Lattices and modular forms.

Gabriele Nebe

Lehrstuhl D für Mathematik

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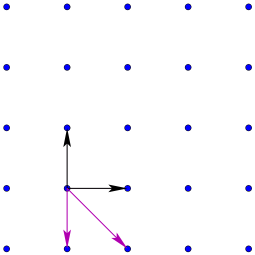
Lattices

Definition.

A lattice L in Euclidean n-space $(\mathbb{R}^n, (\,,\,))$ is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis $B = (b_1, \ldots, b_n)$ of $\mathbb{R}^n$

$$L = \langle b_1, \ldots, b_n \rangle_{\mathbb{Z}} = \{\sum_{i=1}^{n} a_i b_i \mid a_i \in \mathbb{Z}\}.$$

$\mathcal{L}_n := \{L \leq \mathbb{R}^n \mid L \text{ is lattice } \}$ the set of all lattices in $\mathbb{R}^n$.



$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Invariants of lattices.

Gram matrix.

$\mathrm{Gram}(L) = \{g\,\mathrm{Gram}(B)g^{tr} \mid g \in \mathrm{GL}_n(\mathbb{Z})\}$ where

$$\mathrm{Gram}(B) = ((b_i, b_j)) = BB^{tr} \in \mathbb{R}^{n\times n}_{sym}$$

is the Gram matrix of B.

Invariants from Gram matrix.

- $\det(L) = \det(\mathrm{Gram}(B)) = \det(BB^{\mathrm{tr}})$ the determinant of L is the square of the volume of the fundamental parallelotope of B.
- $\min(L) = \min\{(\ell,\ell) \mid 0 \neq \ell \in L\}$ the minimum of L.
- $\mathrm{Min}(L) = \{\ell \in L \mid (\ell,\ell) = \min(L)\}$ the shortest vectors of L.
- $|\mathrm{Min}(L)|$ the kissing number of L.

Properties of lattices.

Dual lattice.

Let $L = \langle b_1, \ldots, b_n \rangle_{\mathbb{Z}} \leq \mathbb{R}^n$ be a lattice. Then the dual lattice

$$L^{\#} := \{x \in \mathbb{R}^n \mid (x, \ell) \in \mathbb{Z} \forall \ell \in L\}$$

is again a lattice in $\mathbb{R}^n$ and the dual basis $B^* = (b_1^*, \ldots, b_n^*)$ with $(b_i^*, b_j) = \delta_{ij}$ is a lattice basis for $L^{\#}$.
$\mathrm{Gram}(B^*) = \mathrm{Gram}(B)^{-1}$.

Integral lattices.

- L is called integral, if $L \subset L^{\#}$ or equivalently $\mathrm{Gram}(B) \in \mathbb{Z}^{n \times n}$.
- L is called even, if $Q(\ell) := \frac{1}{2}(\ell, \ell) \in \mathbb{Z}$ for all $\ell \in L$.
- Even lattices are integral and an integral lattice is even if $(b_i, b_i) \in 2\mathbb{Z}$ for all $i = 1, \ldots, n$.
- L is called unimodular if $L = L^{\#}$.

Orthogonal decomposition.

Definition.

Let $L_1 \leq \mathbb{R}^{n_1}$ and $L_2 \leq \mathbb{R}^{n_2}$ be lattices. Then $L_1 \perp L_2 \leq \mathbb{R}^{n_1} \perp \mathbb{R}^{n_2}$ is called the orthogonal sum of L_1 and L_2. A lattice is orthogonally indecomposable if it cannot be written as orthogonal sum of proper sublattices.

If $G_i \in \mathrm{Gram}(L_i)$ are Gram matrices of L_i, then the block diagonal matrix $\mathrm{diag}(G_1, G_2)$ is a Gram matrix of $L_1 \perp L_2$, but not all Gram matrices of $L_1 \perp L_2$ are block diagonal.

Theorem (M. Kneser).

Every lattice L has a unique orthogonal decomposition $L = L_1 \perp \ldots \perp L_s$ with indecomposable lattices L_i.

Construction of orthogonal decomposition.

Proof.

