1 Self-injective rings

Let $\mathcal{C}$ be a commutative ring. One says that $\mathcal{C}$ is **self-injective** if it is injective as a module over itself, that is, if $\mathrm{Hom}_{\mathcal{C}}(\cdot, \mathcal{C})$ is an exact functor. By Baer's criterion, this is equivalent to saying that any homomorphism $\phi : I \to \mathcal{C}$, where I is an ideal of $\mathcal{C}$, can be extended to all of $\mathcal{C}$, that is, there exists a homomorphism $\psi : \mathcal{C} \to \mathcal{C}$ such that $\psi|_I = \phi$. To understand the structure of self-injective rings better, we consider the set

$$S := \{c \in \mathcal{C} \mid \mathrm{ann}(c) \text{ is essential in } \mathcal{C}\},$$

where an ideal of $\mathcal{C}$ is called **essential** in $\mathcal{C}$ if it has a non-zero intersection with any non-zero ideal of $\mathcal{C}$.

Example: Let p be a prime number, $k \in \mathbb{N}$, and $\mathcal{C} = \mathbb{Z}_{p^k}$. The non-zero ideals of $\mathcal{C}$ are the ideals

$$\langle p^{k-1} \rangle \subsetneq \ldots \subsetneq \langle p^2 \rangle \subsetneq \langle p \rangle \subsetneq \langle 1 \rangle.$$

Since the intersection of any two of them always contains $\langle p^{k-1} \rangle \neq 0$, we conclude that any non-zero ideal of $\mathcal{C}$ is essential in $\mathcal{C}$. Since $\mathrm{ann}(p^l) = \langle p^{k-l} \rangle$ holds for all $0 \le l \le k$, we conclude that $S = \{p, \ldots, p^{k-1}, p^k\} = \langle p \rangle$. $\diamondsuit$

We observe that S is an ideal of $\mathcal{C}$ in this example. Next, we show that this holds in general, and this justifies that S is called the **singular ideal** of $\mathcal{C}$.

Lemma 1 *The set S defined above is an ideal of $\mathcal{C}$.*

Proof: Clearly, $0 \in S$. Let $s, t \in S$. Then $\mathrm{ann}(s)$ and $\mathrm{ann}(t)$ are both essential in $\mathcal{C}$. This implies that also $\mathrm{ann}(s) \cap \mathrm{ann}(t)$ is essential in $\mathcal{C}$. Since

$$\mathrm{ann}(s) \cap \mathrm{ann}(t) \subseteq \mathrm{ann}(s + t),$$

it follows that $\mathrm{ann}(s + t)$ is essential in $\mathcal{C}$, that is, $s + t \in S$. Now let $c \in \mathcal{C}$. Since

$$\mathrm{ann}(s) \subseteq \mathrm{ann}(sc),$$

we obtain that $\mathrm{ann}(sc)$ is essential in $\mathcal{C}$, that is, $sc \in S$. $\square$

Next, we establish some connections between the singular ideal S and other remarkable ideals of $\mathcal{C}$. The **nilradical** N of $\mathcal{C}$ is the set of all nilpotent elements of $\mathcal{C}$. It equals the intersection of all prime ideals of $\mathcal{C}$. A ring is called **reduced** if its nilradical is zero.

Lemma 2 *We always have $N \subseteq S$, and $N = 0$ is equivalent to $S = 0$. If $\mathcal{C}$ is Noetherian, then $N = S$.*

Proof: Let $c \in \mathcal{C}$ be nilpotent. We need to show that $\mathrm{ann}(c)$ is essential in $\mathcal{C}$. Let $I \neq 0$ be an ideal of $\mathcal{C}$ and let $0 \neq d \in I$. There exists a smallest positive integer n such that $c^n d = 0$. Then $0 \neq c^{n-1} d \in \mathrm{ann}(c) \cap I$.

Since $N \subseteq S$, it is clear that $S = 0$ implies $N = 0$. Assume conversely that $\mathcal{C}$ is reduced. Let $0 \neq c \in \mathcal{C}$. We wish to show that $c \notin S$, that is, $\mathrm{ann}(c)$ is not essential in $\mathcal{C}$. It suffices to show that $I := \mathrm{ann}(c) \cap \langle c \rangle = 0$. Indeed, any $d \in I$ has the form $d = ce$ for some e with $c^2 e = 0$. Then $0 = c^2 e^2 = (ce)^2$. Since $\mathcal{C}$ has no non-zero nilpotent elements, we must have $ce = 0$ and thus $d = 0$.

