Congruences between Schur elements

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Schur elements matter

For H a finite group and ℓ a prime number, we set the following notation.

- For χ an irreducible character of H, its Schur element is $S_\chi := |H|/\chi(1)$.
- For $h \geq 0$ an integer, $\mathrm{Irr}_h(H) := \{\chi \in \mathrm{Irr}(H) \mid (S_\chi)_\ell = \ell^h\}$.
- For $r \in (\mathbb{Z}/\ell\mathbb{Z})^\times$, $\mathrm{Irr}_{h,r}(H) := \{\chi \in \mathrm{Irr}_h(H) \mid (S_\chi)_{\ell'} \equiv \pm r \bmod \ell\}$.
- For $\mathcal{G}$ a subgroup of $\mathrm{Gal}(\mathbb{Q}_\ell(\sqrt[|G|]{1})/\mathbb{Q}_\ell)$,
  $\mathrm{Irr}_{h,r,\mathcal{G}}(H) := \{\chi \in \mathrm{Irr}_{h,r}(H) \mid \forall \sigma \in \mathcal{G}\,,\ {}^\sigma\chi = \chi\}$.

A **perfect isometry** between (H, b) and (H', b') is (in particular) a "bijection with signs" $\chi \in \mathrm{Irr}(H, b) \xrightarrow{\sim} \varepsilon'_\chi \chi' \in \mathrm{Irr}(H', b')$ such that $\varepsilon_{\chi'} S_{\chi'}/S_\chi = r \in (\mathbb{Z}/\ell\mathbb{Z})^\times$, *i.e.*, $\mathrm{Irr}_{h,r}(H, b) \xrightarrow{\sim} \mathrm{Irr}_{h,r}(H', b')$.

"AMIDRUNK conjecture" – Let b be an ℓ-block of G. Assume that $O_\ell(G) = 1$ and b of defect not 1. Then, whenever $h \geq 0$, $r \in (\mathbb{Z}/\ell\mathbb{Z})^\times$, and $\mathcal{G} \subseteq \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt[|G|]{1})/\mathbb{Q}_\ell)$ are given,

$$\sum_{\mathcal{E}/G} (-1)^{|\mathcal{E}|} |\mathrm{Irr}_{h,r,\mathcal{G}}(N_G(\mathcal{E}), e)| = 0\,, \ldots$$

Towards generic versions

Notation

- K = an abelian finite extension of $\mathbb{Q}$,
- V = an r-dimensional vector space over K,
- W = a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudos-)reflections,
- $\mathcal{A}$ the set of reflecting hyperplanes for W,

and for H a reflection hyperplane of W ($H \in \mathcal{A}$) :

- e_H the order of the fixator W_H of H in G,
- s_H the corresponding reflection with eigenvalue $e^{2\pi i/e_H}$, the generator of W_H called *distinguished reflection*.

Braid groups

Set $V^{\mathrm{reg}} := V - \bigcup_{H\in\mathcal{A}} H\,.$

The covering $V^{\mathrm{reg}} \to V^{\mathrm{reg}}/W$ is Galois, hence it induces a short exact sequence

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_1(V^{\mathrm{reg}}, x_0) & \longrightarrow & \Pi_1(V^{\mathrm{reg}}/W, x_0) & \longrightarrow & W & \longrightarrow & 1 \\
 & & \ddot{\|} & & \ddot{\|} & & & & \\
1 & \longrightarrow & \mathbf{P}_W & \longrightarrow & \mathbf{B}_W & \longrightarrow & W & \longrightarrow & 1 \\
 & & \| & & \| & & & & \\
1 & \longrightarrow & \langle \mathbf{s}_{H,\gamma}^{e_H} \rangle_{H\in\mathcal{A}} & \longrightarrow & \langle \mathbf{s}_{H,\gamma} \rangle_{H\in\mathcal{A}} & \longrightarrow & \langle s_H \rangle_{H\in\mathcal{A}} & \longrightarrow & 1 \\
 & & & & \text{(braid reflexions)} & & & &
\end{array}$$

Theorem

1. The braid group B_W is generated by the braid reflections $(\mathbf{s}_{H,\gamma})$ (for all H and all γ).
2. The pure braid group P_W is generated by the elements $(\mathbf{s}_{H,\gamma}^{e_H})$
3. If W acts irreducibly on V, the center of B_W and P_W are cyclic, and the center of P_W is generated by $\boldsymbol{\pi}$, the positive loop around x_0.

