Integral forms of algebraic groups.

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Orthogonal groups.

$V \cong \mathbb{Q}^n$, $q : V \to \mathbb{Q}$ a positive definite quadratic form,
$b_q(X,Y) := q(X+Y) - q(X) - q(Y)$
the associated symmetric bilinear form.
A lattice L in V is a finitely generated $\mathbb{Z}$-submodule of full rank.

$$L = \{\sum_{i=1}^{n} z_i B_i \mid z_i \in \mathbb{Z}\}, \ (B_1, \ldots, B_n) \text{ a basis of } V.$$

L is called even, if $q(L) \subset \mathbb{Z}$.
$L^{\#} := \{X \in V \mid b_q(X,Y) \in \mathbb{Z}$ for all $Y \in L\}$ dual lattice.
L is called integral if $L \subset L^{\#}$ and unimodular if $L = L^{\#}$.
If $L \subset L^{\#}$ then $|L^{\#}/L| = \det(L) = \det(b_q(B_i, B_j))$ determinant of L.

Equivalence of lattices.

Let L, M be lattices in (V, q).

- $L \cong M$, L isometric to M, if there is $\varphi \in O(V, q)$ with $\varphi(L) = M$.
$$\mathrm{Class}(L) = \{M \mid M \cong L\}.$$
- L and M are in the same genus if for all primes p there are $\varphi_p \in O(V, q)$ such that
$$\varphi_p(L \otimes \mathbb{Z}_{(p)}) = M \otimes \mathbb{Z}_{(p)}.$$

$\mathrm{Genus}(L) = \mathrm{Class}(L_1) \,\dot\cup\, \ldots \,\dot\cup\, \mathrm{Class}(L_h)$
h is called the class number of L.

Adelic formulation.

$$\mathbb{A} := \{x \in \mathbb{R} \times \prod_p \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for almost all } p\}$$

the adele ring of $\mathbb{Q}$.
The adelic orthogonal group $O(V \otimes \mathbb{A}, q)$ acts on the set of all $\mathbb{Z}$-lattices in V.
$g = (g_p) \in O(V \otimes \mathbb{A}, q)$, then $g(L) = M$ if $M \otimes \mathbb{Z}_p = g_p(L \otimes \mathbb{Z}_p)$ for all primes p.

$$O(V, q) \leq O(V \otimes \mathbb{A}, q).$$

$$\mathrm{Class}(L) = O(V, q) \cdot L, \text{ and } \mathrm{Genus}(L) = O(V \otimes \mathbb{A}, q) \cdot L.$$

Strong approximation.

Theorem.
Assume that L is an even lattice of dimension ≥ 3 such that $\det(L)$ is not a multiple of p.
Then every isometry class of lattices in the genus of L contains a lattice M such that

$$M \otimes \mathbb{Z}_\ell = L \otimes \mathbb{Z}_\ell \text{ for all } \ell \neq p.$$

$$\mathrm{Genus}(L) = \cup\{\mathrm{Class}(M) \mid M \otimes \mathbb{Z}_\ell = L \otimes \mathbb{Z}_\ell \text{ for all } \ell \neq p\}$$

Kneser neighboring method.

$M, N \in \mathrm{Genus}(L)$ are called p-neighbors if $[M : (M \cap N)] = [N : (M \cap N)] = p$. Notation $M \underset{p}{—} N$

Theorem (M. Kneser)
For any $M \in \mathrm{Genus}(L)$ there is a lattice $M \in \mathrm{Class}(M')$ and a chain of successive p-neighbors connecting L and M in the genus of L.

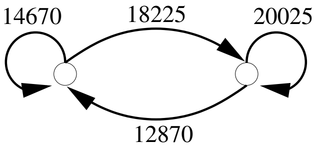
$$L \underset{p}{—} L_1 \underset{p}{—} L_2 \underset{p}{—} \ldots \underset{p}{—} L_k \underset{p}{—} M$$

This yields a very useful algorithm to enumerate the genus. The Kneser p-neighbor graph is the multigraph with vertices $\{\mathrm{Class}(M) \mid M \in \mathrm{Genus}(L)\}$ and adjacency matrix

$$K_p : (\mathrm{Class}(M), \mathrm{Class}(N)) \mapsto |\{L \mid M \underset{p}{—} L, L \cong N\}|.$$

Example for a Kneser p-neighbor graph.