Now suppose that $\mathcal{C}$ is Noetherian and let $c \in S$. We need to show that c is nilpotent. The ideal chain

$$\mathrm{ann}(c) \subseteq \mathrm{ann}(c^2) \subseteq \mathrm{ann}(c^3) \subseteq \ldots$$

becomes stationary, say, $\mathrm{ann}(c^n) = \mathrm{ann}(c^{n+k})$ for all $k \geq 0$. We claim that this implies that $c^n = 0$. Assume conversely that $c^n \neq 0$. Since S is an ideal, we have $c^n \in S$ and hence, $\mathrm{ann}(c^n)$ is essential in $\mathcal{C}$, in particular, $\mathrm{ann}(c^n) \cap \langle c^n \rangle \neq 0$. This means that there exists $d \in \mathcal{C}$ such that $0 \neq c^n d \in \mathrm{ann}(c^n)$, that is, $c^{2n} d = 0$ or $d \in \mathrm{ann}(c^{2n}) = \mathrm{ann}(c^n)$, which implies $c^n d = 0$, a contradiction. □

The **Jacobson radical** J of $\mathcal{C}$ is defined to be the intersection of all maximal ideals of $\mathcal{C}$. The proof of the following useful characterization is left as an exercise.

Lemma 3 *We have*

$$d \in J \quad \Leftrightarrow \quad \forall c \in \mathcal{C} : 1 - cd \text{ is a unit of } \mathcal{C}.$$

So far, we have established the inclusions $N \subseteq S$ and $N \subseteq J$ in general, and $N = S \subseteq J$ in the Noetherian case.

Theorem 1 *Let $\mathcal{C}$ be self-injective. Then $J = S$.*

Proof: "$J \subseteq S$": Let $d \in J$ and let I be an ideal of $\mathcal{C}$ such that $\mathrm{ann}(d) \cap I = 0$. We need to show that this implies that $I = 0$. The homomorphism $\phi : I \to \mathcal{C}$, $c \mapsto cd$ is injective. Hence it induces an isomorphism $\hat{\phi} : I \to \mathrm{im}(\phi)$ with inverse $\hat{\phi}^{-1} : \mathrm{im}(\phi) \to I \subseteq \mathcal{C}$. Consider the resulting map $\varphi : \mathrm{im}(\phi) \to \mathcal{C}$ which satisfies $\varphi(\phi(c)) = c$ for all $c \in I$. Since $\mathcal{C}$ is self-injective, φ can be extended to a homomorphism $\psi : \mathcal{C} \to \mathcal{C}$. Let $\psi(1) =: b$. Then we have for all $c \in I$:

$$cdb = \phi(c)b = \psi(\phi(c)) = \varphi(\phi(c)) = c.$$

We conclude that $1 - db \in \mathrm{ann}(I)$. But since $d \in J$, the element $1 - db$ is a unit and hence $I = 0$ follows.

"$S \subseteq J$": Let $d \in S$ and $c \in \mathcal{C}$. We need to show that $1 - cd$ is a unit of $\mathcal{C}$. Since

$$\mathrm{ann}(d) \cap \mathrm{ann}(1 - cd) = 0$$

and $\mathrm{ann}(d)$ is essential in $\mathcal{C}$, we may conclude that $\mathrm{ann}(1 - cd) = 0$. Thus there exists a homomorphism $\phi : \langle 1 - cd \rangle \to \mathcal{C}$ with $\phi(1 - cd) = 1$. Since $\mathcal{C}$ is self-injective, ϕ can be extended to $\psi : \mathcal{C} \to \mathcal{C}$. Then we have

$$1 = \phi(1 - cd) = \psi(1 - cd) = (1 - cd)\psi(1),$$

which shows that $1 - cd$ is a unit. □

A commutative ring $\mathcal{C}$ is called **von Neumann regular** if for all $c \in \mathcal{C}$, we have $\langle c \rangle = \langle c^2 \rangle$, that is, there exists $d \in \mathcal{C}$ such that $c = c^2 d$. Then $(1 - cd)c = 0 = c(1 - dc)$, that is, d is a weak form of an inverse of c.

