Lattices and spherical designs.

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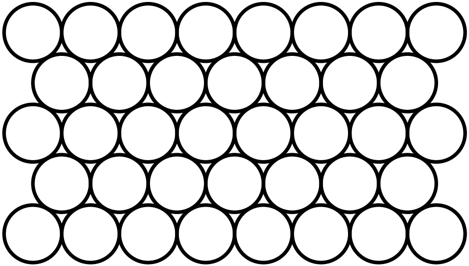
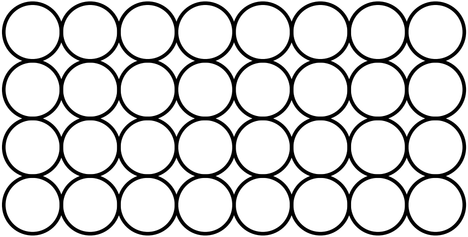
Sphere packings.





Lattice sphere packings.

Definition.

Let $L \in \mathcal{L}_n$. Then $\min(L) = \min\{(\ell,\ell) \mid \ell \in L\}$ is called the minimum of L and

$$\mathrm{Min}(L) := \{x \in L \mid (x,x) = \min(L)\}$$

the set of minimal vectors of L. The density of the sphere packing associated to a lattice L is proportional to the Hermite function

$$\gamma(L) := \frac{\min(L)}{\det(L)^{1/n}}$$

$\gamma_n := \max\{\gamma(L) \mid L \in \mathcal{L}_n\}$ is called the Hermite constant.

Remark.

The Hermite function is invariant under orthogonal transformations and scaling and hence defines a function

$$\gamma : \mathcal{L}_n/(\mathbb{R}_{>0}O_n(\mathbb{R})) \cong \mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})/(\mathbb{R}_{>0}O_n(\mathbb{R})) \to \mathbb{R}_{>0}.$$

Densest and locally densest lattices.

Theorem.

The densest lattices are known up to dimension 8 and in dimension 24.

n	1	2	3	4	5	6	7	8	24
γ_n	1	1.15	1.26	1.41	1.52	1.67	1.81	2	4
L	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{D}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	Λ_{24}
extreme	1	1	1	2	3	6	30	2408	

Definition.

A lattice $L \in \mathcal{L}_n$ is called extreme, if its similarity class realises a local maximum of the Hermite function.

There are only finitely many similarity classes of extreme lattices in $\mathcal{L}_n$. These are known for $n \leq 8$.

Notation and some linear algebra.

- Remember that we consider row vectors $x \in \mathbb{R}^n$ with the usual inner product $(x, y) = \sum_{i=1}^{n} x_i y_i = xy^{tr}$.
- For $0 \neq x \in \mathbb{R}^n$ the orthogonal projection onto $\langle x \rangle$ is given by $\frac{1}{xx^{tr}}\pi_x$ where
$$\pi_x = x^{tr}x \in \mathbb{R}^{n\times n}_{\mathrm{sym}}.$$
- The space $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ carries the Euclidean inner product $(A, B) := \mathrm{trace}(AB)$.
- $A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ defines a quadratic form on $\mathbb{R}^n$ by $\alpha \mapsto p_A(\alpha) = \alpha A \alpha^{tr}$.
- $(A, \pi_x) = p_A(x)$ for all $A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, $0 \neq x \in \mathbb{R}^n$.

Perfection and eutaxie.

Definition.

A lattice $L \in \mathcal{L}_n$ is called perfect, if

$$\langle \pi_x \mid x \in \mathrm{Min}(L) \rangle = \mathbb{R}^{n\times n}_{\mathrm{sym}}.$$

L is called eutactic, if there are $\lambda_x > 0$ for all $x \in \mathrm{Min}(L)$ such that

$$I_n = \sum_{x \in \mathrm{Min}(L)} \lambda_x \pi_x.$$

It is called strongly eutactic if all λ_x can be chosen to be equal.

Remarks.

