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On p-vanishing and sign classes of the symmetric group

Applications of the Murnaghan-Nakayama Formula

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Resumé

Dette speciale omhandler irreducible karakterer i de endelige symmetriske grupper. Der er en naturlig korrespondence mellem partitioner af n og irreducible karakterer af den symmetrisk gruppe S_n. Denne korrepondence er specielt synlig i Murnaghan-Nakayama-formlen. Formlen muliggør at beregne værdien af den irreducible karakter, hørende til partitionen α rekursivt ved at betragte α's Young diagram. Forbindelsen mellen de irreducible karakterer og partitioner viser sig også i den såkaldte hook-formel, som angiver graden af den irreducible karakter hørende til α ved hjælp af "hooklængderne" i α. Hook-formlen kan også ses some en anvendelse af Murnaghan-Nakayama-formlen (eller af forgreningsreglen, som er et specielt tilfælde af den).

Jeg starter med at præsentere et bevis for Murnaghan-Nakayama-formlen og så giver jeg nogle eksempler på anvendelser. Den første anvendelse er en formel for $\chi^\alpha(\pi)$, værdien af den irreducible karakter, hørende til α på π, i det tilfælde hvor for et naturligt tal q, q-vægten af π er større end eller lig α's q-vægt. De andre anvendelser af Murnaghan-Nakayama-formlen er studier af "p-vanishing"- og "sign"-konjugationsklasser i S_n, dvs. konjugationsklasser i S_n hvor alle irreducible karakterer af grad delelig med p antager værdien 0 eller hvor alle irreducible karakterer kun tager værdierne 0,1 eller -1.

Abstract

In this thesis I will study the irreducible characters of the finite symmetric groups. They are labeled in a natural way by the partitions of n. This way of labeling irreducible partitions of S_n becomes even more natural when we study the Murnaghan-Nakayama formula. This formula allows us to find the values of the irreducible character labeled by α in a recursive way by looking at the Young diagram of α. The connection between the irreducible characters of S_n and the partitions of n appears also in the hook formula, which allows us to calculate the degree of the irreducible character labeled by α in terms of the hook lengths of α. The hook formula may actually be seen as an application of the Murnaghan-Nakayama formula (or of the branching rule, which is a special case of it), as the degree of a character is equal to its value on the element 1.

I will start by presenting a proof of the Murnaghan-Nakayama formula and then use it in some applications. The first application is to find a formula for $\chi^\alpha(\pi)$, the value of the irreducible character labeled by α on π, in the case where for some q, the q-weight of π is at least as big as that of α. The other applications of the Murnaghan-Nakayama formula that I will present are to study p-vanishing and sign classes of S_n, that is conjugacy classes of S_n where all irreducible characters of degree divisible by p take the value 0 or where all irreducible characters take value 0,1 or -1 respectively.

Introduction

After having given some definitions and results about representations of a finite group, in particular when the ground field is algebraic closed and has characteristics 0 in section 1, and about partitions in section 2, I begin studying irreducible characters of S_n. Even if in sections 3 and 4 I consider representations both in characteristic 0 and in positive characteristics, starting from section 5 I will only be considering ordinary irreducible characters of S_n, that is irreducible characters of S_n over $\mathbb{C}$.

In section 5 I show that the ordinary irreducible representation of S_n are labeled by the partitions of n in such a way that if $[\alpha]$ is the irreducible representation corresponding to α, where α is a partition of n, I have that $[\alpha]$ is the only common irreducible component of $\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha)$ and $\mathrm{Ind}_{S_{\alpha'}}^{S_n}(AS_{\alpha'})$, where α' is the partition associated to α, for any $H \subset S_n$, IH is the identity representation of H and AH is the sign representation of H and for any β partition of n, S_β is a Young subgroup corresponding to β, that is a subgroup of S_n isomorphic to $S_{\beta_1} \times S_{\beta_2} \times \cdots$. I also show that $[\alpha]$ appears only once in any decomposition of $\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha)$ and $\mathrm{Ind}_{S_{\alpha'}}^{S_n}(AS_{\alpha'})$ in irreducible representations and that $[\alpha]$ is also a representation of S_n over $\mathbb{Q}$.