Hecke algebras

For $W \subset \mathrm{GL}(V)$ a reflection group, let $\{\mathbf{s}, \mathbf{t}, \dots\}$ be a complete set of representatives of conjugacy classes of braid reflections in B_W, with reflections-orders respectively $\{d, e, \dots\}$.
Choose corresponding indeterminates $\{(u_\alpha)_{\alpha=0,\dots d-1}, (v_\beta)_{\beta=0,\dots,e-1}, \dots\}$

Generic Hecke Algebra

The corresponding generic Hecke algebra $\mathcal{H}(W)$ is the quotient of the group algebra of B_W over the Laurent polynomials algebra

$$\mathbb{Z}[(u_\alpha^{\pm 1})_{\alpha=0,\dots d-1}, (v_\beta^{\pm 1})_{\beta=0,\dots,e-1}, \dots]$$

by the ideal generated by

$$(\mathbf{s} - u_0)\cdots(\mathbf{s} - u_{d-1}) \ , \ (\mathbf{t} - v_0)\cdots(\mathbf{t} - v_{e-1}) \,, \dots .$$

Notice that if we specialize $u_\alpha \mapsto \exp(2\pi i\alpha/d)$, $v_\beta \mapsto \exp(2\pi i\beta/e)$, ..., then $\mathcal{H}(W)$ becomes the group algebra of W.

Theorem & Conjecture

1. (Theorem – G. Malle, I. Marin, Mathématiciennes grecques, and al.) $\mathcal{H}(W)$ is free of rank $|W|$ over $\mathbb{Z}[(u_\alpha^{\pm 1}), (v_\beta^{\pm 1}), \dots]$.
2. (Conjecture) There exists a unique linear form

$$t : \mathcal{H}(W) \to \mathbb{Z}[(u_\alpha^{\pm 1})_{\alpha=0,\dots d-1}, (v_\beta^{\pm 1})_{\beta=0,\dots,e-1}, \dots]$$

with the following properties.

   - t is a symmetrizing form on the algebra $\mathcal{H}(W)$,
   - t specializes to the canonical linear form on the group algebra of W,
   - For all $b \in B$, we have

$$t(b^{-1})^\vee = \frac{t(b\pi)}{t(\pi)} .$$

Φ-cyclotomic Hecke algebras

Let x be an indeterminate, let Φ be a cyclotomic polynomial in $K[x]$.

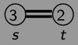
A Φ-cyclotomic Hecke algebra $\mathcal{H}_x(W)$ is (in particular) a specialization "with only one indeterminate" of the generic Hecke algebras such that

- $u_\alpha \mapsto \zeta_\alpha x^{m_\alpha}$, $v_\beta \mapsto \xi_\beta x^{n_\beta}$, ...
  where ζ, ξ,... are roots of 1, and m_α, n_β... are nonnegative rational numbers,
- $\mathcal{H}_x(W)$ is a $\mathbb{Z}_K[x, x^{-1}]$-algebra,
- $\mathcal{H}_x(W)$ is a specialization of the generic Hecke algebra $\mathcal{H}(W)$,
- which is congruent modulo Φ to the group algebra of W.