- $\pi_x = \pi_{-x}$ so if L is perfect, then $|\mathrm{Min}(L)| \geq n(n+1)$.
- $L = M \perp N \Rightarrow L$ is not perfect, since $\pi_x \in \mathrm{End}_{\mathrm{sym}}(\mathbb{R}M) \oplus \mathrm{End}_{\mathrm{sym}}(\mathbb{R}N)$ for all $x \in \mathrm{Min}(L)$.
- If L is eutactic, then $\langle \mathrm{Min}(L) \rangle_{\mathbb{R}} = \mathbb{R}^n$.

A eutaxie criterion using $\mathrm{Aut}(L)$.

Theorem

Let $G \leq \mathrm{Aut}(L)$ be such that the natural representation $G \to O_n(\mathbb{R})$ is real irreducible. Then L is strongly eutactic.

Proof.

- $\mathrm{Min}(L) =: X$ is a union of G-orbits, so

$$f : \mathbb{R}^n \to \mathbb{R}, \alpha \mapsto \sum_{x \in X} (x, \alpha)^2$$

  is a G-invariant quadratic form on $\mathbb{R}^n$.
- Since $\mathbb{R}^n$ is an irreducible G-module, f is a multiple of the inner product $f(\alpha) = c(\alpha, \alpha)$ for some $c \in \mathbb{R}$.
- This implies that $I_n = \sum_{x \in X} c^{-1} \pi_x$ since
- $f(\alpha) = \sum_{x \in X} (x\alpha^{tr})^2 = \sum_{x \in X} \alpha \underbrace{x^{tr} x}_{\pi_x} \alpha^{tr} = c(\alpha, \alpha) = c\alpha I_n \alpha^{tr}$.

Indecomposable root lattices are strongly eutactic.

Corollary

- Let $L \in \mathcal{L}_n$ be an indecomposable root lattice. Then L and its dual $L^{\#}$ are strongly eutactic.
- If $h := |R(L)|/n \in \mathbb{Z}$ denotes the Coxeter number of L then for all $\alpha \in \mathbb{R}^n$
$$\sum_{x \in R(L)} (x, \alpha)^2 = 2h(\alpha, \alpha).$$
- A decomposable root lattice $L = R_1 \perp \ldots \perp R_s$ is strongly eutactic, iff $h(R_1) = \ldots = h(R_s)$.

Proof.
L (and $L^{\#}$) are strongly eutactic, since $S(L)$ is real irreducible. So $\sum_{x \in R(L)} (x, \alpha)^2 = c(\alpha, \alpha)$. Applying the Laplace operator $\Delta_\alpha := \sum_{i=1}^{n} \frac{\partial^2}{\partial^2 \alpha_i}$ we get $4|R(L)| = 2nc$ and hence $c = 2|R(L)|/n$.
Rules: $\Delta_\alpha (x, \alpha)^m = m(m-1)(x, x)(x, \alpha)^{m-2}$ and $\Delta_\alpha (\alpha, \alpha)^m = 2m(2m - 2 + n)(\alpha, \alpha)^{m-1}$

Voronoi's theorem.

Theorem (Voronoi, $\sim$ 1900).

A lattice L is extreme, if and only if it is perfect and eutactic.

Theorem.

All indecomposable root lattices are extreme.

Proof. We have seen that all indecomposable root lattices are strongly eutactic (hence eutactic) since their automorphism group is real irreducible.
Perfection is shown by inspection of their minimal vectors.
$\mathrm{Min}(\mathbb{A}_n) = \{e_i - e_j \mid 1 \leq i \neq j \leq n+1\} \supset \mathrm{Min}(\mathbb{A}_{n-1})$
By induction we assume that $\mathbb{A}_{n-1}$ is perfect.
$\{\pi_{e_i - e_{n+1}} \mid 1 \leq i \leq n\}$ is linearly independent modulo $\langle \pi_x \mid x \in \mathrm{Min}(\mathbb{A}_{n-1}) \rangle$ hence
$\dim(\langle \pi_x \mid x \in \mathrm{Min}(\mathbb{A}_n) \rangle) \geq \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$.

Strongly perfect lattices.

Definition (B. Venkov)

A lattice L is called strongly perfect if $\mathrm{Min}(L)$ is a spherical 5-design, so if for all $p \in \mathbb{R}[x_1, \ldots, x_n]_{deg \leq 5}$

$$\frac{1}{|\mathrm{Min}(L)|} \sum_{x \in \mathrm{Min}(L)} p(x) = \int_S p(t)dt$$

where S is the sphere containing $\mathrm{Min}(L)$.