- Call $x \in L$ indecomposable, if $x \neq y + z$ for $y, z \in L - \{0\}$, $(y, z) = 0$.
- Then any $0 \neq x \in L$ is sum of indecomposables,
- because if x is not itself indecomposable then $x = y + z$ with $(y, z) = 0$ and hence $0 < (y, y) < (x, x)$, $0 < (z, z) < (x, x)$.
- So this decomposition process terminates.
- In particular L is generated by indecomposable vectors.
- Two indecomposable vectors $y, z \in L$ are called connected, if there are indecomposable vectors $x_0 = y, x_1, \ldots, x_t = z$ in L, such that $(x_i, x_{i+1}) \neq 0$ for all i.
- This yields an equivalence relation on the set of indecomposable vectors in L with finitely many classes $K_1, \ldots, K_s$.
- If $L_i := \langle K_i \rangle_{\mathbb{Z}}$ then $L = L_1 \perp \ldots \perp L_s$ is the unique orthogonal decomposition of L in indecomposable sublattices.

Equivalence and automorphism groups.

Equivalence.

The orthogonal group
$O_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid (vg, wg) = (v, w)$ for all $v, w \in \mathbb{R}^n\}$ acts on $\mathcal{L}_n$ preserving all invariants that can be deduced from the Gram matrices like integrality, minimum, determinant, density etc..
Lattices in the same $O_n(\mathbb{R})$-orbit are called isometric.

Automorphism group.

The automorphism group of L is

$$\begin{aligned} &\mathrm{Aut}(L) = \{\sigma \in O_n(\mathbb{R}) \mid \sigma(L) = L\} \\ &\cong \{g \in \mathrm{GL}_n(\mathbb{Z}) \mid g\,\mathrm{Gram}(B) g^{tr} = \mathrm{Gram}(B)\} \end{aligned}$$

$\mathrm{Aut}(L)$ is a finite group and can be calculated efficiently, if the finite set of vectors $\{\ell \in L \mid Q(\ell) \leq \max_{i=1}^n Q(b_i)\}$ can be stored. (Bernd Souvignier, Wilhelm Plesken)

Reflections and automorphisms.

- For a vector $0 \neq v \in \mathbb{R}^n$ the reflection along v is

$$\sigma_v : x \mapsto x - 2\frac{(x,v)}{(v,v)}v = x - \frac{(x,v)}{Q(v)}v.$$

- $\sigma_v \in O_n(\mathbb{R})$.
- If $L \subset L^{\#}$ is an integral lattice and $v \in L$ satisfies $(v,v) \in \{1,2\}$ then $\sigma_v \in \mathrm{Aut}(L)$.
- If L is even then define

$$S(L) := \langle \sigma_v \mid v \in L, Q(v) = 1 \rangle$$

the reflection subgroup of $\mathrm{Aut}(L)$

Root lattices.

Definition.

- An even lattice L is called a root lattice, if $L = \langle \ell \in L \mid Q(\ell) = 1 \rangle$. Then $R(L) := \{\ell \in L \mid Q(\ell) = 1\}$ is called the set of roots of L.
- A root lattice L is called decomposable if $L = M \perp N$ for proper root lattices M and N and indecomposable otherwise.

Theorem.

Let L be an indecomposable root lattice. Then $S(L)$ acts irreducibly on $\mathbb{R}^n$.

Proof. Let $0 \neq U < \mathbb{R}^n$ be an $S(L)$-invariant subspace and $a \in R(L) - U$ Then $\sigma_a(u) = u - (u, a)a \in U$ for all $u \in U$ implies that $(u, a) = 0$ for all $u \in U$ and hence $a \in U^{\perp}$. So $R(L) \subset U \cup U^{\perp}$ and L is decomposable.

Indecomposable root lattices.

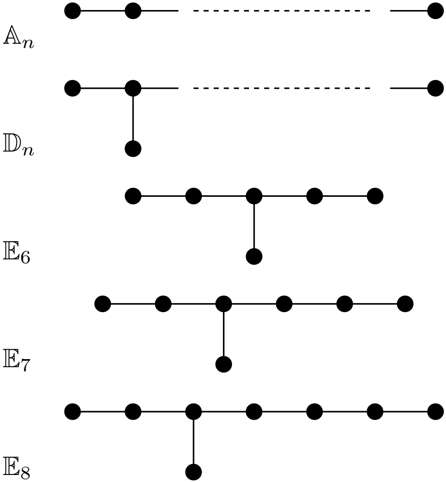
Theorem.