The genus of the even unimodular lattices of dimension 16:
$\mathrm{Class}(E_8 \perp E_8) \cup \mathrm{Class}(D_{16}^+)$



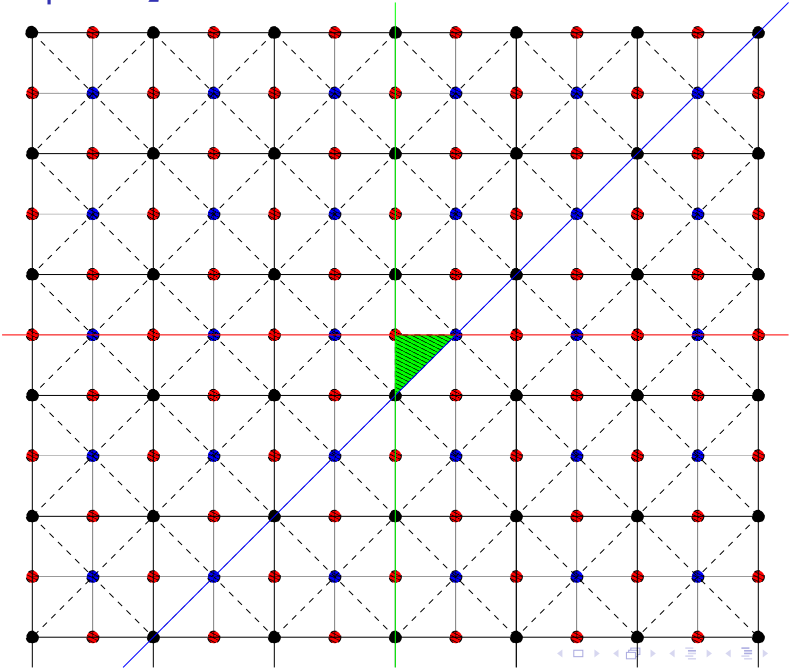
$$K_2 = \begin{pmatrix} 14670 & 18225 \\ 12870 & 20025 \end{pmatrix}$$

The adjacency matrix K_p defines the action of a
Hecke-operator on the subspace of modular forms generated
by the theta-series of lattices in $\mathrm{Genus}(L)$.
Study such operators for other algebraic groups.
For unitary groups, this is work in progress with A. Krieg.

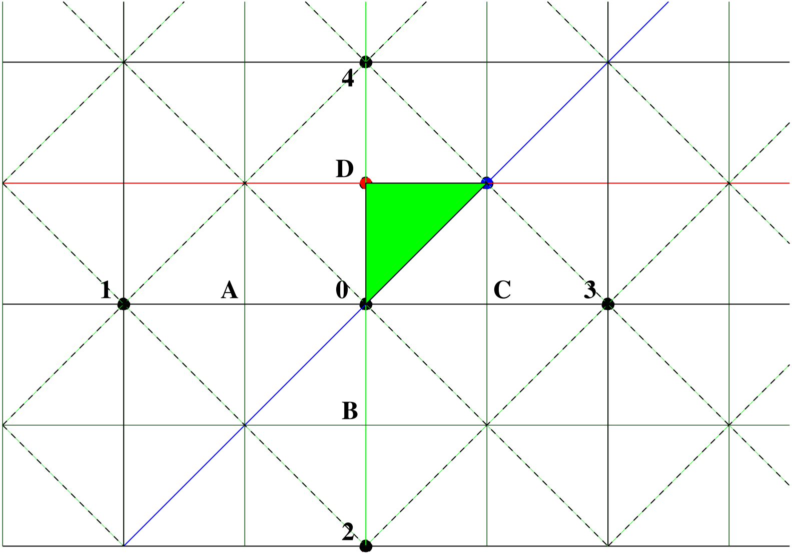
The p-adic affine building.

- The group $O(V \otimes \mathbb{Q}_p, q)$ is an algebraic group over a local field.
- The Bruhat-Tits building is a combinatorial object to organize the compact subgroups of $O(V \otimes \mathbb{Q}_p, q)$ that arise as intersections of stabilizers of p-elementary lattices.
- Points correspond to $\mathrm{End}(L) \cap \mathrm{End}(L^{\#})$ with $pL^{\#} \subset L \subset L^{\#}$.
- If L is an even unimodular lattice, then $\mathrm{Aut}(L \otimes \mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup of $O(V \otimes \mathbb{Q}_p, q)$ for all primes p.
- If L is even and $L^{\#}/L$ has square-free exponent, then $\mathrm{Aut}(L \otimes \mathbb{Z}_p)$ is a maximal compact subgroup of $O(V \otimes \mathbb{Q}_p, q)$ for all primes p and hyperspecial for all but finitely many.