The following are equivalent.

- $X := \mathrm{Min}(L)$ is a 5-design.
- $X := \mathrm{Min}(L)$ is a 4-design.
- $\sum_{x \in X} f(x) = 0$ for all harmonic polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree 2 and 4. harmonic means homogeneous and $\Delta(f) = 0$.

Continued.

The following are equivalent.

- $X := \mathrm{Min}(L)$ is a 5-design.
- $X := \mathrm{Min}(L)$ is a 4-design.
- $\sum_{x\in X} f(x) = 0$ for all harmonic polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree 2 and 4. f harmonic means $\Delta(f) = 0$
- There is some $c \in \mathbb{R}$ such that $\sum_{x\in X}(x,\alpha)^4 = c(\alpha,\alpha)^2$ for all $\alpha \in \mathbb{R}^n$.
- $$\begin{array}{llll} (D4) & \sum_{x\in X}(x,\alpha)^4 & = & \frac{3|X|m^2}{n(n+2)}(\alpha,\alpha)^2 \\ (D2) & \sum_{x\in X}(x,\alpha)^2 & = & \frac{|X|m}{n}(\alpha,\alpha) \end{array}$$

  for all $\alpha \in \mathbb{R}^n$ where $m = \min(L)$.

Strongly perfect lattices are extreme.

Theorem.

Let L be a strongly perfect lattice. Then L is strongly eutactic and perfect and hence extreme.

Proof. (a) The 2-design property is equivalent to L being strongly eutactic, because by (D2)

$$\sum_{x \in X} \underbrace{(x,\alpha)^2}_{\alpha \pi_x \alpha^{tr}} = \frac{m|X|}{n} \underbrace{(\alpha,\alpha)}_{\alpha I_n \alpha^{tr}}$$

for all $\alpha \in \mathbb{R}^n$ where $X = \mathrm{Min}(L)$, $m = \min(L)$.

Strongly perfect lattices are extreme.

Theorem.

Let L be a strongly perfect lattice. Then L is strongly eutactic and perfect and hence extreme.

Proof. (b) 4-design implies perfection: $A \in \mathbb{R}^{n\times n}_{sym}$ defines $p_A : \alpha \mapsto \alpha A \alpha^{tr}$.

$$U := \langle \pi_x \mid x \in X \rangle = \mathbb{R}^{n\times n}_{\mathrm{sym}} \Leftrightarrow U^{\perp} = \{0\}.$$

So assume that $A \in U^{\perp}$, so

$$0 = \mathrm{trace}(x^{tr} x A) = \mathrm{trace}(x A x^{tr}) = x A x^{tr} = p_A(x) \text{ for all } x \in X$$

By the design property we then have

$$\int_S p_A^2(t)dt = \frac{1}{|X|} \sum_{x \in X} p_A(x)^2 = 0$$

and hence $A = 0$.

Strongly perfect lattices.

Theorem.

Let L be strongly perfect. Then $\min(L)\min(L^{\#}) \geq (n+2)/3$.

Proof. Let $\alpha \in \mathrm{Min}(L^{\#})$. Then

$$(D4)-(D2) = \sum_{x\in X} \underbrace{(x,\alpha)^2((x,\alpha)^2-1)}_{\geq 0} = \frac{|X|m}{n}(\alpha,\alpha)\underbrace{\left(\frac{3m(\alpha,\alpha)}{n+2}-1\right)}_{\Rightarrow \geq 0}$$

Remember
$$\begin{array}{lll} (D4) & \sum_{x\in X}(x,\alpha)^4 & = \frac{3|X|m^2}{n(n+2)}(\alpha,\alpha)^2 \\ (D2) & \sum_{x\in X}(x,\alpha)^2 & = \frac{|X|m}{n}(\alpha,\alpha) \end{array}$$

The strongly perfect root lattices.

Theorem.

The strongly perfect root lattices are $\mathbb{A}_1, \mathbb{A}_2, \mathbb{D}_4, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$.