- Let $L = \langle R(L) \rangle$ be a root lattice.
- Then L has a basis $B \in R(L)^n$ such that $(b_i, b_j) \in \{0, -1\}$ for all $i \neq j$.
- The Gram matrix of this basis is visualised by a Dynkin diagram, a graph with n vertices corresponding to the n basis elements and with an edge (i, j) if $(b_i, b_j) = -1$.
- The Dynkin diagram is connected, if L is indecomposable.

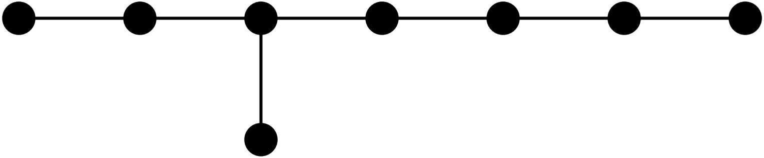
Theorem.

Let $L \in \mathcal{L}_n$ be an indecomposable root lattice. Then L is isometric to one of $\mathbb{A}_n$, $\mathbb{D}_n$, if $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ if $n = 6, 7, 8$ respectively.

Dynkin diagrams of indecomposable root lattices.



Gram matrix for $\mathbb{E}_8$.



yields the following Gram matrix

$$\mathrm{Gram}(\mathbb{E}_8) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

The indecomposable root lattices.

- Let $r, s \in R(\mathbb{E}_8)$ with $(r, s) = -1$.
  Then $\mathbb{E}_7 = r^\perp \cap \mathbb{E}_8$ and $\mathbb{E}_6 = \langle r, s\rangle^\perp \cap \mathbb{E}_8$.
- If $(e_1, \ldots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$ then
  $\mathbb{D}_n = \langle e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\rangle_{\mathbb{Z}}$.
- $\mathbb{A}_{n-1} \leq (e_1 + \ldots + e_n)^\perp \cong \mathbb{R}^{n-1}$ has basis
  $(e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n)$.
- $h := |R(L)|/n \in \mathbb{Z}$ is called the Coxeter number of an indecomposable root lattice L.

L	$\|R(L)\|$	h	$\det(L)$	$S(L)$	$\mathrm{Aut}(L)$
$\mathbb{A}_n$	$n(n+1)$	$n+1$	$n+1$	S_{n+1}	$\pm S_{n+1}$
$\mathbb{D}_n$	$2n(n-1)$	$2(n-1)$	4	$C_2^{n-1} : S_n$	$C_2 \wr S_n$
$\mathbb{E}_6$	72	12	3	$PSp_4(3).2$	$C_2 \times PSp_4(3).2$
$\mathbb{E}_7$	126	18	2	$2.Sp_6(2)$	$2.Sp_6(2)$
$\mathbb{E}_8$	240	30	1	$2.O_8^+(2).2$	$2.O_8^+(2).2$

The Leech lattice.

The Leech lattice.

There is a unique even unimodular lattice Λ_{24} of dimension 24 without vectors of norm 2. $\mathrm{Aut}(\Lambda_{24}) = 2.Co_1$ is the sporadic quasisimple Conway group.

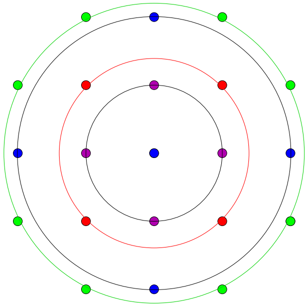
A construction of the Leech lattice.

- $\mathbb{E}_8$ has a hermitian structure over $\mathbb{Z}[\alpha]$ where $\alpha^2 - \alpha + 2 = 0$.
- The 3-dimensional $\mathbb{Z}[\alpha]$-lattice P_6 with hermitian Grammatrix $E = \begin{pmatrix} 2 & \alpha & -1 \\ \overline{\alpha} & 2 & \alpha \\ -1 & \overline{\alpha} & 2 \end{pmatrix}$ is known as the Barnes-lattice.
- Then the Leech lattice Λ_{24} is $\mathbb{E}_8 \otimes_{\mathbb{Z}[\alpha]} P_6$ with euclidean inner product $(x, y) = \mathrm{Tr}(h(x, y))$.

Theta-series of lattices.

- The theta series $\theta_L = \sum_{\ell \in L} q^{Q(\ell)}$.
- Assume that L is an even lattice and let $L_a := \{\ell \in L \mid Q(\ell) = a\}$. Then L_a is a finite $\mathrm{Aut}(L)$-set and $\theta_L = \sum_{a=0}^{\infty} |L_a| q^a$.
- $L = \sqrt{2}\mathbb{Z}^2$ the square lattice with Gram matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$:

  $\theta_L = 1 + 4q^1 + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + \ldots$
  $\mathrm{Aut}(L) \cong D_8$ (the symmetry group of a square)



Example: the hexagonal lattice.