In section 6 I find a method to compute the irreducible characters of S_n, even if this method isn't really useful when n is large, as it would require a really large number of calculations.

In section 7 I prove the determinantal form, i.e. that $[\alpha] = |[\alpha_i + j - i]|$, where

$$[n_1] \cdots [n_k] = \mathrm{Ind}_{S_{n_1} \times \cdots \times S_{n_k}}^{S_{n_1+\ldots+n_k}} (I(S_{n_1} \times \cdots \times S_{n_k}))$$

when all $n_i \geq 0$ and is 0 otherwise, which gives an easier method to find the values of the irreducible characters of S_n on all conjugacy classes, as thanks to this formula I then only need to find the characters of the representations $[\alpha_1 + \pi(1) - 1] \cdots [\alpha_k + \pi(k) - k]$ for $\pi \in S_k$, for some big enough k (k needs to be such that $\alpha_j = 0$ for $j > k$), which are either induced characters from the identity representation of some Young subgroup or they are 0.

This is used in section 8 to prove first the branching rule (one part of which is a particular case of the Murnaghan-Nakayama formula) and then the Murnaghan-Nakayama formula, which says that

$$\chi^\alpha(\pi) = \sum_{(i,j)\in\alpha : h_{i,j}^\alpha = k} (-1)^{l_{i,j}^\alpha} \chi^{\alpha \setminus R_{i,j}^\alpha}(\rho)$$

where for any partition β, χ^β is the character of the irreducible representation labeled by β, and π, ρ and k are such that $\pi \in S_n$, $\rho \in S_{n-k}$ and the cycle partition of ρ is obtained by the cycle partition of π by removing a part equal to k (in particular I need to have that π has at least one k-cycle).

The proof of the Murnaghan-Nakayama formula I present in sections from 3 to 8 follows a book from James and Kerber ([2]).

I then prove in section 9 some results on cores, quotients and β-sets of a partition, which, together with the Murnaghan-Nakayama formula are used to prove in section 10 a formulas for $\chi^\alpha(\pi)$ in the case where the q-weight of π is at least as big as that of α, where q is any positive integer. I focus mainly on the case where $w_q(\pi) = w_q(\alpha)$ (as if $w_q(\pi) > w_q(\alpha)$ I have that $\chi^\alpha(\pi) = 0$) and in this case if $\pi = \rho\sigma$, where ρ consists cycles of lengths $q\lambda$, where $\lambda \vdash w_q(\pi)$ and σ and ρ acts on distinct sets of elements, I prove that

$$\chi^\alpha(\pi) = \delta_q(\alpha) f_\lambda^{\alpha^{(q)}} \chi^{\alpha_{(q)}}(\sigma)$$

where $\delta_q(\alpha)$, $\alpha^{(q)}$ and $\alpha_{(q)}$ are the q-sign, the q-quotient and the q-core of α respectively and $f_\lambda^{\alpha^{(q)}}$ depends on $\alpha^{(q)}$ and λ only and I find a formula for $f_\lambda^{\alpha^{(q)}}$. This section generalizes formula 2.7.25 from the book of James and Kerber [2], in which ρ consists of cycles all of length q.

In section 11 I present a proof of the hook-formula, which gives the degree of an irreducible representation of S_n and which says that if f^α is the degree of $[\alpha]$ then

$$f^\alpha = \frac{n!}{\prod_{(i,j)\in\alpha} h_{i,j}^\alpha}.$$

The proof of the hook-formula which is presented is taken from some lecture notes written by Olsson ([5]).

The Murnaghan-Nakayama and the hook formulas are then used in section 12 to find informations about p-vanishing conjugacy classes of S_n, that is those conjugacy classes of S_n which are 0 on all irreducible characters of S_n of degree divisible by p, where p is a prime. In particular if I let $n = a_0 + a_1 p + \ldots + a_k p^k$ be the p-adic decomposition of n, I prove that when $w_{p^i}(\pi) = a_i + a_{i+1}p + \ldots + a_k p^{k-i}$ for all $0 \leq i \leq k$, that is when π is of p-adic type, then π is p-vanishing. This part of this section is based on results by Malle, Navarro and Olsson, which may be found in section 4 of [3]. Afterward I prove some new results trying to prove that p-vanishing elements are of p-adic type. This is not true for $p = 2, 3$, even if in this I completely classify p-vanishing elements and show that they are really close to be of p-adic type. In the case where $p \neq 2, 3$ I have a conjecture that p-vanishing elements are exactly elements of p-adic type. Even if I haven't been able to completely prove this conjecture I have been able to prove different results which support it. The work in this last part has been originated by a question of Navarro about which conjugacy classes of S_n are 2-vanishing.

