Hecke operators for codes.
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This talk introduces Hecke operators for codes and therewith answers a question raised in 1977 by Michel Broué.
A **lattice** $L$ in Euclidean $N$-space $E := (\mathbb{R}^N, (,))$ is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis $B = (b_1, \ldots, b_N)$ of $E$

$$L = \langle b_1, \ldots, b_N \rangle_\mathbb{Z} = \{ \sum_{i=1}^{N} a_i b_i | a_i \in \mathbb{Z} \}.$$ 

The **dual lattice** of $L$ is

$$L^* := \{ v \in E | (v, \ell) \in \mathbb{Z} \forall \ell \in L \}.$$ 

$L$ is called **integral**, if $L \subset L^*$ or equivalently $(\ell, m) \in \mathbb{Z}$ for all $\ell, m \in L$. 

$L$ is called **even**, if $(\ell, \ell) \in 2\mathbb{Z}$ for all $\ell \in L$. 

$L$ is called **unimodular**, if $L = L^*$. 

The **theta series** of a lattice $L$ is

$$\vartheta_L = \sum_{\ell \in L} q^{(\ell, \ell)}$$

where $q = \exp(\pi i z)$. 

The hexagonal lattice.

\[ \vartheta_L = 1 + 6q^2 + 6q^6 + 6q^8 + 12q^{14} + 6q^{18} + 6q^{24} + 12q^{26} + 6q^{32} + \ldots \]
Theorem. (Theta transformation formula)

\[ \vartheta_{L^*}(z) = \left( \frac{z}{i} \right)^{-k} \sqrt{\det(L)} \vartheta_L \left( -\frac{1}{z} \right) \quad \text{(where } 2k = N = \dim(L) \text{)} \]

Hecke's theorem. If \( L = L^* \) then \( \vartheta_L \in \mathcal{M}_k(\Theta) \) where

\[ \Theta = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \]

If \( L = L^* \) and \( L \) is even, then \( \vartheta_L \in \mathcal{M}_k(\text{SL}_2(\mathbb{Z})) \) where

\[ \text{SL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \]

We have

\[ \mathcal{M}(\Theta) := \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Theta) = \mathbb{C}[\vartheta_{\mathbb{Z}^2}, \vartheta_{E_8}] \]

and

\[ \mathcal{M}(\text{SL}_2(\mathbb{Z})) = \bigoplus_{k=0}^{\infty} \mathcal{M}_{4k}(\Theta) = \mathbb{C}[\vartheta_{E_8}, \vartheta_{\Lambda_{24}}] \]
Construction A.

Let $p$ be a prime and $(b_1, \ldots, b_N)$ be a basis of $E$ such that

$$(b_i, b_j) = \begin{cases} 
0 & \text{if } i \neq j \\
1/p & \text{if } i = j 
\end{cases}$$

Let $C \leq \mathbb{F}_p^N = \mathbb{Z}^N/p\mathbb{Z}^N$ be a code. Then the code lattice $L_C$ is

$$L_C := \{ \sum_{i=1}^{N} a_i b_i \mid (a_1 \pmod{p}, \ldots, a_N \pmod{p}) \in C \}$$

Example. $L_{i_2} = \mathbb{Z}^2$, $L_{e_8} = E_8$ and

$\mathcal{M}(\Theta) = \mathbb{C}[\vartheta_{L_{i_2}}, \vartheta_{L_{e_8}}]$, $\mathcal{M}(\text{SL}_2(\mathbb{Z})) = \mathbb{C}[\vartheta_{L_{e_8}}, \vartheta_{L_{g_{24}}}]$

Remark. (a) $L_C^* = L_{C^\perp}$, so $L_C$ is unimodular, if $C$ is self-dual.

(b) $L_C$ is even unimodular, if $p = 2$ and $C$ is a Type II code.

(c) $\vartheta_{L_C} = \text{cwe}_C(\vartheta_0, \ldots, \vartheta_{p-1})$ where $\vartheta_a = \vartheta_{(a+p\mathbb{Z})b_1} = \sum_{n=-\infty}^{\infty} q^{(a+pn)^2/p}$. 
Parallels between lattices and codes.

