1 Self-injective rings

Let $\mathcal{C}$ be a commutative ring. One says that $\mathcal{C}$ is self-injective if it is injective as a module over itself, that is, if $\text{Hom}_{\mathcal{C}}(\cdot, \mathcal{C})$ is an exact functor. By Baer’s criterion, this is equivalent to saying that any homomorphism $\phi : I \to \mathcal{C}$, where $I$ is an ideal of $\mathcal{C}$, can be extended to all of $\mathcal{C}$, that is, there exists a homomorphism $\psi : \mathcal{C} \to \mathcal{C}$ such that $\psi|_I = \phi$. To understand the structure of self-injective rings better, we consider the set

$$S := \{ c \in \mathcal{C} \mid \text{ann}(c) \text{ is essential in } \mathcal{C} \},$$

where an ideal of $\mathcal{C}$ is called essential in $\mathcal{C}$ if it has a non-zero intersection with any non-zero ideal of $\mathcal{C}$.

**Example:** Let $p$ be a prime number, $k \in \mathbb{N}$, and $\mathcal{C} = \mathbb{Z}_{p^k}$. The non-zero ideals of $\mathcal{C}$ are the ideals

$$\langle p^{k-1} \rangle \subset \ldots \subset \langle p^2 \rangle \subset \langle p \rangle \subset \langle 1 \rangle.$$

Since the intersection of any two of them always contains $\langle p^{k-1} \rangle \neq 0$, we conclude that any non-zero ideal of $\mathcal{C}$ is essential in $\mathcal{C}$. Since $\text{ann}(p^l) = \langle p^{k-l} \rangle$ holds for all $0 \leq l \leq k$, we conclude that $S = \{ p, \ldots, p^{k-1}, p^k \} = \langle p \rangle$. We observe that $S$ is an ideal of $\mathcal{C}$ in this example. Next, we show that this holds in general, and this justifies that $S$ is called the singular ideal of $\mathcal{C}$.

**Lemma 1** The set $S$ defined above is an ideal of $\mathcal{C}$.

**Proof:** Clearly, $0 \in S$. Let $s, t \in S$. Then $\text{ann}(s)$ and $\text{ann}(t)$ are both essential in $\mathcal{C}$. This implies that also $\text{ann}(s) \cap \text{ann}(t)$ is essential in $\mathcal{C}$. Since

$$\text{ann}(s) \cap \text{ann}(t) \subseteq \text{ann}(s + t),$$

it follows that $\text{ann}(s + t)$ is essential in $\mathcal{C}$, that is, $s + t \in S$. Now let $c \in \mathcal{C}$. Since $\text{ann}(s) \subseteq \text{ann}(sc)$, we obtain that $\text{ann}(sc)$ is essential in $\mathcal{C}$, that is, $sc \in S$. □

Next, we establish some connections between the singular ideal $S$ and other remarkable ideals of $\mathcal{C}$. The nilradical $N$ of $\mathcal{C}$ is the set of all nilpotent elements of $\mathcal{C}$. It equals the intersection of all prime ideals of $\mathcal{C}$. A ring is called reduced if its nilradical is zero.

**Lemma 2** We always have $N \subseteq S$, and $N = 0$ is equivalent to $S = 0$. If $\mathcal{C}$ is Noetherian, then $N = S$.  

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Proof: Let \( c \in \mathcal{C} \) be nilpotent. We need to show that \( \text{ann}(c) \) is essential in \( \mathcal{C} \). Let \( I \neq 0 \) be an ideal of \( \mathcal{C} \) and let \( 0 \neq d \in I \). There exists a smallest positive integer \( n \) such that \( c^n d = 0 \). Then \( 0 \neq c^{n-1} d \in \text{ann}(c) \cap I \).

