Algebraic Geometry (WS 2025)

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(24.1) Varieties as spaces with functions. a) We keep the earlier setting. Any affine, quasi-affine, projective or quasi-projective variety is (for the time being) simply called a **(K-)variety (over L)**.

A variety V is a topological space together with a sheaf $\mathcal{O}_V$ of K-algebras of L-valued functions, whose restriction maps are given by restriction of functions. The pair $(V, \mathcal{O}_V)$ is called a **space with functions**. In particular, this naturally yields the definition of morphisms of varieties.

b) Let $(U, \mathcal{O}_U)$ and $(V, \mathcal{O}_V)$ be spaces with functions. Then a continuous map $\varphi \colon U \to V$ is called a **morphism** (of spaces with functions), if for any $W \subseteq V$ open and any $f \in \mathcal{O}_V(W)$ we have $\varphi_W^*(f) := (f \circ \varphi)|_{\varphi^{-1}(W)} \in \mathcal{O}_U(\varphi^{-1}(W))$.

Thus the **comorphism** $\varphi^* \colon \mathcal{O}_V \Rightarrow \mathcal{O}_U$ induces homomorphisms of K-algebras $\varphi_W^* \colon \mathcal{O}_V(W) \to \mathcal{O}_U(\varphi^{-1}(W))$, which commute with the restriction of functions, that is for $W' \subseteq W \subseteq V$ open we have $\varphi_{W'}^* \circ \rho_{W' \subseteq V}^{W} = \rho_{\varphi^{-1}(W') \subseteq U}^{\varphi^{-1}(W)} \circ \varphi_W^*$.

In other words, the assignment $\varphi^* \colon \mathcal{O}_V \Rightarrow \mathcal{O}_U$ behaves like a (contravariant) natural transformation (that is a morphism of sheaves, where we consider sheaves as functors). In particular, id_U is a morphism, whose comorphism $(\mathrm{id}_U)^* \colon \mathcal{O}_U \Rightarrow \mathcal{O}_U$ induces the identity isomorphism on all sections of $\mathcal{O}_U$. Moreover, if $(W, \mathcal{O}_W)$ is a space with functions, and $\psi \colon V \to W$ is a morphism, then $\varphi\psi \colon U \to W$ is a morphism again, where the associated comorphisms fulfill $(\varphi\psi)^* = \psi^*\varphi^*$.

A morphism $\varphi \colon U \to V$ is called an **isomorphism** (of spaces with functions), if there is a morphism $\psi \colon V \to U$ such that $\varphi\psi = \mathrm{id}_U$ and $\psi\varphi = \mathrm{id}_V$. This is equivalent to saying that $\varphi \colon U \to V$ is a homeomorphism, such that the associated comorphisms fulfill $\psi^*\varphi^* = (\mathrm{id}_U)^*$ and $\varphi^*\psi^* = (\mathrm{id}_V)^*$, where the latter in turn is equivalent to saying that $\varphi_W^* \colon \mathcal{O}_V(W) \to \mathcal{O}_U(\varphi^{-1}(W))$ is an isomorphism with inverse $\psi_{\psi(W)}^* \colon \mathcal{O}_U(\psi(W)) \to \mathcal{O}_V(W)$, for any $W \subseteq V$ open.

(24.2) Prevarieties. a) We keep the earlier notation. A space with (K-algebras of L-valued) functions $(V, \mathcal{O}_V)$ is called a **prevariety**, if **i)** V is a connected topological space, and **ii)** there is an open covering $\{V_i; i \in \mathcal{I}\}$ of V, where $\mathcal{I}$ is a finite index set, such that the space with functions $(V_i, \mathcal{O}_V|_{V_i})$ is (isomorphic to) an irreducible affine variety, for $i \in \mathcal{I}$.

Proposition. i) Any prevariety V is Noetherian, hence V is quasi-compact.
ii) If $V \neq \emptyset$, then V is irreducible.