<table>
<thead>
<tr>
<th>Code</th>
<th>Lattice</th>
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<tr>
<td>self-dual code</td>
<td>unimodular lattice</td>
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<tr>
<td>doubly-even self-dual code</td>
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<td>weight enumerator</td>
<td>theta series</td>
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<td>invariant polynomial</td>
<td>modular form</td>
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<td>MacWilliams identity</td>
<td>Theta transformation formula</td>
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<td>Hecke’s theorem</td>
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<td>Molien’s theorem</td>
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<td>Hamming code $e_8$</td>
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<td>Golay code $g_{24}$</td>
<td>Leech lattice $\Lambda_{24}$</td>
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<td>Runge’s $\Phi$-operator</td>
<td>Siegel’s $\Phi$-operator</td>
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<tr>
<td>Kneser-Hecke operators</td>
<td>Hecke operators</td>
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Motivation.
Determine linear relations between $\text{cwe}_m(C)$ for $C \in M_N(T) = \{C \leq V^N \mid C \text{ of Type } T\}$.

$M_{16}(\text{II}) = [e_8 \perp e_8] \cup [d_{16}^+]$ and these two codes have the same genus 1 and 2 weight enumerator, but $\text{cwe}_3(e_8 \perp e_8)$ and $\text{cwe}_3(d_{16}^+)$ are linearly independent.

$h(M_{24}(\text{II})) = 9$ and only the genus 6 weight enumerators are linearly independent, there is one relation for the genus 5 weight enumerators.

$h(M_{32}(\text{II})) = 85$ and here the genus 10 weight enumerators are linearly independent, whereas there is a unique relation for the genus 9 weight enumerators.
Three different approaches:

1) Determine all the codes and their weight enumerators. If \( \dim(C) = n = N/2 \) there are \( \prod_{i=0}^{d-1} \frac{(2^n - 2^i)}{(2^d - 2^i)} \) subspaces of dimension \( d \) in \( C \).

\( N = 32, d = 10 \) yields more than \( 10^{18} \) subspaces.

2) Use Molien’s theorem:

\[
\text{Inv}_N(C_m(\Pi)) = \langle \text{cwe}_m(C) \mid C \in M_N(\Pi) \rangle
\]

and if \( a_N := \text{dim}(\text{Inv}_N(C_m(\Pi))) \) then

\[
\sum_{N=0}^{\infty} a_N t^N = \frac{1}{|C_m(\Pi)|} \sum_{g \in C_m(\Pi)} (\det(1 - tg))^{-1}
\]

Problem: \( C_{10}(\Pi) \leq \text{GL}_{1024}(\mathbb{C}) \) has order \( > 10^{69} \).

3) Use Hecke operators.
Fix a Type $T = (\mathbb{F}_q, \mathbb{F}_q, \beta, \Phi)$ of self-dual codes over a finite field with $q$ elements.

$$M_N(T) = \{C \leq \mathbb{F}_q^N \mid C \text{ of Type } T\} = [C_1] \cup \ldots \cup [C_h]$$

where $[C]$ denotes the permutation equivalence class of the code $C$. Then $n := \frac{N}{2} = \dim(C)$ for all $C \in M_N(T)$.

$C, D \in M_N(T)$ are called neighbours, if $\dim(C) - \dim(C \cap D) = 1$, $C \sim D$.

$$\mathcal{V} = \mathbb{C}[C_1] \oplus \ldots \oplus \mathbb{C}[C_h] \cong \mathbb{C}^h$$

$$K_N(T) \in \text{End}(\mathcal{V}), \quad K_N(T) : [C] \mapsto \sum_{D \in M_N(T), D \sim C} [D].$$

Kneser-Hecke operator.
(adjacency matrix of neighbouring graph)
Example. \( M_{16}(II) = [e_8 \perp e_8] \cup [d^+_{16}] \)

\[
K_{16}(II) = \begin{pmatrix} 78 & 49 \\ 70 & 57 \end{pmatrix}
\]
\( \mathcal{V} \) has a Hermitian positive definite inner product defined by

\[
\langle [C_i], [C_j] \rangle := |\text{Aut}(C_i)| \delta_{ij}.
\]

**Theorem.** (N. 2006)
The Kneser-Hecke operator \( K \) is a self-adjoint linear operator.

\[
\langle v, Kw \rangle = \langle Kv, w \rangle \text{ for all } v, w \in \mathcal{V}.
\]

**Example.** \( \frac{7}{10} = \frac{|\text{Aut}(e_8 \perp e_8)|}{|\text{Aut}(d_{16}^+)|} \) hence

\[
diag(7, 10)K_{16}(\text{II})^{\text{Tr}} = K_{16}(\text{II}) \text{diag}(7, 10).
\]
\[ \text{cwe}_m : \mathcal{V} \to \mathbb{C}[X], \sum_{i=1}^{h} a_i[C_i] \mapsto \sum_{i=1}^{h} a_i \text{cwe}_m(C_i) \]

is a linear mapping with kernel

\[ \mathcal{V}_m := \ker(\text{cwe}_m). \]