Since \( N \subseteq S \), it is clear that \( S = 0 \) implies \( N = 0 \). Assume conversely that \( \mathcal{C} \) is reduced. Let \( 0 \neq c \in \mathcal{C} \). We wish to show that \( c \not\in S \), that is, \( \text{ann}(c) \) is not essential in \( \mathcal{C} \). It suffices to show that \( I := \text{ann}(c) \cap \langle c \rangle = 0 \). Indeed, any \( d \in I \) has the form \( d = ce \) for some \( e \) with \( c^2 e = 0 \). Then \( 0 = c^2 e^2 = (ce)^2 \). Since \( \mathcal{C} \) has no non-zero nilpotent elements, we must have \( ce = 0 \) and thus \( d = 0 \).

Now suppose that \( \mathcal{C} \) is Noetherian and let \( c \in S \). We need to show that \( c \not\in S \), that is, \( \text{ann}(c) \) is not essential in \( \mathcal{C} \). It suffices to show that \( I := \text{ann}(c) \cap \langle c \rangle = 0 \). Indeed, any \( d \in I \) has the form \( d = ce \) for some \( e \) with \( c^2 e = 0 \). Then \( 0 = c^2 e^2 = (ce)^2 \). Since \( \mathcal{C} \) has no non-zero nilpotent elements, we must have \( ce = 0 \) and thus \( d = 0 \).

The Jacobson radical \( J \) of \( \mathcal{C} \) is defined to be the intersection of all maximal ideals of \( \mathcal{C} \). The proof of the following useful characterization is left as an exercise.

**Lemma 3** We have

\[
d \in J \iff \forall c \in \mathcal{C} : 1 - cd \text{ is a unit of } \mathcal{C}.
\]

So far, we have established the inclusions \( N \subseteq S \) and \( N \subseteq J \) in general, and \( N = S \subseteq J \) in the Noetherian case.

**Theorem 1** Let \( \mathcal{C} \) be self-injective. Then \( J = S \).

**Proof:** “\( J \subseteq S \)”: Let \( d \in J \) and let \( I \) be an ideal of \( \mathcal{C} \) such that \( \text{ann}(d) \cap I = 0 \). We need to show that this implies that \( I = 0 \). The homomorphism \( \phi : I \to \mathcal{C} \), \( c \mapsto cd \) is injective. Hence it induces an isomorphism \( \hat{\phi} : I \to \text{im}(\phi) \) with inverse \( \hat{\phi}^{-1} : \text{im}(\phi) \to I \subseteq \mathcal{C} \). Consider the resulting map \( \varphi : \text{im}(\phi) \to \mathcal{C} \) which satisfies \( \varphi(\phi(c)) = c \) for all \( c \in I \). Since \( \mathcal{C} \) is self-injective, \( \varphi \) can be extended to a homomorphism \( \psi : \mathcal{C} \to \mathcal{C} \). Let \( \psi(1) =: b \). Then we have for all \( c \in I \):

\[
cdb = \phi(c)b = \psi(\phi(c)) = \varphi(\phi(c)) = c.
\]

We conclude that \( 1 - db \in \text{ann}(I) \). But since \( d \in J \), the element \( 1 - db \) is a unit and hence \( I = 0 \) follows.
"$S \subseteq J$": Let $d \in S$ and $c \in C$. We need to show that $1 - cd$ is a unit of $C$. Since $\text{ann}(d) \cap \text{ann}(1 - cd) = 0$

and $\text{ann}(d)$ is essential in $C$, we may conclude that $\text{ann}(1 - cd) = 0$. Thus there exists a homomorphism $\phi : \langle 1 - cd \rangle \to C$ with $\phi(1 - cd) = 1$. Since $C$ is self-injective, $\phi$ can be extended to $\psi : C \to C$. Then we have

$$1 = \phi(1 - cd) = \psi(1 - cd) = (1 - cd)\psi(1),$$

which shows that $1 - cd$ is a unit. □

A commutative ring $C$ is called von Neumann regular if for all $c \in C$, we have $\langle c \rangle = \langle c^2 \rangle$, that is, there exists $d \in C$ such that $c = c^2d$. Then $(1 - cd)c = 0 = c(1 - dc)$, that is, $d$ is a weak form of an inverse of $c$.