Proof. Let L be a strongly perfect root lattice.
We use the necessary condition that $\min(L)\min(L^{\#}) \geq \frac{n+2}{3}$ to exclude the other cases.

- $L = \mathbb{A}_n$, then $\min(L^{\#}) = \frac{n}{n+1}$ and

$$\min(\mathbb{A}_n)\min(\mathbb{A}_n^{\#}) = \frac{2n}{n+1} \geq \frac{n+2}{3} \Leftrightarrow (n-1)(n-2) \leq 0$$

- $L = \mathbb{D}_n$. Then $\min(L^{\#}) = 1$ and

$$\min(\mathbb{D}_n)\min(\mathbb{D}_n^{\#}) = 2 \geq \frac{n+2}{3} \Leftrightarrow n \leq 4$$

- $\min(\mathbb{E}_6)\min(\mathbb{E}_6^{\#}) = 2\frac{4}{3}$, $\min(\mathbb{E}_7)\min(\mathbb{E}_7^{\#}) = 2\frac{3}{2}$, $\min(\mathbb{E}_8)^2 = 4 > \frac{10}{3}$.

$\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{D}_4$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ are strongly perfect.

Need to show that for all $\alpha \in \mathbb{R}^n$

$$p(\alpha) = \sum_{x \in X} (x, \alpha)^4 - \frac{3|X|m^2}{n(n+2)} (\alpha, \alpha)^2 = 0$$

Introduce a Euclidean inner product by

$$< f, g >:= g(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})(f) \text{ for } f, g \in \mathbb{R}[x_1, \dots, x_n]_{\deg = t}$$

If $g(\alpha) = (\alpha, \alpha)^{t/2}$, then $< f, g >= \Delta^{t/2}(f)$ and if $g(\alpha) = (x, \alpha)^t$ then $< f, g >= f(x)$.
Hence $< p, p >= \sum_{x,y \in X} (x, y)^4 - \frac{3|X|^2 m^4}{n(n+2)}$.

Corollary.

A lattice $L \in \mathcal{L}_n$ with $\min(L) = m$ is strongly perfect, if and only if

$$\sum_{x,y \in \mathrm{Min}(L)} (x, y)^4 = \frac{3|\mathrm{Min}(L)|^2 m^4}{n(n+2)}.$$

Some applications of representation theory.

- Recall that the automorphism group of a lattice L is $G := \mathrm{Aut}(L) = \{\sigma \in O_n(\mathbb{R}) \mid \sigma(L) = L\}$.
- G acts on $L_a = \{\ell \in L \mid \frac{1}{2}(\ell, \ell) = Q(\ell) = a\}$.
- In particular $\mathrm{Min}(L)$ is a union of G orbits.
- $$\alpha \mapsto \sum_{x \in \mathrm{Min}(L)} (x, \alpha)^d$$
  is a G-invariant polynomial of degree d.
- $\mathrm{Inv}_d(G) := \{p \in \mathbb{R}[x_1, \ldots, x_n] \mid p$ is $G -$ invariant $, \deg(p) = d\}$ is a finite-dimensional vector space of which the dimension is calculated from the character table.
- Since $-1 \in G$ the space $\mathrm{Inv}_d(G) = 0$ for odd d.
- $(\alpha, \alpha)^d \in \mathrm{Inv}_{2d}(G)$.

No harmonic invariants.

Theorem.

Let $G = \mathrm{Aut}(L)$ and assume that $\langle(\alpha, \alpha)^d\rangle = \mathrm{Inv}_{2d}(G)$ for all $d = 1, \ldots, t$. Then all G-orbits and all non-empty layers of L are spherical $2t$-designs.

Corollary.

- If $\mathbb{R}^n$ is an irreducible $\mathbb{R}G$-module then $\mathrm{Inv}_2(G) = \langle(\alpha, \alpha)\rangle$ and L is strongly eutactic.
- If additionally $\mathrm{Inv}_4(G) = \langle(\alpha, \alpha)^2\rangle$, then L is strongly perfect.

The Barnes-Wall lattices of dimension 2^d.