Theorem 2 *Let $\mathcal{C} \neq \{0\}$. The following are equivalent:*

1. *$\mathcal{C}$ is von Neumann regular.*
2. *$\mathcal{C}$ is reduced and has Krull dimension zero.*
3. *For every maximal ideal $\mathfrak{m}$ of $\mathcal{C}$, the localization $\mathcal{C}_\mathfrak{m}$ is a field.*

Proof: "$1 \Rightarrow 2$": "$N = 0$": Let $c^n = 0$, where n is the minimal positive integer with this property. Let d be such that $c = c^2 d$. If n is even, then we have $0 = c^n d^{\frac{n}{2}} = c^{\frac{n}{2}}$ contradicting the minimality of n. Thus $n = 2m + 1$ and $0 = c^{2m+1} d^m = c^{m+1}$. The minimality of n implies that $m = 0$, that is, $c = 0$.

"Krull-dim$(\mathcal{C}) = 0$": Let $\mathfrak{p}$ be a prime ideal of $\mathcal{C}$. We need to show that $\mathfrak{p}$ is maximal. The ring $\bar{\mathcal{C}} := \mathcal{C}/\mathfrak{p}$ is a domain. Let $0 \neq [c] \in \bar{\mathcal{C}}$. By assumption, we have $c = c^2 d$ for some d, that is, $[c][1 - dc] = 0$ holds in $\bar{\mathcal{C}}$. This implies that $[c]$ is a unit in $\bar{\mathcal{C}}$. Thus we have shown that $\bar{\mathcal{C}}$ is a field, that is, $\mathfrak{p}$ is maximal.

"$2 \Rightarrow 3$": Suppose that $N(\mathcal{C}) = 0$ and Krull-dim$(\mathcal{C}) = 0$ and let $\mathfrak{m}$ be a maximal ideal of $\mathcal{C}$. Since $N(\mathcal{C}_\mathfrak{m}) = N(\mathcal{C})_\mathfrak{m}$ and Krull-dim$(\mathcal{C}_\mathfrak{m}) \leq$ Krull-dim$(\mathcal{C})$, we conclude that also $\mathcal{C}_\mathfrak{m}$ is reduced and of Krull dimension zero. However, it is also a local ring, and thus $\mathcal{C}_\mathfrak{m}$ contains only one prime ideal, namely the maximal ideal $\mathfrak{m}_\mathfrak{m}$. Since the nilradical equals the intersection of all prime ideals, we obtain $\mathfrak{m}_\mathfrak{m} = 0$. Thus $\mathcal{C}_\mathfrak{m}$ is a field.

"$3 \Rightarrow 1$": Let $c \in \mathcal{C}$ be given. Let $\mathfrak{m}$ be a maximal ideal. We have

$$(\langle c \rangle / \langle c^2 \rangle)_\mathfrak{m} \cong \langle c \rangle_\mathfrak{m} / \langle c^2 \rangle_\mathfrak{m} \cong \langle \tfrac{c}{1} \rangle / \langle \tfrac{c^2}{1} \rangle.$$

Since $F := \mathcal{C}_\mathfrak{m}$ is a field, $\frac{c}{1} \in F$ and $(\frac{c}{1})^2 \in F$ generate the same ideal in F. Thus we have shown that $(\langle c \rangle / \langle c^2 \rangle)_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$. Since being zero is a local property, this implies that $\langle c \rangle / \langle c^2 \rangle = 0$, that is, $\langle c \rangle = \langle c^2 \rangle$. □

Theorem 3 *Let $\mathcal{C}$ be self-injective. Then $\mathcal{C}/S$ is von Neumann regular.*

For the proof of this theorem, we need some preparation. For any ideal I of $\mathcal{C}$, we may consider

$$\mathcal{M} = \{I' \mid I' \text{ is an ideal of } \mathcal{C} \text{ and } I \cap I' = 0\}.$$

By Zorn's Lemma, the set $\mathcal{M}$ contains an element that is maximal with respect to inclusion. Let I^c be such a maximal element. Then I^c is called a **complement** of I, and this implies that $I \oplus I^c$ is essential in $\mathcal{C}$.