Finally in section 13 I study p-vanishing classes of S_n, that is conjugacy classes of S_n which always take value 0,1 or -1 on all irreducible characters. Some results from this section are taken from an article from Olsson ([6]) or are generalizations of results from this article, while I have proved other results in this section myself.

Through all of my thesis $\mathbb{N}$ always contains 0.

1 Basics on Group Representation Theory

In this section we want to give an overview about results on representations of finite groups. The results from this section have been taken from [8], where proofs of these results can be also found. Let V be a vector space over a field K and let G be a group.

Definition 1 (Representation)**.** *A* representation *of G over V is a homomorphism*

$$\rho : G \to \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ *is the group of automorphisms of V.*

If $\mathrm{Dim}_K(V) = n$ we say that ρ has *degree n*.

A basic example of representation is given by the *identity representation*, which has degree 1 and for which $\rho(g) = \mathrm{id}$ for all $g \in G$.

An other example of representation is the *regular representation*, which has degree $|G|$. Let V be the vector space with basis $\{e_g : g \in G\}$. The regular representation is given by extending

$$\rho(g)(e_h) = e_{gh}$$

by linearity.

A representation is called *irreducible* if $V \neq 0$ and no proper subspace of V is stable under G.

If ρ_V and ρ_W are two representations of G over V and W respectively we can define the direct sum of ρ_V and ρ_W by

$$\begin{array}{rccc} \rho_V \oplus \rho_W : & G & \to & \mathrm{GL}(V \oplus W) \\ & g & \mapsto & \left(\begin{array}{cc} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{array} \right). \end{array}$$

It is easy to see that $\rho_V \oplus \rho_W$ is also a representation of G.

Let now ρ be a representation of G over V and ϕ be a representation of H over W. We can define the tensor product of ρ and ϕ

$$\rho \otimes \phi : G \times H \to \mathrm{GL}(V \otimes W),$$

by

$$\rho \otimes \phi(g,h)(v \otimes w) = \rho(g)(v) \otimes \phi(h)(w), \forall g \in G, h \in H, v \in V, w \in W.$$

It can be seen that $\rho \otimes \phi$ is a representation of $G \times H$ (see section 3.2 of [8]).

In all of the following we will assume that V is finite dimensional.

Definition 2 (Character of a representation)**.** *Let ρ be a representation. The character of ρ is given by*

$$\chi(g) = \mathrm{Tr}(\rho(g)), \qquad g \in G.$$

A character is called irreducible if it is the character of an irreducible representation. Also it is clear by the properties of the trace that isomorphic representations have the same character.

We will now assume that K has characteristic 0 and is algebraic closed and that G is finite. We then have that (theorem 2 and proposition 1 of [8])

Theorem 1. *Every representation is a direct sum of irreducible representations.*

Theorem 2. *If χ is the character of a representation of degree n we have that*

i) $\chi(1) = n$,

ii) $\chi(g^{-1}) = \overline{\chi(g)}$ *for* $g \in G$,

iii) $\chi(hgh^{-1}) = \chi(g)$ *for* $g, h \in G$.

Even if K doesn't need to be contained in $\mathbb{C}$, $\chi(g) \in \mathbb{C}$ for any $g \in G$, as it is the sum of the eigenvalues of $\rho(g)$ (χ is the character of ρ) and if λ_i is an eigenvalue for g then $\lambda_i^{|G|}$ is an eigenvalue for $g^{|G|} = 1$ and so $\lambda_i^{|G|} = 1$ and then $\lambda_i \in \mathbb{Q}^{al} \subset \mathbb{C}$, where $\mathbb{Q}^{al}$ is the algebraic closure of $\mathbb{Q}$.