Examples

- $\Phi = x^2 + x + 1$ (a "3-Weyl group" for $G = \mathrm{GL}_6$),

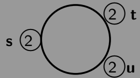
$$W_3 = B_2(3) = \boldsymbol{\mu}_3 \wr \mathfrak{S}_2 \quad \longleftrightarrow$$



$$\mathcal{H}(W_3) = \left\langle S, T ; \begin{Bmatrix} STST = TSTS \\ (S-1)(S-x)(S-x^2) = 0 \\ (T-x^3)(T+1) = 0 \end{Bmatrix} \right\rangle$$

- $\Phi = x^2 + 1$ (a "4-Weyl group" for $G = \mathrm{O}_8$, $W = D_4$),

$$W_4 = G(4,2,2) \quad \longleftrightarrow$$



$$\mathcal{H}(W_4) = \left\langle S, T, U ; \begin{Bmatrix} STU = TUS = UST \\ (S-x^2)(S-1) = 0 \end{Bmatrix} \right\rangle$$

Some conjectures implying congruences

Let $\mathbb{G}$ be the type ("reflection coset") of a "finite reductive group" $(\mathbf{G}, F)$, and let $(\mathbb{L}, \lambda)$ be a Φ–cuspidal pair of $\mathbb{G}$.

Conjectures (from last century)

There exists a Φ-cyclotomic algebra $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$ attached to the relative Weyl group $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ (a complex reflection group), where

- $t : \mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda) \to \mathbb{Z}_K[x, x^{-1}]$ is the specialization of the canonical form,
- the Schur elements $\mathbf{S}_\chi^{(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))} \in \mathbb{Z}_K[x, x^{-1}]$ attached to characters χ of $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$ are defined by

$$t = \sum_{\chi \in \mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))} \frac{1}{\mathbf{S}_\chi^{(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))}} \chi \,,$$

which satisfies Conjectures **Conj-1** and **Conj-2** below.

Conj-1. There is a bijection $\mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)) \xrightarrow{\sim} \mathrm{Irr}(\mathbb{G},(\mathbb{L},\lambda))$, $\chi \mapsto \rho_\chi$ together with a collection of signs (ε_χ) such that, as a character of a $\mathbf{G}^F$–object–$\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)_{|x=q}$, we have

$$R_{\mathbf{L}}^{\mathbf{G}}(\lambda)_{|x=q} = \bigoplus_{\chi \in \mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))} \varepsilon_\chi \rho_{\chi_{|x=q}} \otimes \chi_{|x=q} \,.$$

Conj-2. $\mathrm{tr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))_{|x=q} \,;\, R_{\mathbb{L}}^{\mathbb{G}}(\lambda)_{|x=q}) = \mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\lambda))_{|x=q}\, t_{|x=q}$, hence

$$\mathrm{tr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))_{|x=q} \,;\, R_{\mathbb{L}}^{\mathbb{G}}(\lambda)_{|x=q}) = \sum_{\chi \in \mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)))} \frac{\mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\lambda))_{|x=q})}{(\mathbf{S}_\chi)_{|x=q}} \chi_{|x=q} \text{, and}$$

$$\mathrm{Deg}_{\varepsilon_\chi \rho_\chi} = \frac{\mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\lambda))}{\mathbf{S}_\chi^{(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))}} \,.$$

Since

$$\mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\lambda)) = \varepsilon_{\mathbb{G}}\varepsilon_{\mathbb{L}}\frac{|\mathbb{G}:\mathbb{L}|}{|\mathbb{G}:\mathbb{L}|_x}\mathrm{Deg}(\lambda)\,,$$

Conj-2 implies

$$\frac{\mathbf{S}^{(\mathbb{G})}_{\varepsilon_\chi\rho_\chi}}{\mathbf{S}^{(\mathbb{L})}_{\lambda}\mathbf{S}^{(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))}_{\chi}} = \varepsilon_{\mathbb{G}}\varepsilon_{\mathbb{L}}|\mathbb{G}:\mathbb{L}|_x \text{ hence } \frac{\mathbf{S}^{(\mathbb{G})}_{\varepsilon_\chi\rho_\chi}}{\mathbf{S}^{(\mathbb{L})}_{\lambda}\mathbf{S}^{(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))}_{\chi}} \equiv 1 \mod \Phi\,.$$