Proof: Let $c \in \mathcal{C}$ be given. Let $I := \operatorname{ann}(c)$ and let I^c be a complement of I. Consider the homomorphism $\phi : I^c \to \mathcal{C}$, $d \mapsto cd$, which is injective. Hence $\hat{\phi} : I^c \to \operatorname{im}(\phi)$ is an isomorphism with inverse $\hat{\phi}^{-1} : \operatorname{im}(\phi) \to I^c \subseteq \mathcal{C}$. Consider the resulting map $\varphi : \operatorname{im}(\phi) \to \mathcal{C}$ which satisfies $\varphi(\phi(d)) = d$ for all $d \in I^c$. Since $\mathcal{C}$ is self-injective, we can extend φ to a homomorphism $\psi : \mathcal{C} \to \mathcal{C}$ with $b := \psi(1)$. Then we have for all $d \in I^c$:

$$cdb = \phi(d)b = \psi(\phi(d)) = \varphi(\phi(d)) = d.$$

This implies $c^2db = cd$ for all $d \in I^c$. Hence we have $\operatorname{ann}(c^2b - c) \supseteq I^c$, and $\operatorname{ann}(c^2b - c) \supseteq \operatorname{ann}(c) = I$ holds anyhow. Thus $\operatorname{ann}(c^2b - c) \supseteq I \oplus I^c$, and since $I \oplus I^c$ is essential in $\mathcal{C}$, we conclude that also $\operatorname{ann}(c^2b - c)$ is essential in $\mathcal{C}$. This means that $c^2b - c \in S$, that is, $[c] = [c^2b]$ in $\mathcal{C}/S$. $\square$

Examples: The ring $\mathbb{Z}$ is not self-injective, since the homomorphism $\phi : \langle 2 \rangle \to \mathbb{Z}$ with $\phi(2) = 1$ cannot be extended to $\mathbb{Z}$. More generally, any reduced ring $\mathcal{C}$ of Krull dimension at least 1 is not self-injective: $N = 0$ implies $S = 0$, and $\text{Krull-dim}(\mathcal{C}) \geq 1$ implies that $\mathcal{C}$ is not von Neumann regular, hence $\mathcal{C}$ cannot be self-injective according to Theorem 3. $\diamondsuit$

In contrast to these "counter"-examples, a variety of self-injective rings will be found in the next section.

2 Quasi-Frobenius rings

Let $\mathcal{C} \neq \{0\}$ be a commutative ring. So far, we have studied the annihilators of elements of $\mathcal{C}$. Now we study the annihilators of ideals of $\mathcal{C}$. We begin by stating the most basic properties without proof.

Lemma 4 *Let I, I_1, I_2 be ideals of $\mathcal{C}$.*

1. $I_1 \subseteq I_2 \Rightarrow \operatorname{ann}(I_1) \supseteq \operatorname{ann}(I_2)$.

2. $I \subseteq \mathrm{ann}(\mathrm{ann}(I))$.

3. $\mathrm{ann}(I) = \mathrm{ann}(\mathrm{ann}(\mathrm{ann}(I)))$.

4. $\mathrm{ann}(I_1 + I_2) = \mathrm{ann}(I_1) \cap \mathrm{ann}(I_2)$.

Lemma 5 *Consider the following assertions:*

(P_1) *$\mathcal{C}$ is self-injective.*

(P_2) $\mathrm{ann}(\mathrm{ann}(I)) = I$ *for all ideals I of $\mathcal{C}$.*

(C_1) $\mathrm{ann}(I_1) + \mathrm{ann}(I_2) = \mathrm{ann}(I_1 \cap I_2)$ *for all ideals I_1, I_2 of $\mathcal{C}$.*

(C_2) $\mathrm{ann}(\mathrm{ann}(I)) = I$ *for all finitely generated ideals I of $\mathcal{C}$.*

Then $(P_i) \to (C_j)$ holds for all $i, j \in \{1, 2\}$.

Proof: "$(P_2) \Rightarrow (C_2)$" is trivial.