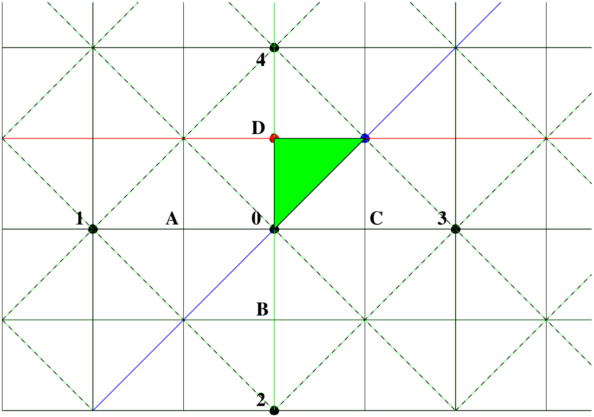
Example: $\tilde{C}_2$.



$\tilde{C}_2$: neighbors in apartment.



Neighbors: explicit bases.

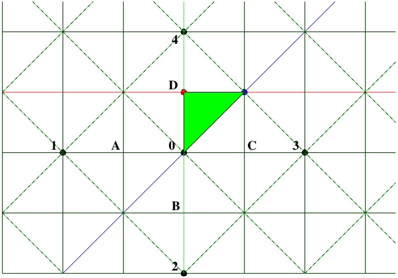


Basis (e, f, e', f'): $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$L = \langle e, f, e', f' \rangle$, $M = \langle e, f, pe', f' \rangle$, $N = \langle pe, f, pe', f' \rangle$.

$\ldots N^{\#} \supset M^{\#} \supset L = L^{\#} \supset M \supset N = pN^{\#} \supset pM^{\#} \supset pL \supset \ldots$

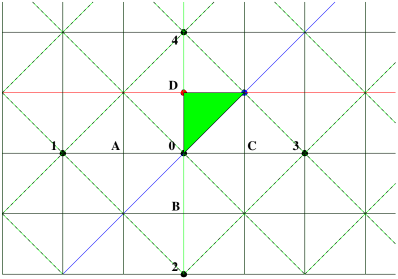
Fundamental reflections.



$L = \langle e, f, e', f' \rangle$, $M = \langle e, f, pe', f' \rangle$, $N = \langle pe, f, pe', f' \rangle$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/p \\ 0 & 0 & p & 0 \end{pmatrix}$$

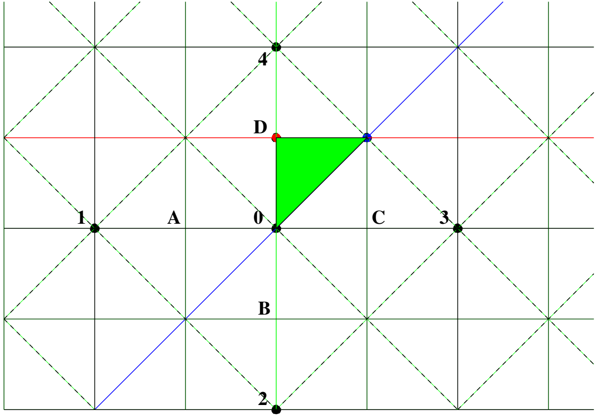
Neighbors: reflections.



$$A = \begin{pmatrix} 0 & 1/p & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & p & 0 & 0 \\ 1/p & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & 1/p & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/p \\ 0 & 0 & p & 0 \end{pmatrix}$$

Neighbors: lattices.



$$L0 = L = \langle e, f, e', f' \rangle,$$

$$L1 = LA = \langle e, f, pe', 1/pf' \rangle,\ L3 = LC = \langle e, f, 1/pe', pf' \rangle,$$

$$L2 = LB = \langle 1/pe, pf, e', f' \rangle,\ L4 = LD = \langle pe, 1/pf, e', f' \rangle.$$

Global situation.

Isometrie classes of lattices yield a "coloring" of the hyperspecial points in the genus of L.