Moreover, since $\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda) \equiv KW_{\mathbb{G}}(\mathbb{L},\lambda)) \mod \Phi$, we have

$$\frac{\mathbf{S}^{(\mathbb{G})}_{\varepsilon_\chi\rho_\chi}}{\mathbf{S}^{(\mathbb{L})}_{\lambda} S_{\chi_{W_{\mathbb{G}}(\mathbb{L},\lambda)}}} \equiv 1 \mod \Phi\,.$$

Assuming extendibility of λ

Ask Britta = Assume that λ extends to a character of $N_{\mathbf{G}^F}(\mathbf{L},\lambda)$, still denoted by λ .

Then the characters of $N_{\mathbf{G}^F}(\mathbf{L},\lambda)$ may be denoted by $\lambda.\chi$

- where χ runs over $\mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L},\lambda))$,
- and where $(\lambda.\chi)(1) = \mathrm{Deg}(\lambda)_{|_{x=q}}\chi(1) = \lambda(1)\chi(1)$.

Then the above equality becomes :

$$\frac{\mathbf{S}_{\varepsilon_\chi \rho_\chi}}{S_{\lambda.\chi}^{(N_{\mathbf{G}^F}(\mathbb{L},\lambda))}} \equiv 1 \mod \Phi \,.$$

Now comes ℓ

Let ℓ be a prime number which does not divide q, and divides $\Phi(q)$.

By the above congruence, it follows that

$$\frac{\varepsilon_\chi S_{\rho_\chi^{\mathbf{G}^F}}}{S_{\lambda.\chi}} \equiv 1 \mod \ell \,,$$

Note that the above congruence means the following two facts.

1. if $(S_{\lambda.\chi})_\ell = \ell^h$ then $(S_{\rho_\chi^{\mathbf{G}^F}})_\ell = \ell^h$.
2. $\left(\frac{\varepsilon_\chi S_{\rho_\chi^{\mathbf{G}^F}}}{S_{\lambda.\chi}}\right)_{\ell'} \equiv 1 \mod \ell \,,$

hence we have a bijection a bijection

$$\mathrm{Irr}_{h,1}(N_{\mathbf{G}^F}(\mathbf{L},\lambda)) \xrightarrow{\sim} \mathrm{UnCha}_{h,1}(\mathbf{G}^F;(\mathbf{L},\lambda)) \quad , \quad \lambda\cdot\chi \mapsto \rho_\chi^{\mathbf{G}^F} \,.$$

If time permits : Extending this to Lusztig's series

Let s be an ℓ-element of $\mathbf{G}^{*F^*}$.
We know that there exists a bijection

$$\mathrm{UnCha}(C_{\mathbf{G}^*}(s))^{F^*} \xrightarrow{\sim} \mathcal{E}(\mathbf{G}^F,(s)) \quad , \quad \rho \mapsto \xi_\rho$$

such that

$$\mathrm{Deg}(\xi_\rho) = \pm \frac{|\mathbf{G}^F|_{q'}}{|C_{\mathbf{G}^*}(s)^{F^*}|_{q'}} \mathrm{Deg}(\rho)\,.$$

Since

$$\frac{|\mathbf{G}^F|_q}{|C_{\mathbf{G}^*}(s)^{F^*}|_q} \equiv \pm 1 \mod \ell\,,$$

it follows that

$$\left(\frac{S_{\xi_\rho}}{S_\rho}\right)_\ell \equiv \pm 1 \mod \ell\,.$$

Sadness, Seriousness, ... and Friendly smile

The late Jan Saxl, Three Fields Medalists,...

and two dissipated guys.