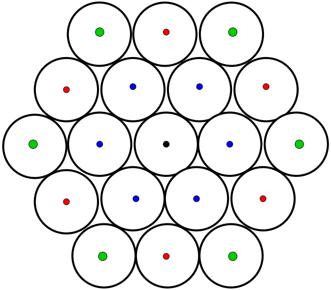
The hexagonal lattice.

Basis $B = ((1,1), (\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}))$, $\mathrm{Gram}(B) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\det(L) = 3$, $\min(L) = 2$, $\gamma(L) = \frac{2}{\sqrt{3}} \sim 1.1547$ (density .91)

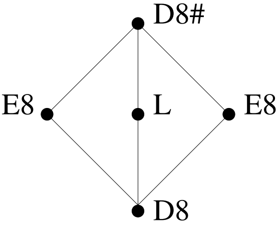
$\theta_L = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{16} + \ldots$

$\mathrm{Aut}(L) \cong D_{12}$ (the symmetry group of a regular hexagon)



Example: the $\mathbb{E}_8$-lattice.

- Let $(e_1, \ldots, e_8)$ be an orthonormal basis of $\mathbb{R}^8$ and consider $L := \mathbb{Z}^8 = \langle e_1, \ldots, e_8 \rangle_{\mathbb{Z}} = L^{\#}$.
- Let $\mathbb{D}_8 := \{\ell \in L \mid (\ell, \ell) \in 2\mathbb{Z}\}$ be the even sublattice of L.
- $\theta_{\mathbb{D}_8} = 1 + 112q + 1136q^2 + 3136q^3 + 9328q^4 + 14112q^5 + \ldots$
- Then $\mathbb{D}_8^{\#}/\mathbb{D}_8 = \langle e_1 + \mathbb{D}_8, v + \mathbb{D}_8 \rangle \cong C_2 \times C_2$, where $v = \frac{1}{2}\sum_{i=1}^{8} e_i$.
- $(v, v) = \frac{8}{4} = 2$ and $\mathbb{E}_8 = \langle \mathbb{D}_8, v \rangle$ is an even unimodular lattice.
- $\theta_{\mathbb{E}_8} = \theta_{\mathbb{D}_8} + \theta_{v+\mathbb{D}_8} = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + \ldots = 1 + 240(q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + \ldots)$



Theta series as holomorphic functions.

In the following we will consider even lattices L and the associated integral quadratic form $Q : L \to \mathbb{Z}, \ell \mapsto \frac{1}{2}(\ell, \ell) = \frac{1}{2}\sum_{j=1}^{n} \ell_j^2$.

Theorem.

Define $q(z) := \exp(2\pi i z)$ and $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ the upper half plane. The function

$$\theta_L : \mathbb{H} \to \mathbb{C}, \; z \mapsto \theta_L(z) = \sum_{\ell \in L} \exp(2\pi i z)^{Q(\ell)} = \sum_{a=0}^{\infty} |L_a| q(z)^a$$

is a holomorphic function on the upper half plane $\mathbb{H}$.
It satisfies $\theta_L(z) = \theta_L(z+1)$.

The theta series of the dual lattice.

Poisson summation formula.

For any well behaved function $f : \mathbb{R}^n \to \mathbb{C}$ and any lattice $L \in \mathfrak{L}_n$

$$\det(L)^{1/2} \sum_{x \in L} f(x) = \sum_{y \in L^{\#}} \hat{f}(y)$$

where $\hat{f}(y) = \int_{\mathbb{R}^n} f(x) \exp(-2\pi i(x,y))dx$ is the Fourier transform of f.

Theorem.

Let $L \in \mathfrak{L}_n$. Then $\theta_L(\frac{-1}{z}) = \left(\frac{z}{i}\right)^{n/2} \det(L)^{-1/2} \theta_{L^{\#}}(z)$.

Proof.

Proof of $\theta_L(\frac{-1}{z}) = \left(\frac{z}{i}\right)^{n/2} \det(L)^{-1/2}\theta_{L^\#}(z)$.