- Let $d \in \mathbb{N}$, $m := \lfloor \frac{d}{2} \rfloor$, $A := \mathbb{F}_2^d$ and $(e_a \mid a \in A)$ an orthogonal basis of $\mathbb{R}^{2^d}$ with $(e_a, e_a) = 2^{-m}$.
- For an affine subspace $X = a + U$, $a \in A, U \leq A$ let $\chi_X := \sum_{x \in X} e_x \in \mathbb{R}^{2^d}$.
- Then $(\chi_X, \chi_X) = 2^{-m}|X| = 2^{k-m}$, where $k = \dim(X) := \dim(U)$.
- Let $\mathcal{A}(d,k)$ denote the set of all affine subspaces of A of dimension k.
- For $X \in \mathcal{A}(d, 2k)$ the norm $(\chi_X, \chi_X) = 2^{2k-m}$.
- Define the Barnes-Wall lattice

$$\mathrm{BW}_d := \langle 2^{m-k} \chi_X \mid k = 0, \ldots, m, X \in \mathcal{A}(d, 2k) \rangle_{\mathbb{Z}}.$$

The Barnes-Wall lattices of dimension 2^d.

Some properties of BW_d.

- $\min(\mathrm{BW}_d) = 2^m$, where $m = \lfloor \frac{d}{2} \rfloor$, $\det(\mathrm{BW}_d) = \begin{cases} 2^m & d \text{ even} \\ 1 & d \text{ odd} \end{cases}$.
- $\mathrm{BW}_1 = \mathbb{Z}^2$, $\mathrm{BW}_2 = \mathbb{D}_4$, $\mathrm{BW}_3 = \mathbb{E}_8$.
- BW_4 densest known lattice in dimension 16.
- BW_5 extremal even unimodular lattice.
- $\mathrm{Aut}(\mathrm{BW}_d) \cong 2_+^{1+2d}.\Omega_{2d}^+(2)$ if $d > 3$.
- For $d \geq 2$ the Barnes-Wall lattice BW_d is a strongly perfect lattice, in fact Christine Bachoc has shown that all $\mathrm{Aut}(\mathrm{BW}_d)$-orbits form spherical 6-designs using coding theory.

The Thompson-Smith lattice of dimension 248.

- Let G =Th denote the sporadic simple Thompson group.
- Then G has a 248-dimensional rational representation $\rho : G \to O(248, \mathbb{Q})$.
- Since G is finite, $\rho(G)$ fixes a lattice $L \leq \mathbb{Q}^{248}$.
- Modular representation theory tells us that for all primes p the $\mathbb{F}_p G$-module L/pL is simple.
- Therefore $L = L^{\#}$ and L is even (otherwise $L_0/2L < L/2L$ would be a proper G-invariant submodule).
- $\mathrm{Inv}_{2d}(G) = \langle (\alpha, \alpha)^d \rangle$ for $d = 1, 2, 3$. So all layers of L form spherical 6-designs and in particular L is strongly perfect.
- $\min(L) \min(L^{\#}) = \min(L)^2 \geq \frac{248+2}{3} > 83.3$, so $\min(L) \geq 10$.
- There is a $v \in L$ with $(v, v) = 12$, so $\min(L) \in \{10, 12\}$.

Classification of strongly perfect lattices.

Theorem.

- All strongly perfect lattices of dimension ≤ 12 are known (Nebe/Venkov).
- All integral strongly perfect lattices of minimum 2 and 3 are known (Venkov).
- All lattices $L \in \mathcal{L}_{14}$ such that L and $L^{\#}$ are strongly perfect (dual strongly perfect) are known (Nebe/Venkov).
- Elisabeth Nossek will classify the dual strongly perfect lattices in dimension 13,15,….
- All integral lattices L of minimum ≤ 5 such that $\mathrm{Min}(L)$ is a 6-design are known (Martinet).
- All lattices L of dimension ≤ 24 such that $\mathrm{Min}(L)$ is a 6-design are known (Nebe/Venkov).

Certain extremal lattices are strongly perfect.

Theorem.