The last condition in the theorem says that any character is constant over conjugacy classes.

We will now define a bilinear form on the set of characters of a group. Let χ, ψ be characters of G. Then (χ, ψ) is defined by

$$(\chi, \psi) = 1/|G| \sum_{g \in G} \chi(g)\psi(g^{-1}) = 1/|G| \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

It easy to see from the definition that $(\chi, \psi) = (\psi, \chi)$ for any two characters χ and ψ of G.

It can be seen that (theorem 3 of [8])

Theorem 3 (Character relations of the first kind)**.** *i) If χ is an irreducible character we have that $(\chi, \chi) = 1$.*

ii) If χ and ψ are irreducible characters of non-equivalent representations we have that $(\chi, \psi) = 0$.

From this theorem it follows easily that (theorem 4 of [8])

Corollary 4. *Let ρ and ϕ be representation and assume that ϕ is irreducible. Let χ and ψ be the characters of ρ and ϕ respectively. If $\rho = \oplus_i \rho_i$, where ρ_i are irreducible representations, we have that (χ, ψ) is equal to the number of ρ_i which are equivalent to ϕ.*

Corollary 5. *If $\rho_1 = \sum_i m_i\phi_i$ and $\rho_2 = \sum_i n_i\phi_i$ are representations of G, where the ϕ_i are pairwise non-equivalent irreducible representations of G, and χ_j is the character of ρ_j we have that*

$$(\chi_1, \chi_2) = \sum_i m_i n_i.$$

This last corollary follows easily from the previous one and from the fact that from the definitions of the direct sum of two representations and of the character of a representation we have that the character of $\rho \oplus \phi$ is the sum of the character of ρ and the character of ϕ.

In particular we have that (theorem 5 of [8])

Theorem 6. *A representation with character χ is irreducible if and only if $(\chi, \chi) = 1$.*

We can also define

Definition 3 (Intertwining number)**.** *Let ρ and ϕ be representations of G over V and W respectively. The intertwining number of ρ and ϕ is defined by*

$$i(\rho, \phi) = \mathrm{D}im_K \left(\mathrm{Hom}_G(V, W)\right).$$

From lemma 2 of [8] we have that that the intertwining number $i(\rho, \phi)$ is equal to (χ, ψ), where χ is the character of ρ and ψ is the character of ϕ, when the characteristic of the field K is 0, but the intertwining number can also be defined when the characteristic of K is different from 0.

By this formula it can be easily shown that if ρ is an irreducible representation of G and ϕ is an irreducible representation of H then $\rho \otimes \phi$ is an irreducible representation of $G \times H$, as if χ is the character of ρ and ψ is the character of ϕ then the character of $\rho \otimes \phi$ at (g, h) is given by $\chi(g)\psi(h)$.

In the following we will write $\chi \otimes \psi$ for the character of $\rho \otimes \phi$. Actually it can be proved that all irreducible representations of $G \times H$ are of this form (theorem 10 of [8]).

In particular if ρ is the regular representation and χ is its character we have by proposition 5 of [8] that $\chi(1) = |G|$ and $\chi(g) = 0$ if $g \neq 1$, so that if ϕ is any irreducible representation, ψ is its character and n is the degree of ϕ, then $(\chi, \psi) = n$, so we have corollary 1 to proposition 5 of [8]

Theorem 7. *Any irreducible representation is contained in the regular representation with multiplicity equal to its degree.*

From this last theorem we have in particular that all the irreducible representations appear in (any) decomposition of the regular representation in irreducible representations.

Also it can be proved that (theorem 7 of [8])

Theorem 8. *The number of distinct irreducible characters of a group G is equal to the number of conjugacy classes of G.*

Let now H be a subgroup of G. We will now show how we can construct representations of H by representations of G and vice-versa.