Both sides are holomorphic functions on $\mathbb{H}$, so it suffices to prove the identity for $z = it$ and $t \in \mathbb{R}_{>0}$.
The Fourier transform of

$$f(x) = \exp(\frac{-2\pi}{t}Q(x)) \text{ is } \hat{f}(y) = \sqrt{t}^n \exp(-2\pi t Q(y)).$$

Hence Poisson summation yields

$$\theta_L(\frac{-1}{it}) = \sum_{x\in L} f(x) = \det(L)^{-1/2} \sum_{y\in L^\#} \hat{f}(y) = \det(L)^{-1/2} t^{n/2} \theta_{L^\#}(it).$$

Poisson summation:

$$\det(L)^{1/2} \sum_{x\in L} f(x) = \sum_{y\in L^\#} \hat{f}(y)$$

The space of modular forms.

The group of biholomorphic mappings of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ is the group of Möbius transformations

$$z \mapsto A(z) := \frac{az+b}{cz+d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

For all $k \in \mathbb{Z}$ this yields an action $|_k$ of $\mathrm{SL}_2(\mathbb{R})$ on the space of meromorphic functions $f : \mathbb{H} \to \mathbb{C}$ defined by

$$f|_k A(z) := (cz+d)^{-k} f(\frac{az+b}{cz+d}).$$

Definition.

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called modular form of weight k, $f \in M_k$, if

$$f|_k A = f \text{ for all } A \in \mathrm{SL}_2(\mathbb{Z})$$

and f is holomorphic at $i\infty$.
f is called cuspform, $f \in M_k^0$, if additionally $\lim_{t\to\infty} f(it) = 0$.

Fourier expansion.

Remember: $f|_k A(z) := (cz+d)^{-k} f(\frac{az+b}{cz+d})$.

$$\mathrm{SL}_2(\mathbb{Z}) = \langle T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$$

where S acts on $\mathbb{H}$ by $z \mapsto -\frac{1}{z}$ and T by $z \mapsto z+1$.

Theorem.

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k, if $f(z) = f(z+1)$ and $f(\frac{-1}{z}) = (-z)^k f(z)$ and f is holomorphic at $i\infty$.

Theorem.

Let $f \in M_k$ for some k. Then $f(z) = f|_k T(z) = f(z+1)$ and hence f has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} c_n \exp(2\pi i z)^n = \sum_{n=0}^{\infty} c_n q(z)^n$$

The form f is a cuspform, if $c_0 = 0$.

Even unimodular lattices have dimension 8d.

Theorem.

Let $L = L^{\#} \in \mathcal{L}_n$ be even. Then $n \in 8\mathbb{Z}$.

Proof. Assume not. Replacing L by $L \perp L$ or $L \perp L \perp L \perp L$, if necessary, we may assume that $n = 4 + 8m$. Then by Poisson summation

$$\theta_L(Sz) = \theta_L(\frac{-1}{z}) = (\frac{z}{i})^{n/2}\theta_L(z) = -z^{n/2}\theta_L(z)$$

and since θ_L is invariant under T, we hence get

$$\theta_L((TS)(z)) = -z^{n/2}\theta_L(z)$$

where $(TS)(z) = \frac{-1}{z} + 1 = \frac{z-1}{z}$. $(TS)^2(z) = \frac{-z}{z-1} + 1 = \frac{-1}{z-1}$. Since $(TS)^3 = 1$ we calculate

$$\theta_L(z) = \theta_L((TS)^3 z) = \theta_L((TS)(TS)^2 z) = -(\tfrac{1}{z-1})^{n/2}\theta_L((TS)^2 z)$$
$$= (\tfrac{1}{z-1})^{n/2}(\tfrac{z-1}{z})^{n/2}\theta_L((TS)z) = (\tfrac{1}{z})^{n/2}\theta_L((TS)z) = -\theta_L(z)$$

a contradiction.

Theta series of even unimodular lattices are modular forms

Theorem.

If $L = L^{\#} \in \mathcal{L}_n$ is even, then $\theta_L(z) \in M_k$ with $k = \frac{n}{2}$.
In particular the weight of θ_L is half of the dimension of L and hence a multiple of 4.

Proof. $\theta_L(z) = \theta_L(z+1)$ because L is even.
From the Poisson summation formula we get

$$\theta_L(\frac{-1}{z}) = (\frac{z}{i})^{n/2} \det L^{-1/2} \theta_{L^{\#}}(z) = z^{n/2} \theta_{L^{\#}}(z)$$

since n is a multiple of 8 and $\det(L) = 1$.