Then

\[ \mathcal{V} =: \mathcal{V}_{-1} \geq \mathcal{V}_0 \geq \mathcal{V}_1 \geq \ldots \geq \mathcal{V}_n = \{0\}. \]

is a filtration of \( \mathcal{V} \) yielding the orthogonal decomposition

\[ \mathcal{V} = \bigoplus_{m=0}^{n} \mathcal{V}_m \text{ where } \mathcal{V}_m = \mathcal{V}_{m-1} \cap \mathcal{V}_m^\perp. \]

\[ \mathcal{V}_0 = \{ \sum_{i=1}^{h} a_i[C_i] \mid \sum_{i=1}^{h} a_i = 0 \} \]

and

\[ \mathcal{V}_0^\perp = \mathcal{V}_0 = \langle \sum_{i=1}^{h} \frac{1}{|\text{Aut}(C_i)|}[C_i] \rangle. \]
**Theorem.** (N. 2006)
The space $\mathcal{Y}_m = \mathcal{Y}_m(N)$ is the $K_N(T)$-eigenspace to the eigenvalue $\nu_N^{(m)}(T)$ with $\nu_N^{(m)}(T) > \nu_N^{(m+1)}(T)$ for all $m$.

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<tr>
<th>Type</th>
<th>$\nu_N^{(m)}(T)$</th>
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<tr>
<td>$q_1^E$</td>
<td>$(q^{n-m} - q - q^m + 1)/(q - 1)$</td>
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<tr>
<td>$q_1^{E'}$</td>
<td>$(q^{n-m-1} - q^m)/(q - 1)$</td>
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<tr>
<td>$q^E$</td>
<td>$(q^{n-m} - q^m)/(q - 1)$</td>
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<tr>
<td>$q_1^{E'}$</td>
<td>$(q^{n-m-1} - q^m)/(q - 1)$</td>
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<tr>
<td>$q^H$</td>
<td>$(q^{n-m+1/2} - q^m - q^{1/2} + 1)/(q - 1)$</td>
</tr>
<tr>
<td>$q_1^H$</td>
<td>$(q^{n-m-1/2} - q^m - q^{1/2} + 1)/(q - 1)$</td>
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</tbody>
</table>

**Corollary.** The neighbouring graph is connected.

Proof. The maximal eigenvalue $\nu_0$ of the adjacency matrix is simple with eigenspace $\mathcal{Y}_0$. 
Example: $M_{16}(\Pi) = [e_8 \perp e_8] \cup [d_{16}^+]$

$(2^{8-m-1} - 2^m : m = 0, 1, 2, 3) = (127, 62, 28, 8)$

$$K_{16}(\Pi) = \begin{pmatrix} 78 & 49 \\ 70 & 57 \end{pmatrix}$$

has eigenvalues 127 and 8 with eigenvectors $(7, 10)$ and $(1, -1)$. Hence

$$\mathcal{V}_0 = \langle 7[e_8 \perp e_8] + 10[d_{16}^+] \rangle$$

$$\mathcal{V}_1 = \mathcal{V}_2 = 0$$

$$\mathcal{V}_3 = \langle [e_8 \perp e_8] - [d_{16}^+] \rangle.$$
\[ M_{24}(\Pi) = [e^3_8] \cup [e_8 d_{16}] \cup [e^2_7 d_{10}] \cup [d^3_8] \cup [d_{24}] \cup [d^2_{12}] \cup [d^4_6] \cup [d^6_4] \cup [g_{24}] \]
\[ K_{24}(\Pi) = \]
\[
\begin{pmatrix}
213 & 147 & 344 & 343 & 0 & 0 & 0 & 0 & 0 \\
70 & 192 & 896 & 490 & 7 & 392 & 0 & 0 & 0 \\
10 & 14 & 504 & 490 & 0 & 49 & 980 & 0 & 0 \\
1 & 3 & 192 & 447 & 0 & 36 & 1152 & 216 & 0 \\
0 & 990 & 0 & 0 & 133 & 924 & 0 & 0 & 0 \\
0 & 60 & 480 & 900 & 1 & 206 & 400 & 0 & 0 \\
0 & 0 & 72 & 216 & 0 & 3 & 1108 & 648 & 0 \\
0 & 0 & 0 & 45 & 0 & 0 & 720 & 1218 & 64 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1771 & 276 \\
\end{pmatrix}
\]

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\[ \langle 99[e^3_8] - 297[e_8 d_{16}] - 3465[d^3_8] + 7[d_{24}] + 924[d^2_{12}] + 4928[d^4_6] - 2772[d^6_4] + 576[g_{24}] \rangle = \ker(\text{cwe}_5) = \nu_5 \]
The Dimension of $\mathcal{Y}_m(N)$ for doubly-even binary self-dual codes.