The following theorem (proposition 7 of [8]) shows how characters tables of two conjugacy classes of G relates

Theorem 9. *[Characters relations of the second kind] If $g, h \in G$, C_g and C_h are the conjugacy classes of g and h in G and $\chi_1, \ldots, \chi_n$ are all the irreducible characters of G, we have that*

$$\sum_i \chi_i(g)\overline{\chi_i(h)} = \frac{|G|}{|C_g|}\delta_{C_g,C_h}.$$

Definition 4 (Restriction of a representation)**.** *Let ρ be a representation of G over V. The restriction of ρ to H is defined by*

$$\begin{array}{rccc} \operatorname{Res}^G_H(\rho): & H & \to & \mathrm{GL}(V) \\ & h & \mapsto & \rho(h). \end{array}$$

It is easy to see that if χ is the character of a representation ρ of G and $\operatorname{Res}^G_H(\chi)$ is the character of $\operatorname{Res}^G_H(\rho)$, then $\chi(h) = \operatorname{Res}^G_H(\chi)(h)$ for any $h \in H$ and that ρ and $\operatorname{Res}^G_H(\rho)$ have the same degree.

We will now describe how it is possible to obtain a representation of G from a representation of H. Let ϕ be a representation of H over W. For any $\sigma \in G/H$ let W_σ be a copy of W. Set

$$V = \bigoplus_{\sigma \in G/H} W_\sigma.$$

Then if any element of V can be written like $\sum_{\sigma \in G/H} w_\sigma$, where each $w_\sigma \in W_\sigma$, in a unique way. Also let S be a system of representatives of the left cosets of H in G. Let $g \in G$. Then we can write $g = hs$ for some $h \in H$ and $s \in S$ in a unique way. Define for any $\sigma \in G/H$ and any $w_\sigma \in W_\sigma$

$$\rho(g)(w_\sigma) = \phi(h)(w_\sigma) \in W_{s\sigma}.$$

It can be shown that extending ρ by linearity to V does actually define a representation which doesn't depend on S (up to equivalence). For a proof of this see section 3.3 of [8].

Definition 5 (Induced representation)**.** *ρ as it has just been defined is called the* induced representation *of ϕ and is denoted by $Ind_H^G(\phi)$.*

It is easy to see by the definition that if the degree of ϕ is n, then the degree of $Ind_H^G(\phi)$ is equal to $n \cdot |G/H|$.

Also it can be proved that (theorem 12 of [8])

Theorem 10. *Let ψ be the character of ϕ and $Ind_H^G(\psi)$ be the character of $Ind_H^G(\phi)$. Let S be a system of representatives of left cosets of H in G. Then for any $g \in G$*

$$Ind_H^G(\psi)(g) = \sum_{\substack{s \in S \\ s^{-1}gs \in H}} \psi(s^{-1}gs) = \frac{1}{|H|} \sum_{\substack{r \in G \\ r^{-1}gr \in H}} \psi(r^{-1}gr).$$

If ρ and ϕ are two representations of G over V and W respectively we can define a representation of G over $V \otimes W$ by extending by linearity $\rho \otimes \phi(g)(v \otimes w) = \rho(g)(v) \otimes \phi(g)(w)$. It can be shown that $\rho \otimes \phi$ is actually a representation and that if χ is the character of ρ and ψ is the character of ϕ we have that the character of $\rho \otimes \phi$ is given by $\chi\psi$.

Theorem 11. *If ρ is a representation of G and ϕ is a representation of H, where H is a subgroup of G we have that*

$$\rho \otimes \mathrm{Ind}_H^G(\phi) = Ind_H^G(\mathrm{Res}_H^G(\rho) \otimes \phi).$$

The last theorem can be proved by showing that the two representations have the same character (remark (3) to theorem 13 of [8]).

If ρ is a representation of H, where H is a subgroup of G and $g \in G$ it is easy to see that $\rho^{(g)}(ghg^{-1}) := \rho(h)$ defines a representation on ${}^gH = gHg^{-1}$.