**Theorem 2** Let $C \neq \{0\}$. The following are equivalent:

1. $C$ is von Neumann regular.
2. $C$ is reduced and has Krull dimension zero.
3. For every maximal ideal $m$ of $C$, the localization $C_m$ is a field.

**Proof:** "1 $\Rightarrow$ 2": "$N = 0$": Let $c^n = 0$, where $n$ is the minimal positive integer with this property. Let $d$ be such that $c = c^2d$. If $n$ is even, then we have $0 = c^nd^{n/2} = c^{n/2}$ contradicting the minimality of $n$. Thus $n = 2m + 1$ and $0 = c^{2m+1}d^{n} = c^{m+1}$. The minimality of $n$ implies that $m = 0$, that is, $c = 0$.

"Krull-dim($C$) = 0": Let $p$ be a prime ideal of $C$. We need to show that $p$ is maximal. The ring $\bar{C} := C/p$ is a domain. Let $0 \neq [c] \in \bar{C}$. By assumption, we have $c = c^2d$ for some $d$, that is, $[c][1 - dc] = 0$ holds in $\bar{C}$. This implies that $[c]$ is a unit in $\bar{C}$. Thus we have shown that $\bar{C}$ is a field, that is, $p$ is maximal.

"2 $\Rightarrow$ 3": Suppose that $N(C) = 0$ and Krull-dim($C$) = 0 and let $m$ be a maximal ideal of $C$. Since $N(C_m) = N(C)_m$ and Krull-dim($C_m$) $\leq$ Krull-dim($C$), we conclude that also $C_m$ is reduced and of Krull dimension zero. However, it is also a local ring, and thus $C_m$ contains only one prime ideal, namely the maximal ideal $m_m$. Since the nilradical equals the intersection of all prime ideals, we obtain $m_m = 0$. Thus $C_m$ is a field.

"3 $\Rightarrow$ 1": Let $c \in C$ be given. Let $m$ be a maximal ideal. We have

$$(\langle c \rangle/\langle c^2 \rangle)_m \cong \langle c \rangle_m/\langle c^2 \rangle_m \cong \langle c \rangle/\langle c^2 \rangle_1.$$ 

Since $F := C_m$ is a field, $\xi \in F$ and $\langle \xi \rangle^2 \in F$ generate the same ideal in $F$. Thus we have shown that $\langle c \rangle/\langle c^2 \rangle_m = 0$ for all maximal ideals $m$. Since being zero is a local property, this implies that $\langle c \rangle/\langle c^2 \rangle = 0$, that is, $\langle c \rangle = \langle c^2 \rangle$. □
Theorem 3 Let $\mathcal{C}$ be self-injective. Then $\mathcal{C}/S$ is von Neumann regular.

For the proof of this theorem, we need some preparation. For any ideal $I$ of $\mathcal{C}$, we may consider

\[ \mathcal{M} = \{ I' \mid I' \text{ is an ideal of } \mathcal{C} \text{ and } I \cap I' = 0 \} . \]

By Zorn’s Lemma, the set $\mathcal{M}$ contains an element that is maximal with respect to inclusion. Let $I^c$ be such a maximal element. Then $I^c$ is called a complement of $I$, and this implies that $I \oplus I^c$ is essential in $\mathcal{C}$.