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The Molien series of $\mathcal{C}_m(\Pi)$ is

$$1 + t^8 + a(m)t^{16} + b(m)t^{24} + c(m)t^{32} + \ldots$$

where

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\( \dim(\mathcal{Y}_m(N)) \) for binary self-dual codes.

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The Molien series of $C_m(I)$ is

$$1 + t^2 + t^4 + t^6 + 2t^8 + 2t^{10} + \sum_{N=12}^{\infty} a_N(m) t^N$$

where

$$a_N(m) := \dim \langle \text{cwe}_m(C) : C = C^\perp \leq \mathbb{F}_2^N \rangle$$

is given in the following table:

<table>
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<th>$m, N$</th>
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A group theoretic interpretation of the Kneser-Hecke operator.

In modular forms theory, Hecke operators are double cosets of the modular group. So I tried to find a similar interpretation for the Kneser-Hecke operator.

Let $T = (R, V, \beta, \Phi)$ be a Type. Then the invariant ring $\text{Inv}(C_m(T)) = \langle \text{cwe}_m(C) \mid C \text{ of Type } T \rangle$

**The finite Siegel $\Phi$-operator**

$$\Phi_m : \text{Inv}(C_m(T)) \to \text{Inv}(C_{m-1}(T)), \text{cwe}_m(C) \mapsto \text{cwe}_{m-1}(C)$$

defines a surjective graded $\mathbb{C}$-algebra homomorphism between invariant rings of complex matrix groups of different degree. $\Phi$ is given by the variable substitution:

$$x(v_1, \ldots, v_m) \mapsto \begin{cases} x(v_1, \ldots, v_{m-1}) & \text{if } v_m = 0 \\ 0 & \text{else} \end{cases}$$
Explanation:

cwe_{m-1}(C) is obtained from cwe_m(C) by counting only those matrices

\[
\begin{pmatrix}
  c_1^{(1)} & c_2^{(1)} & \ldots & c_j^{(1)} & \ldots & c_N^{(1)} \\
  c_1^{(2)} & c_2^{(2)} & \ldots & c_j^{(2)} & \ldots & c_N^{(2)} \\
  \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
  c_1^{(m)} & c_2^{(m)} & \ldots & c_j^{(m)} & \ldots & c_N^{(m)} \\
\end{pmatrix}
\]

in which the last row is zero.

This is expressed by the variable substitution

\[x(v_1,\ldots,v_m) \mapsto \begin{cases}
  x(v_1,\ldots,v_{m-1}) & \text{if } v_m = 0 \\
  0 & \text{else}
\end{cases}\]
\[(p, q)_m := p \left( \frac{\partial}{\partial x} \right)(\bar{q}) \text{ for } p, q \in \mathbb{C}[x_v : v \in V^m]_N \]
defines a positive definite Hermitian form on the homogeneous component \( \mathbb{C}[x_v : v \in V^m]_N \).

The monomials of degree \( N \) form an orthogonal basis and

\[
( \prod_{v \in V^m} x_v^{n_v}, \prod_{v \in V^m} x_v^{n_v})_m = \prod_{v \in V^m} (n_v!).
\]

Then \( \Phi_m : \ker(\Phi_m)^\perp \rightarrow \text{Inv}(C_{m-1}(T)) \) is an isomorphism with inverse

\[
\varphi_m : \text{Inv}(C_{m-1}(T)) \rightarrow \text{Inv}(C_m(T)), x_{(v_1, \ldots, v_{m-1})} \mapsto R(x_{(v_1, \ldots, v_{m-1}, 0)})
\]

where \( R(p) = \frac{1}{|C_m(T)|} \sum_{g \in C_m(T)} p(gx) \) is the **Reynolds operator** (the orthogonal projection onto the invariant ring).