Theorem 12 (Frobenius' reciprocity law)**.** *If ρ is a representation of G and ϕ is a representation of H, which is a subgroup of G we have that*

$$i(\rho, \mathrm{Ind}_H^G(\phi)) = i(\mathrm{Res}_H^G(\rho), \phi)$$

and so if χ and ψ are the characters of ρ and ϕ respectively we have that

$$i(\chi, \mathrm{Ind}_H^G(\psi)) = i(\mathrm{Res}_H^G(\chi), \psi).$$

Theorem 13 (Mackey's subgroup theorem)**.** *Let H and K be subgroups of G and let ρ be a representation of H. If S is a set of representatives of the double cosets KgH in G we have that*

$$\mathrm{Res}_K^G(\mathrm{Ind}_H^G(\rho)) = \sum_{s\in S} \mathrm{Ind}_{K\cap {}^sH}^K(\mathrm{Res}_{K\cap {}^sH}^{{}^sH}(\rho^{(s)})).$$

Theorem 14 (Mackey's intertwining number theorem)**.** *Let H and K be subgroup of G and ρ and ϕ be representations of H and K respectively. If S is a set of representatives of the double cosets HgK in G we have that*

$$i(\mathrm{Ind}_H^G(\rho), \mathrm{Ind}_K^G(\phi)) = \sum_{s\in S} i(\mathrm{Res}_{H\cap {}^sK}^H(\rho), \mathrm{Res}_{H\cap {}^sK}^{{}^sK}(\phi^{(s)})).$$

For a proof of the theorems 12 and 13 see theorem 13 and proposition 22 of [8] respectively. For the proof of theorem 14 in the case where $H = K$ and $\rho = \phi$ see proposition 23 of [8]. The proof in the general case can be obtained similarly.

2 Some definitions about partitions

In this section we will give some basics definitions about partitions.

Definition 6 (Partition)**.** *A sequence of non-negative integers*

$$\alpha = (\alpha_1, \alpha_2, \ldots)$$

is a partition *of n if*

$$\begin{array}{rl} \text{i)} & \alpha_i \geq \alpha_{i+1} \quad \forall i \geq 1, \\ \text{ii)} & \sum_i \alpha_i = n. \end{array}$$

If α is a partition of n we write $\alpha \vdash n$. Also α_i are called the parts of α. As $\alpha_i \in \mathbb{N}$ for all i and $\sum_i \alpha_i$ converges if α is a partition we have that if α is a partition we can find $h \in \mathbb{N}$ such that $\alpha_i = 0$ if $i > h$. For such an h we can write

$$\alpha = (\alpha_1, \ldots, \alpha_h).$$

In order to get uniqueness in writing $\alpha = (\alpha_1, \ldots, \alpha_h)$ we can choose h to be minimal, that is choose h such that $\alpha_h \neq 0$ and $\alpha_i = 0$ for all $i > h$. As α is a partition we have that with this choice of h, $\alpha_i \neq 0$ for any $i \leq h$.

For example the partitions of 5 are given by

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).$$

For $n \in \mathbb{N}$, $p(n)$ is the number of partitions of n.

If α is a partition of n and, for $i = 1, \cdots, n$, we define a_i to be the number of parts of α equal to i (all parts of α are $\leq n$ as $\alpha \vdash n$), we can also write

$$\alpha = \left(n^{a_n}, \cdots, 1^{a_1}\right).$$

Usually i^{a_i} is left out if $a_i = 0$.

In this notation the partitions of 5 are given by

$$\left(5^1\right), \left(4^1, 1^1\right), \left(3^1, 2^1\right), \left(3^1, 1^2\right), \left(2^2, 1^1\right), \left(2^1, 1^3\right), \left(1^5\right).$$

Sometimes a mix of these two ways to write a partition is used, especially when we are only focusing on how many parts equal to one or more certain numbers a partition has. For example if $\alpha_i = \alpha_{i+1} = \ldots = \alpha_{i+s-1}$ but $\alpha_{i-1}, \alpha_{i+s} \neq \alpha_i$ we could write

$$(\alpha_1, \ldots, \alpha_h) = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i^s, \alpha_{i+s}, \ldots, \alpha_h).$$

Let α be a partition of n.

Definition 7 (Young diagram)**.** *The* Young diagram *for α consists of an array where each the i-th row contains α_i nodes and such that the rows are left-justified.*

In the following we will indicate the Young diagram of α still by α.

For example the Young diagram of the partition $(3,2)$ is given by

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \end{array}$$

If α is a partition we can define a new partition α' which is called the associated partition of α in the following way

Definition 8 (Associated partition)**.** *Let α be a partition. For any i let α'_i be the number of parts of α which are bigger or equal to i. $\alpha' = (\alpha'_1, \alpha'_2, \ldots)$ is the* partition associated with *α.*

For example $(3,2)' = (2,2,1)$.