The graded ring of modular forms.

Remember: $f|_k A(z) := (cz+d)^{-k} f(\frac{az+b}{cz+d})$.
Since $|_k$ is multiplicative

$$(f|_k A)(g|_m A) = (fg)|_{k+m} A$$

for all $A \in \mathrm{SL}_2(\mathbb{R})$ the space of all modular forms is a graded ring

$$\mathcal{M} := \bigoplus_{k=0}^{\infty} M_k.$$

Theorem.

$M_k = \{0\}$ if k is odd.

Proof: Let $A = -I_2 \in \mathrm{SL}_2(\mathbb{Z})$ and $f \in M_k$. Then
$f|_k A(z) = (-1)^k f(z) = f(z)$ for all $z \in \mathbb{H}$ and hence $f = 0$ if k is odd.

The ring of theta-series.

If L is an even unimodular lattice of dimension n, then n is a multiple of 8 and hence $\theta_L \in M_{n/2}$ is a modular of weight $k = n/2 \in 4\mathbb{Z}$.

$$\theta_L \in \mathcal{M}' := \bigoplus_{k=0}^{\infty} M_{4k}.$$

$E_4 := \theta_{E_8} \in M_4$ is the normalized Eisenstein series of weight 4. Put

$$\Delta := \frac{1}{720}(\theta_{E_8}^3 - \theta_{\Lambda_{24}}) = q - 24q^2 + 252q^3 - 1472q^4 + \ldots \in M_{12}$$

Theorem.

$\mathcal{M}' = \mathbb{C}[E_4, \Delta]$.

Theta series of certain lattices.

$\mathcal{M}' = \mathbb{C}[E_4, \Delta]$.

Corollary.

Let L be an even unimodular lattice of dimension n.

- If $n = 8$ then $\theta_L = \theta_{\mathbb{E}_8} = E_4 = 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m$.
- If $n = 16$ then $\theta_L = \theta_{\mathbb{E}_8 \perp \mathbb{E}_8} = E_4^2 = 1 + 480q + 61920q^2 + 1050240q^3 + \dots$.
- For $n = 24$ let $c_1 = |L_1|$ be the number of roots in L. Then $\theta_L = 1 + c_1 q + (196560 - c_1) q^2 + \dots$.
- Let L be an even unimodular lattice of dimension 80 with minimum 8. Then $|\operatorname{Min}(L)| =$ 1 250 172 000.

Extremal modular forms.

$$\mathcal{M}' = \bigoplus_{k=0}^{\infty} M_{4k} = \mathbb{C}[E_4, \Delta]$$

$E_4 = \theta_{\mathbb{E}_8} = 1 + 240q + \ldots \in M_4, \qquad \Delta = 0 + q + \ldots \in M_{12}$.

Basis of M_{4k}:

$$\begin{array}{llllll} E_4^k = & 1+ & 240kq+ & *q^2+ & \ldots & \\ E_4^{k-3}\Delta = & & q+ & *q^2+ & \ldots & \\ E_4^{k-6}\Delta^2 = & & & q^2+ & \ldots & \\ \vdots & & & & & \\ E_4^{k-3a}\Delta^a = & & \ldots & & q^a+ & \ldots \end{array}$$

where $a = \lfloor \frac{n}{24} \rfloor = \lfloor \frac{k}{3} \rfloor$.

Definition.

This space contains a unique form

$$f^{(k)} := 1 + 0q + 0q^2 + \ldots + 0q^a + f^{(k)}_{a+1} q^{a+1} + f^{(k)}_{a+2} q^{a+2} + \ldots$$

$f^{(k)}$ is called the extremal modular form of weight $4k$.

Extremal even unimodular lattices.

Theorem (Siegel).

$f^{(k)}_{a+1} > 0$ for all k and $f^{(k)}_{a+2} < 0$ for large k ($k \geq 5200$).

Corollary.

Let L be an n-dimensional even unimodular lattice. Then

$$\min(L) \leq 2 + 2\lfloor \frac{n}{24} \rfloor .$$

Lattices achieving this bound are called extremal.

Extremal even unimodular lattices L$\leq \mathbb{R}^n$

n	8	16	24	32	48	56	72	80
min(L)	2	2	4	4	6	6	8	8
number of extremal lattices	1	2	1	$\geq 10^6$	≥ 3	many	?	≥ 2