Example. $p = 3$, $L = \begin{pmatrix} 18 & 1 \\ 1 & 36 \end{pmatrix} \perp \begin{pmatrix} 18 & 1 \\ 1 & 54 \end{pmatrix}$

Big class number and the lattices in the picture above are

$$L1 = \begin{pmatrix} 18 & 1 \\ 1 & 36 \end{pmatrix} \perp \begin{pmatrix} 162 & 1 \\ 1 & 6 \end{pmatrix}$$

$$L2 = \begin{pmatrix} 2 & 1 \\ 1 & 324 \end{pmatrix} \perp \begin{pmatrix} 18 & 1 \\ 1 & 54 \end{pmatrix}$$

$$L3 = \begin{pmatrix} 18 & 1 \\ 1 & 36 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 486 \end{pmatrix}$$

$$L4 = \begin{pmatrix} 162 & 1 \\ 1 & 4 \end{pmatrix} \perp \begin{pmatrix} 18 & 1 \\ 1 & 54 \end{pmatrix}$$

and these are pairwise non-isometric lattices.

The mass formula.

If $L_1, \ldots, L_h$ is a system of representatives of isometry classes of lattices in the genus of L then

$$\sum_{i=1}^{h} |\operatorname{Aut}(L_i)|^{-1} = \operatorname{mass}(\operatorname{Genus}(L))$$

where $\operatorname{mass}(\operatorname{Genus}(L))$ can be read off from the local stabilizers $\operatorname{Stab}_{O(V\otimes\mathbb{Q}_p,q)}(L \otimes \mathbb{Z}_p)$ (local densities).
Idea: The isometry classes of lattices are the $O(V,q)$-orbits in the $O(V \otimes \mathbb{A}, q)$-orbit $\operatorname{Genus}(L)$.

$$\operatorname{Aut}(L) = \{g \in O(V,q) \mid g(L) = L\}$$

is the stabilizer of L in $O(V,q)$.

Proof of mass formula.

for finite sets: Finite group G acting on finite set M with orbits $m_1 G, \ldots, m_h G$ and stabilizers $S_i := \mathrm{Stab}_G(m_i)$. Then

$$|M| = \sum_{i=1}^{h} |m_i G| = \sum_{i=1}^{h} \frac{|G|}{|S_i|}$$

and hence

$$\sum_{i=1}^{h} \frac{1}{|S_i|} = \frac{|M|}{|G|}.$$

In our situation this reads as

$$\sum_{i=1}^{h} \frac{1}{|\operatorname{Aut}(L_i)|} = \frac{|O(V \otimes \mathbb{A}, q) \cdot L|}{|O(V, q)|} = \frac{|O(V \otimes \mathbb{A}, q)|}{|\operatorname{Aut}(L \otimes \mathbb{A})||O(V, q)|}.$$

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In our situation this reads as

$$\sum_{i=1}^{h} \frac{1}{|\operatorname{Aut}(L_i)|} = \mu(O(V,q) \backslash O(V \otimes \mathbb{A}, q) / \operatorname{Aut}(L \otimes \mathbb{A})).$$

Linear algebraic groups.

- Let $G \leq GL_m$ be a reductive linear algebraic group defined over some number field K.
- Let O_K be the ring of integers in K.
- Then an integral form G of G is given as $G = \mathrm{Stab}_G(\Lambda)$ for some O_K-lattice Λ in K^m.
- Two integral forms G and G' are called isomorphic, if they are conjugate under $G(K)$.
- They lie in the same genus, if they are conjugate under $G(\mathbb{A}_K)$ where $\mathbb{A}_K$ is the adèle ring of K.
- Interesting genera are the models studied by Dick Gross: $O_K = \mathbb{Z}$ and $G(\mathbb{Z}_p)$ is hyperspecial for all primes p

The Type of a maximal integral form.

- With Arjeh Cohen and Wilhelm Plesken we generalized this notion to the notion of maximal integral forms, where $\mathsf{G}(O_\wp)$ is maximal parahoric for all primes $\wp \trianglelefteq O_K$.
- Then the Type of the genus of a maximal integral form is the sequence of maximal parahoric subgroups

$$(\mathsf{G}(O_\wp) \mid \wp \trianglelefteq O_K)$$

Cayley octonions.