Let $L = L^{\#} \in \mathcal{L}_n$ be an even unimodular lattice and $p \in \mathbb{R}[x_1, \ldots, x_n]$, $\deg(p) = t > 0$, $\Delta(p) = 0$. Then

$$\theta_{L,p} := \sum_{\ell \in L} p(\ell) q^{Q(\ell)} = \sum_{j=1}^{\infty} (\sum_{\ell \in L_j} p(\ell)) q^j \in M^0_{n/2+t}.$$

If $2m = \min(L)$ then $\theta_{L,p}$ is divisible by $\Delta^m \in M^0_{12m}$.
In particular if $n/2 + t < 12m$, then $\theta_{L,p} = 0$ and hence $\sum_{\ell \in L_j} p(\ell) = 0$ for all j and all harmonic polynomials of degree t.

Theorem.

Let L be an extremal even unimodular lattice of dimension $n = 24a + 8b$ with $b = 0, 1, 2$.

- All nonempty L_j are $(11 - 4b)$-designs.
- If $b = 0$ or $b = 1$ then L is strongly perfect and hence extreme.
- All extremal even unimodular lattices of dimension 32 are extreme.

Proof of Theorem for $n = 24a$.

- L extremal means $\min(L) = 2 + 2a$.
- Let p be a harmonic polynomial of degree $t > 0$.
- Then $\theta_{L,p} = \star q^{1+a} + \ldots$, hence $\theta_{L,p}$ is a multiple of Δ^{1+a}.
- Δ^{1+a} has weight $12(1 + a) = 12a + 12$.
- $\theta_{L,p}$ has weight $12a + t$.
- So if $t \leq 11$ then $\theta_{L,p} = 0$.
- This means that $\sum_{\ell \in L_j} p(\ell) = 0$ for all j, so the nonempty L_j are spherical 11-designs.

A similar proof applies to $n = 24a + 8b$ for $b = 1, 2$.

Even unimodular lattices of dimension 24.

Remember.

Let $L = L^{\#} \in \mathcal{L}_n$ be an even unimodular lattice and $p \in \mathbb{R}[x_1, \ldots, x_n]$, $\deg(p) = t > 0$, $\Delta(p) = 0$. Then

$$\theta_{L,p} := \sum_{\ell \in L} p(\ell) q^{Q(\ell)} = \sum_{j=1}^{\infty} (\sum_{\ell \in L_j} p(\ell)) q^j \in M^0_{n/2+t}.$$

If $2m = \min(L)$ then $\theta_{L,p}$ is divisible by $\Delta^m \in M^0_{12m}$.

Application for $n = 24$.

Know that $M^0_{14} = \{0\}$ so if L is an even 24-dimensional unimodular lattice and p a harmonic polynomial of degree 2, then $\theta_{L,p} = 0$.
In particular all even unimodular 24-dimensional lattices are strongly eutactic.

Venkov's classication of the even unimodular lattices of dimension 24.

Theorem (Venkov).

Let L be an even unimodular lattice of dimension 24.

- The root system $R(L)$ is either empty or has full rank.
- The indecomposable components of $R(L)$ have the same Coxeter number.

Proof. Assume that $R(L) \neq \emptyset$. Since L is strongly eutactic

$$\sum_{x \in R(L)} (x,\alpha)^2 = \frac{|R(L)|}{12}(\alpha,\alpha) \text{ for all } \alpha \in \mathbb{R}^{24}$$

In particular $R(L)^{\perp} = \{0\}$.
If $R(L) = R_1 \perp \ldots \perp R_s$, $n_i = \dim(R_i)$, and $\alpha \in \langle R_i \rangle_{\mathbb{R}}$, then

$$\sum_{x \in R(L)} (x,\alpha)^2 = \sum_{x \in R_i} (x,\alpha)^2 = \frac{2|R_i|}{n_i}(\alpha,\alpha).$$

Hence $h(R_i) = \frac{|R_i|}{n_i} = \frac{|R(L)|}{24}$ is independent of i.

The even unimodular lattices of dimension 24.