"$(P_2) \Rightarrow (C_1)$": The previous lemma implies that

$$\mathrm{ann}(\mathrm{ann}(I_1) + \mathrm{ann}(I_2)) = \mathrm{ann}(\mathrm{ann}(I_1)) \cap \mathrm{ann}(\mathrm{ann}(I_2)).$$

Using (P_2), this means that

$$\mathrm{ann}(\mathrm{ann}(I_1) + \mathrm{ann}(I_2)) = I_1 \cap I_2.$$

Taking annihilators on both sides and using once more (P_2), we obtain

$$\mathrm{ann}(I_1) + \mathrm{ann}(I_2) = \mathrm{ann}(I_1 \cap I_2).$$

"$(P_1) \Rightarrow (C_1)$": The inclusion "$\subseteq$" is straightforward. For the converse, let $c \in \mathrm{ann}(I_1 \cap I_2)$. Define $\phi : I_1 + I_2 \to \mathcal{C}$ by setting $\phi(c_1 + c_2) = cc_1$. To see that this is well-defined, suppose that $c_1 + c_2 = c_1' + c_2'$. Then $c_1 - c_1' = c_2' - c_2 \in I_1 \cap I_2$. Thus $c(c_1 - c_1') = 0$, that is, $cc_1 = cc_1'$.

Since $\mathcal{C}$ is self-injective, there exists a homomorphism $\psi : \mathcal{C} \to \mathcal{C}$ that extends ϕ. Set $b := \psi(1)$. Then we have for all $c_1 \in I_1$, $c_2 \in I_2$:

$$b(c_1 + c_2) = \psi(c_1 + c_2) = \phi(c_1 + c_2) = cc_1.$$

In particular, we get (setting $c_1 = 0$) that $bc_2 = 0$ for all $c_2 \in I_2$, that is, $b \in \mathrm{ann}(I_2)$, and (setting $c_2 = 0$) that $bc_1 = cc_1$ for all $c_1 \in I_1$, that is, $c - b \in \mathrm{ann}(I_1)$. Finally, we can write $c = (c - b) + b \in \mathrm{ann}(I_1) + \mathrm{ann}(I_2)$.

"$(P_1) \Rightarrow (C_2)$": First, consider the special case of a principal ideal $I = \langle c_1 \rangle$. Let $d \in \text{ann}(\text{ann}(I)) = \text{ann}(\text{ann}(c_1))$. We need to show that $d \in I$. Define a homomorphism $\phi : I \to \mathcal{C}$ via $\phi(c_1) = d$. For well-definedness, we need to show that $\text{ann}(c_1) \subseteq \text{ann}(d)$. However, if $c \in \text{ann}(c_1)$, then $\text{ann}(c) \supseteq \text{ann}(\text{ann}(c_1)) \ni d$ and thus $cd = 0$, that is, $c \in \text{ann}(d)$.

Since $\mathcal{C}$ is self-injective, ϕ can be extended to a homomorphism $\psi : \mathcal{C} \to \mathcal{C}$. Set $b := \psi(1)$. Then we have

$$bc_1 = \psi(c_1) = \phi(c_1) = d,$$

which shows that $d \in \langle c_1 \rangle = I$.

Second, consider the general case $I = \langle c_1, \ldots, c_n \rangle$. Then we have

$$\text{ann}(\text{ann}(I)) = \text{ann}(\bigcap_{i=1}^{n} \text{ann}(c_i)).$$

Using the already proven implication "$(P_1) \Rightarrow (C_1)$", we obtain

$$\text{ann}(\text{ann}(I)) = \sum_{i=1}^{n} \text{ann}(\text{ann}(c_i)).$$

From the principal ideal case above, we know that $\text{ann}(\text{ann}(c_i)) = \langle c_i \rangle$ and thus we may conclude

$$\text{ann}(\text{ann}(I)) = \sum_{i=1}^{n} \langle c_i \rangle = I.$$

□

For proving the subsequent main result, we use the following fact without proof: A commutative ring is Artinian if and only if it is Noetherian and has Krull dimension zero.

Theorem 4 *The following are equivalent:*

1. *$\mathcal{C}$ is Noetherian and self-injective.*
2. *$\mathcal{C}$ is Noetherian and satisfies* $\text{ann}(\text{ann}(I)) = I$ *for all ideals I of $\mathcal{C}$.*
3. *$\mathcal{C}$ is Artinian and satisfies* $\text{ann}(\text{ann}(I)) = I$ *for all ideals I of $\mathcal{C}$.*

If the equivalent conditions are satisfied, then $\mathcal{C}$ is called a **quasi-Frobenius** *ring.*