Proof: Let $c \in \mathcal{C}$ be given. Let $I := \text{ann}(c)$ and let $I^c$ be a complement of $I$. Consider the homomorphism $\phi : I^c \to \mathcal{C}$, $d \mapsto cd$, which is injective. Hence $\hat{\phi} : I^c \to \text{im}(\phi)$ is an isomorphism with inverse $\hat{\phi}^{-1} : \text{im}(\phi) \to I^c \subseteq \mathcal{C}$. Consider the resulting map $\varphi : \text{im}(\phi) \to \mathcal{C}$ which satisfies $\varphi(\phi(d)) = d$ for all $d \in I^c$. Since $\mathcal{C}$ is self-injective, we can extend $\varphi$ to a homomorphism $\psi : \mathcal{C} \to \mathcal{C}$ with $b := \psi(1)$. Then we have for all $d \in I^c$:

\[ cd = \phi(d)b = \psi(\phi(d)) = \varphi(\phi(d)) = d. \]

This implies $c^2db = cd$ for all $d \in I^c$. Hence we have $\text{ann}(c^2b - c) \supseteq I^c$, and $\text{ann}(c^2b - c) \supseteq \text{ann}(c) = I$ holds anyhow. Thus $\text{ann}(c^2b - c) \supseteq I \oplus I^c$, and since $I \oplus I^c$ is essential in $\mathcal{C}$, we conclude that also $\text{ann}(c^2b - c)$ is essential in $\mathcal{C}$. This means that $c^2b - c \in S$, that is, $[c] = [c^2b]$ in $\mathcal{C}/S$. $\square$

Examples: The ring $\mathbb{Z}$ is not self-injective, since the homomorphism $\phi : \langle 2 \rangle \to \mathbb{Z}$ with $\phi(2) = 1$ cannot be extended to $\mathbb{Z}$. More generally, any reduced ring $\mathcal{C}$ of Krull dimension at least 1 is not self-injective: $N = 0$ implies $S = 0$, and $\text{Krull-dim}(\mathcal{C}) \geq 1$ implies that $\mathcal{C}$ is not von Neumann regular, hence $\mathcal{C}$ cannot be self-injective according to Theorem 3.

In contrast to these “counter”-examples, a variety of self-injective rings will be found in the next section.

2 Quasi-Frobenius rings

Let $\mathcal{C} \neq \{ 0 \}$ be a commutative ring. So far, we have studied the annihilators of elements of $\mathcal{C}$. Now we study the annihilators of ideals of $\mathcal{C}$. We begin by stating the most basic properties without proof.

Lemma 4 Let $I, I_1, I_2$ be ideals of $\mathcal{C}$.

1. $I_1 \subseteq I_2 \Rightarrow \text{ann}(I_1) \supseteq \text{ann}(I_2)$.
2. $I \subseteq \text{ann}(\text{ann}(I))$.
3. $\text{ann}(I) = \text{ann}(\text{ann}(\text{ann}(I)))$.
4. $\text{ann}(I_1 + I_2) = \text{ann}(I_1) \cap \text{ann}(I_2)$.

Lemma 5 Consider the following assertions:

$(P_1)$ $C$ is self-injective.

$(P_2)$ $\text{ann}(\text{ann}(I)) = I$ for all ideals $I$ of $C$.

$(C_1)$ $\text{ann}(I_1) + \text{ann}(I_2) = \text{ann}(I_1 \cap I_2)$ for all ideals $I_1, I_2$ of $C$.

$(C_2)$ $\text{ann}(\text{ann}(I)) = I$ for all finitely generated ideals $I$ of $C$.

Then $(P_i) \rightarrow (C_j)$ holds for all $i, j \in \{1, 2\}$.

Proof: “$(P_2) \Rightarrow (C_2)$” is trivial.

“(P_2) $\Rightarrow$ (C1)” : The previous lemma implies that

$$\text{ann}(\text{ann}(I_1) + \text{ann}(I_2)) = \text{ann}(\text{ann}(I_1)) \cap \text{ann}(\text{ann}(I_2)).$$