Proof. i) Let $V \supseteq W_1 \supseteq W_2 \supseteq \cdots$ be a descending chain of closed subsets. Then for $i \in \mathcal{I}$ we have $V_i \supseteq (V_i \cap W_1) \supseteq (V_i \cap W_2) \supseteq \cdots$. Since V_i is Noetherian, each of the latter chains stabilises after finitely many steps. Thus, since $\mathcal{I}$ is finite and $V = \bigcup_{i\in\mathcal{I}} V_i$, so does the given chain.

ii) Let $\emptyset \neq V = \bigcup_{j=1}^{r} W_j$ be the irreducible components of V, where $r \in \mathbb{N}$, and assume that $r \geq 2$. Since $W_1 \subseteq V$ and $\bigcup_{i\geq 2} W_i \subseteq V$ are closed, and V is connected, we may assume that there is $v \in W_1 \cap W_2 \neq \emptyset$. Let $v \in V_i$, for some $i \in \mathcal{I}$. Then $\emptyset \neq V_i \cap W_j \subseteq W_j$ is open, hence dense, for both $j \in \{1, 2\}$. Thus we have $W_1 \cup W_2 \subseteq \overline{V}_i \subseteq V$. Since V_i is irreducible, so is its closure $\overline{V}_i$. Thus $W_1 \not\subseteq W_2 \not\subseteq W_1$ cannot possibly be irreducible components, a contradiction. $\sharp$

An open subset $\emptyset \neq U \subseteq V$, such that $(U, \mathcal{O}_V|_U)$ is an (irreducible) affine variety, is called an **affine open** subset of V. Since the (irreducible) affine open subsets form a basis of the Zariski topology on an irreducible affine variety, we conclude that the affine open subsets form a basis of the topology on V.

Morphisms of prevarieties are their morphisms as spaces with functions, giving rise to the full subcategory of prevarieties, within the category of spaces with functions. In particular, any irreducible affine, quasi-affine, projective or quasi-projective variety is a prevariety.

(24.3) Remark: Comparison with manifolds. Recall that a topological space $\mathcal{V}$ is called **Hausdorff**, if for any $x \neq y \in \mathcal{V}$ there are open neighbourhoods $\mathcal{U}_x$ and $\mathcal{U}_y$ of x and y, respectively, such that $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$.

Now, for $n \in \mathbb{N}$, we may define an n-dimensional **complex manifold** $\mathcal{M}$ along the above lines, as being a connected Hausdorff space, together with a sheaf of $\mathbb{C}$-valued functions $\mathcal{H}_\mathcal{M}$, such that there is an open covering $\{\mathcal{M}_i; i \in \mathcal{I}\}$ of $\mathcal{M}$, where $\mathcal{I}$ is an index set, such that $(\mathcal{M}_i, \mathcal{H}_\mathcal{M}|_{\mathcal{M}_i})$ is isomorphic to $(\mathbb{C}^n, \mathcal{H}_{\mathbb{C}^n})$, for $i \in \mathcal{I}$, where $\mathcal{H}_{\mathbb{C}^n}$ is the sheaf of holomorphic functions on $\mathbb{C}^n$.

In particular, for $n = 1$ we recover **Riemann surfaces**. Similarly, we may define an n-dimensional **differentiable manifold** by being connected Hausdorff, and prescribing $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ on an open covering, where $\mathcal{C}^\infty_{\mathbb{R}^n}$ is the sheaf of smooth $\mathbb{R}$-valued functions on $\mathbb{R}^n$. But note that, since open subsets of affine open subsets are not necessarily affine again, in the algebraic setting we cannot proceed with 'smooth transition maps' between 'charts', as is usually done for manifolds.

(24.4) Restricted sheaves. a) Let V be a topological space, and let $\mathcal{O}_V$ be a presheaf of functions on V, whose restriction maps are given by restriction of functions. For $W \subseteq V$ the **restricted** presheaf $\mathcal{O}_V|_W$ is defined as follows:

For $U \subseteq W$ open let $\mathcal{O}_V|_W(U)$ be the set of all functions $f : U \to L$, such that for all $v \in U$ there exist an open neighborhood $v \in U_v \subseteq V$ and a section $f_v \in \mathcal{O}_V(U_v)$, such that $U_v \cap W \subseteq U$ and $f_v|_{U_v \cap W} = f|_{U_v \cap W}$.

Then the natural inclusion map $\iota_W^V : W \to V$, being continuous, induces a comorphism $(\iota_W^V)^* : \mathcal{O}_V \Rightarrow \mathcal{O}_V|_W$:

Let $U \subseteq V$ be open and $f \in \mathcal{O}_V(U)$. Then we have $U' := (\iota_W^V)^{-1}(U) = U \cap W$. Letting $U_v = U$ and $f_v = f$, for all $v \in U'$, we get $(\iota_W^V)_U^*(f) = f|_{U \cap W} = f|_{U'}$, showing that $f|_{U'} \in \mathcal{O}_V|_W(U')$. $\sharp$
