^E	q_1^E	q_{II}^E	q_1^H	q^H
a	0	$q-1$	0	0	$\sqrt{q}-1$	$\sqrt{q}-1$
e	0	0	1	1	$1/2$	$-1/2$

Conclusion.

- formal notion of Type $T = (R, V, \beta, \Phi)$.
- self-dual code C of Type T.
- genus-m complete weight enumerators
- the associated Clifford-Weil group $\mathcal{C}_m(T)$, a finite complex matrix group of degree $|V|^m$ such that

$$\mathrm{Inv}_N(\mathcal{C}_m(T)) = \langle \mathrm{cwe}_m(C) \mid C = C^{\perp} \leq V^N \text{ of Type } T \rangle$$

- In particular the scalar subgroup $\mathcal{C}_m(T) \cap \mathbb{C}^* \,\mathrm{id}$ is cyclic of order

$$\min\{N \mid \text{ there is a code } C \leq V^N \text{ of Type } T\}.$$

- $\mathcal{C}_m(T)$ has a nice group theoretic structure.
- $\Phi_m : \mathrm{Inv}(\mathcal{C}_m(T)) \to \mathrm{Inv}(\mathcal{C}_{m-1}(T))$
- if R is a field then:
- As in modular forms theory, the invariant ring of $\mathcal{C}_m(T)$ can be investigated using Hecke operators.
- The Hecke algebra is generated by the incidence matrix of the Kneser neighbouring graph.
- Obtain linear relations between weight enumerators.