The possible root systems are found combinatorically from the classification of indecomposable root systems and their Coxeter numbers:

$$\begin{array}{l} \emptyset,\ 24\mathbb{A}_1,\ 12\mathbb{A}_2,\ 8\mathbb{A}_3,\ 6\mathbb{A}_4,\ 4\mathbb{A}_6,\ 3\mathbb{A}_8,\ 2\mathbb{A}_{12},\ \mathbb{A}_{24}, \\ 6\mathbb{D}_4,\ 4\mathbb{D}_6,\ 3\mathbb{D}_8,\ 2\mathbb{D}_{12},\ \mathbb{D}_{24},\ 4\mathbb{E}_6,\ 3\mathbb{E}_8, \\ 4\mathbb{A}_5 \perp \mathbb{D}_4,\ 2\mathbb{A}_7 \perp 2\mathbb{D}_5,\ 2\mathbb{A}_9 \perp \mathbb{D}_6,\ \mathbb{A}_{15} \perp \mathbb{D}_9, \\ \mathbb{E}_8 \perp \mathbb{D}_{16}, 2\mathbb{E}_7 \perp \mathbb{D}_{10},\ \mathbb{E}_7 \perp \mathbb{A}_{17},\ \mathbb{E}_6 \perp \mathbb{D}_7 \perp \mathbb{A}_{11} \end{array}$$

Theorem.

For each of the 24 possible root systems there is a unique even unimodular lattice in dimension 24 having this root system.

Proof of Theorem for $R \neq \emptyset$.

Proof. Let $M := \langle R(L) \rangle_{\mathbb{Z}} \subset L = L^{\#} \subset M^{\#}$. The inner product induces a bilinear form

$$b_M : M^{\#}/M \times M^{\#}/M \to \mathbb{Q}/\mathbb{Z}, (x + M, y + M) \mapsto (x, y) + \mathbb{Z}$$

with associated quadratic form

$$q_M : M^{\#}/M \to \mathbb{Q}/\mathbb{Z}, x + M \mapsto Q(x) + \mathbb{Z} = \frac{1}{2}(x, x) + \mathbb{Z}.$$

The even unimodular lattices L that contain M correspond to totally isotropic self-dual subgroups

$$(L/M)^{\perp} = L/M \leq M^{\#}/M \text{ with } q_M(L/M) = \{0\}.$$

$R(L) = R(M)$ iff for all $\ell \in L - M$,

$$\min(\ell + M) = \min\{2Q(\ell + m) \mid m \in M\} \geq 4.$$

Example. Root system $6\mathbb{A}_4$.

$\mathbb{A}_4^{\#}/\mathbb{A}_4 = \langle x \rangle \cong \mathbb{F}_5$.

$\mathbb{A}_4^{\#} = \langle \mathbb{A}_4, x \rangle$ with $\min(ax + \mathbb{A}_4) = \begin{cases} 4/5 & \text{for } a = 1, -1 \\ 6/5 & \text{for } a = 2, -2 \end{cases}$.

Unimodular overlattices of $6\mathbb{A}_4$ correspond to self-dual codes $C = C^{\perp} \leq \mathbb{F}_5^6$.

$$C_1 : \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}, \quad C_2 : \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 & 1 \end{bmatrix}$$

yield the lattices

$$L_1 = \langle 6\mathbb{A}_4, x_1 + 2x_4, x_2 + 2x_5, x_3 + 2x_6 \rangle \cong 4\mathbb{E}_8$$

$$L_2 = \langle 6\mathbb{A}_4, x_1 + 2x_4 + x_5 + 2x_6, x_2 + x_4 + 2x_5 + 3x_6, x_3 + 3x_4 + 2x_5 + x_6 \rangle$$

with $R(L_2) = 6\mathbb{A}_4$.

24-dimensional even unimodular lattices.

Theorem.

For each of the 24 possible root systems there is a unique even unimodular lattice in dimension 24 having this root system.

Remark.

The uniqueness of the Leech lattice, the unique even unimodular lattice of dimension 24 with no roots is proven differently. It follows for instance from the uniqueness of the Golay code, but also by applying the mass formula:

$$\sum_{i=1}^{h} |\operatorname{Aut}(L_i)|^{-1} = m_{2k} = \frac{|B_k|}{2k} \prod_{j=1}^{k-1} \frac{B_{2j}}{4j}$$

where $L_1, \ldots, L_h$ represent the isometry classes of even unimodular lattices in $\mathbb{R}^{2k}$.

$$m_{24} = \frac{1027637932586061520960267}{129477933340026851560636148613120000000}$$