Let $\mathcal{C}_{\mathbb{Q}} := \langle 1 = e_0, e_1, \ldots, e_7 \rangle$ be the split alternative algebra of Cayley octonions.

$$\begin{aligned} e_i^2 &= -e_0 && \text{for } i = 1, \ldots, 7, \\ e_i e_j &= -e_j e_i = e_k && \text{if } (i,j,k) = (1+\ell, 2+\ell, 4+\ell) \text{ for some } \ell, \\ e_0 e_j &= e_j e_0 = e_j && \text{for all } j. \end{aligned}$$

$G_2 := \mathrm{Aut}(\mathcal{C})$ linear algebraic group defined over $\mathbb{Q}$.
The norm form N with

$$N(\sum_{i=0}^{7} a_i e_i) := \sum_{i=0}^{7} a_i^2$$

is a multiplicative positive definite quadratic form on $\mathcal{C}_{\mathbb{Q}}$.
$G_2 \hookrightarrow \mathrm{Stab}_{O(N)}(e_0) \cong O_7$.

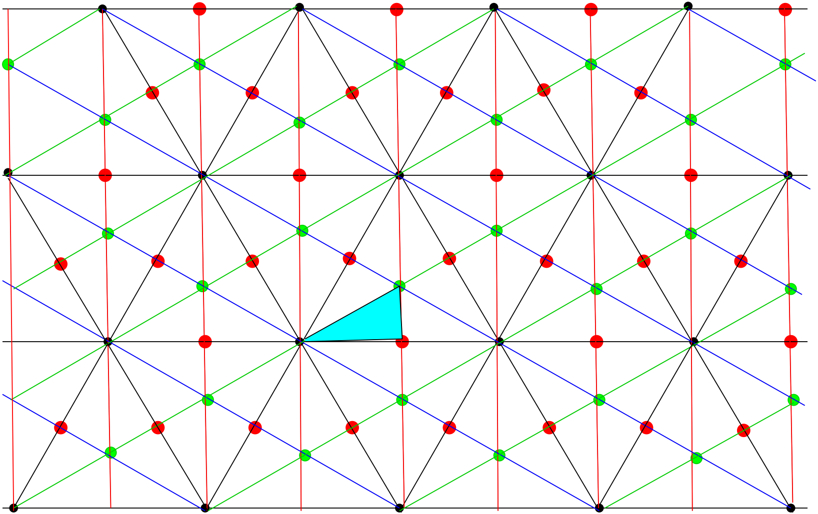
Integral forms of G_2.

- Integral forms of G_2 arise as stabilizers of Cayley orders, these are lattices that are closed under multiplication.
- Maximal Cayley order

$$\mathcal{M} = \langle e_0, e_1, e_2, e_3, h, e_1 h, e_2 h, e_3 h \rangle_{\mathbb{Z}} \text{ where } h = \frac{1}{2}(e_0 + e_1 + e_2 - e_4)$$

- $\mathrm{Aut}(\mathcal{M}) \cong G_2(2)$.
- $\mathcal{M}$ defines the unique model of G_2, its genus has class number 1.

An apartment in the building of G_2.

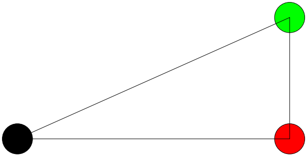


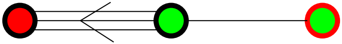
The local picture for $p \neq 2$.

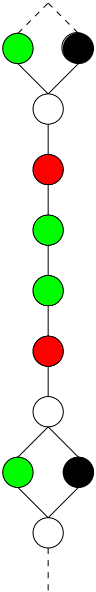
Cayley orders for the maximal

parahoric subgroups

for primes not dividing 2







Maximal finite subgroups of G_2.

- What are interesting maximal integral forms for G_2 ?
- Of course the model from above. Other models over totally real numberfields. Which fields ?
- Theorem. (Arjeh Cohen) Let F be a maximal finite Lie primitive subgroup of $G_2(\mathbb{C})$. Then F is one of $G_2(2), 2^3.\mathrm{GL}_3(2), \mathrm{PSL}_2(8), \mathrm{PSL}_2(13)$.
- In all cases there is a minimal defining field K and
- each of these maximal finite groups defines a unique maximal integral form over O_K.

The type of these integral forms.

F	$G_2(2)$	$2^3.\mathrm{GL}_3(2)$	$\mathrm{PSL}_2(8)$	$\mathrm{PSL}_2(13)$
K	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}[\zeta_9 + \zeta_9^{-1}]$	$\mathbb{Q}[\sqrt{13}]$
h	1	1	8	$>$ 14372
G (O_2)	G_2	A_2	$A_1 + A_1$	G_2
G $(O_\wp)$	hyperspecial for $2 \notin \wp$