Note that \( R \) is not a ring homomorphism.
This yields an orthogonal decomposition of the space of degree $N$ invariants of $C_m(T)$

$$\text{Inv}_N(C_m(T)) = \ker(\Phi_m) \perp \varphi_m^{-1}(\text{Inv}_N(C_{m-1}(T))) = \ker(\Phi_m) \perp \varphi_m^{-1}(\ker(\Phi_{m-1}) \perp \varphi_{m-1}^{-1}(\text{Inv}_N(C_{m-2})(T)))) =$$

$$Y_m \perp Y_{m-1} \perp \ldots \perp Y_0$$

such that for all $0 \leq k \leq m$ the mapping

$$\text{cwe}_m : \mathcal{Y}_k \rightarrow Y_k.$$  

is an isomorphism of vector spaces.

$$\mathcal{V} = \mathcal{Y}_n \perp \ldots \perp \mathcal{Y}_{m+1} \perp \mathcal{Y}_m \perp \mathcal{Y}_{m-1} \perp \ldots \perp \mathcal{Y}_0$$

$$\text{cwe}_m \downarrow \quad \ldots \quad \downarrow \quad \downarrow \quad \downarrow \quad \ldots \quad \downarrow$$

$$\text{Inv}_N(C_m(T)) = 0 \perp \ldots \perp 0 \perp \mathcal{Y}_m \perp \mathcal{Y}_{m-1} \perp \ldots \perp Y_0$$
The Kneser-Hecke operator $K_{N}(T)$ acts on $\text{Inv}_{N}(C_{m}(T))$ as $\delta_{m}(K_{N}(T))$ having $Y_{m} \perp Y_{m-1} \perp \ldots \perp Y_{0}$ as the eigenspace decomposition.

\[ C_{m}(T) = \underbrace{S.(\ker(\lambda) \times \ker(\lambda))}_{\mathcal{E}_{m}(T)} \cdot G_{m}(T) \]

Choose a suitable subgroup $\mathcal{U}_{1}$ of $\mathcal{E}_{m}(T)$ that corresponds to a 1-dimensional subspace of $(\ker(\lambda) \times \ker(\lambda))$ and let

\[ p_{1} := \frac{1}{q} \sum_{u \in \mathcal{U}_{1}} u \in \mathbb{C}^{q^{m} \times q^{m}} \]

be the orthogonal projection onto the fixed space of $\mathcal{U}_{1}$ and let

\[ H_{m}(T) := C_{m}(T)p_{1}C_{m}(T) = \bigcup_{U \in X} p_{U}C_{m}(T) \]

then this double coset acts on $\text{Inv}_{N}(C_{m}(T))$ via

\[ \Delta_{N}(H_{m}(T)) : f \mapsto \frac{1}{|X|} \sum_{U \in X} f(xp_{U}) \]
Theorem. (N. 2006)

\[(q - 1)\delta_m(K_N(T)) = q^{n-m-e}((q - 1)\Delta_N(H_m(T)) + \text{id}) - (q^m + a) \text{id}\]

where \(n = N/2\) and \(e, a\) are as follows:

<table>
<thead>
<tr>
<th>(T)</th>
<th>(q^E)</th>
<th>(q^E)</th>
<th>(q^E)</th>
<th>(q^E)</th>
<th>(q^H)</th>
<th>(q^H)</th>
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<tbody>
<tr>
<td>(a)</td>
<td>0</td>
<td>(q - 1)</td>
<td>0</td>
<td>0</td>
<td>(\sqrt{q} - 1)</td>
<td>(\sqrt{q} - 1)</td>
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<tr>
<td>(e)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>(1/2)</td>
<td>(-1/2)</td>
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• formal notion of Type $T = (R, V, \beta, \Phi)$.
• self-dual code $C$ of Type $T$.
• automorphisms and equivalences of codes of a given Type
• mass formula, classifications with Kneser’s neighbouring method.
• the associated Clifford-Weil group $C_m(T)$, a finite complex matrix group of degree $|V|^m$ such that

$$\text{Inv}_N(C_m(T)) = \langle \text{cwe}_m(C) \mid C = C^\perp \leq V^N \text{ of Type } T \rangle$$

• In particular the scalar subgroup $C_m(T) \cap \mathbb{C}^* \text{id}$ is cyclic of order

$$\min\{N \mid \text{there is a code } C \leq V^N \text{ of Type } T\}.$$

• $C_m(T)$ has a nice group theoretic structure.
• $\Phi_m : \text{Inv}(C_m(T)) \to \text{Inv}(C_{m-1}(T))$
• if $R$ is a field then:
• As in modular forms theory, the invariant ring of $C_m(T)$ can be investigated using Hecke operators.
• The Hecke algebra is generated by the incidence matrix of the Kneser neighbouring graph.
• Obtain linear relations between weight enumerators.