Using $(P_2)$, this means that

$$\text{ann}(\text{ann}(I_1) + \text{ann}(I_2)) = I_1 \cap I_2.$$ 

Taking annihilators on both sides and using once more $(P_2)$, we obtain

$$\text{ann}(I_1) + \text{ann}(I_2) = \text{ann}(I_1 \cap I_2).$$

“(P1) $\Rightarrow$ (C1)” : The inclusion “$\subseteq$” is straightforward. For the converse, let $c \in \text{ann}(I_1 \cap I_2)$. Define $\phi : I_1 + I_2 \rightarrow C$ by setting $\phi(c_1 + c_2) = cc_1$. To see that this is well-defined, suppose that $c_1 + c_2 = c_1' + c_2'$. Then $c_1 - c_1' = c_2' - c_2 \in I_1 \cap I_2$. Thus $c(c_1 - c_1') = 0$, that is, $cc_1 = cc_1'$. Since $C$ is self-injective, there exists a homomorphism $\psi : C \rightarrow C$ that extends $\phi$. Set $b := \psi(1)$. Then we have for all $c_1 \in I_1$, $c_2 \in I_2$:

$$b(c_1 + c_2) = \psi(c_1 + c_2) = \phi(c_1 + c_2) = cc_1.$$ 

In particular, we get (setting $c_1 = 0$) that $bc_2 = 0$ for all $c_2 \in I_2$, that is, $b \in \text{ann}(I_2)$, and (setting $c_2 = 0$) that $bc_1 = cc_1$ for all $c_1 \in I_1$, that is, $c - b \in \text{ann}(I_1)$. Finally, we can write $c = (c - b) + b \in \text{ann}(I_1) + \text{ann}(I_2)$. 

"\((P_1) \Rightarrow (C_2)\)"\: First, consider the special case of a principal ideal \(I = \langle c_1 \rangle\). Let \(d \in \text{ann}(\text{ann}(I)) = \text{ann}(\text{ann}(c_1))\). We need to show that \(d \in I\). Define a homomorphism \(\phi : I \to \mathcal{C}\) via \(\phi(c_1) = d\). For well-definedness, we need to show that \(\text{ann}(c_1) \subseteq \text{ann}(d)\). However, if \(c \in \text{ann}(c_1)\), then \(\text{ann}(c) \supseteq \text{ann}(\text{ann}(c_1)) \ni d\) and thus \(cd = 0\), that is, \(c \in \text{ann}(d)\).

Since \(\mathcal{C}\) is self-injective, \(\phi\) can be extended to a homomorphism \(\psi : \mathcal{C} \to \mathcal{C}\). Set \(b := \psi(1)\). Then we have

\[bc_1 = \psi(c_1) = \phi(c_1) = d,\]

which shows that \(d \in \langle c_1 \rangle = I\).

Second, consider the general case \(I = \langle c_1, \ldots, c_n \rangle\). Then we have

\[\text{ann}(\text{ann}(I)) = \text{ann}\left(\bigcap_{i=1}^{n} \text{ann}(c_i)\right).\]

Using the already proven implication "\((P_1) \Rightarrow (C_1)\)"\, we obtain

\[\text{ann}(\text{ann}(I)) = \sum_{i=1}^{n} \text{ann}(\text{ann}(c_i)).\]

From the principal ideal case above, we know that \(\text{ann}(\text{ann}(c_i)) = \langle c_i \rangle\) and thus we may conclude

\[\text{ann}(\text{ann}(I)) = \sum_{i=1}^{n} \langle c_i \rangle = I.\]

\[\square\]

For proving the subsequent main result, we use the following fact without proof: A commutative ring is Artinian if and only if it is Noetherian and has Krull dimension zero.

**Theorem 4** The following are equivalent:

1. \(\mathcal{C}\) is Noetherian and self-injective.
2. \(\mathcal{C}\) is Noetherian and satisfies \(\text{ann}(\text{ann}(I)) = I\) for all ideals \(I\) of \(\mathcal{C}\).
3. \(\mathcal{C}\) is Artinian and satisfies \(\text{ann}(\text{ann}(I)) = I\) for all ideals \(I\) of \(\mathcal{C}\).

If the equivalent conditions are satisfied, then \(\mathcal{C}\) is called a quasi-Frobenius ring.