Looking at the Young diagram of α it can be easily sen that the Young diagram of α' is obtained by reflecting the Young diagram of α across the diagonal, from which it easily follows that if α is a partition of n then also

α' is a partition of n and that $(\alpha')' = \alpha$. If $\alpha' = \alpha$ we say that α is a *self associated* partition.

If α is a Young diagram and (i, j) is such that $j \leq \alpha(i)$ we call the j-th node on the i-th row of α the (i, j)-node of α.

We will now give some definitions about hooks of a partition.

Definition 9 (Hook)**.** *If (i, j) is a node of α we denote by $H^{\alpha}_{i,j}$ the (i, j)-hook of α, that is the set of nodes of α of the form (i, j') for some $j' \geq j$ or (i', j) for some $i' \geq i$.*

Definition 10 (Hook-length)**.** *Let (i, j) be a node of α. The* hook-length *$h^{\alpha}_{i,j}$ of (i, j) is equal to the number of nodes in $H^{\alpha}_{i,j}$.*

Definition 11 (Leg-length)**.** *Let (i, j) be a node of α. The* leg-length *$l^{\alpha}_{i,j}$ of (i, j) is equal to the number of nodes of α of the form (i', j), $i' > i$.*

Definition 12 (Arm-length)**.** *Let (i, j) be a node of α. The* arm-length *$a^{\alpha}_{i,j}$ of (i, j) is equal to the number of nodes of α of the form (i, j'), $j' > j$.*

It follows easily from the definition that $a^{\alpha}_{i,j} = \alpha_i - j$ and that $l^{\alpha}_{i,j} = \alpha'_j - i$, where α' is the partition associated to α.

Definition 13 (Rim of α)**.** *Let α be a Young diagram. The* rim *of α, R^{α}, is the set of nodes (i, j) of α such that $(i + 1, j + 1)$ is not in α.*

Definition 14. *If (i, j) is a node of α, $R^{\alpha}_{i,j}$ is the set of nodes of the rims of the form (i', j') with $i' \geq i$ and $j' \geq j$.*

It can be easily seen that $|R^{\alpha}_{i,j}| = h^{\alpha}_{i,j}$ and that $R^{\alpha}_{i,j}$ consists of the partition of the rim between (i, α_i) and (α'_j, j) (see lemma 1.1 of [4]).

Definition 15 (Improper partition)**.** *A sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ is called an* improper partition *of n if*

i) $\lambda_i \in \mathbb{N}\ \forall i \geq 1$,

ii) $\sum_i \lambda_i = n$.

If λ is an improper partition of n we write $\lambda \models n$.

Also in the case of an improper partition if $\lambda_i = 0$ for all $i > h$ we can write $\lambda = (\lambda_1, \dots, \lambda_h)$.

Let now $\pi \in S_n$. We can write

$$\pi = \Pi_{j=1}^{c(\pi)} \left(c_j, \pi(c_j), \dots, \pi^{\alpha_j(\pi)-1}(c_j)\right)$$

where the $(c_j, \pi(c_j), \dots, \pi^{\alpha_j(\pi)-1}(c_j))$ are disjoint cycles, each $i \in \{1, \dots, n\}$ appears in exactly one of those cycles and the $\alpha_j(\pi)$ are non-increasing. Also let $a_i(\pi)$ be the number of the $\alpha_j(\pi)$ which are equal to i.

Definition 16.

$$\alpha(\pi) = (\alpha_1(\pi), \dots, \alpha_{c(\pi)}(\pi))$$

is called the cycle partition *of* π *and*

$$a(\pi) = (a_1(\pi), \dots, a_n(\pi))$$

is called the cycle type *of* π.

Even if the cycles $\left(c_j, \dots, \pi^{\alpha_j(\pi)-1}(c_j)\right)$ which appear in the previous decomposition of π are not uniquely determined (we can start the j-th cycle with $\pi(c_j)$ instead of c_j or we can switch the i-th and the j-th cycle if they have the same length), it is easy to see that $c(\pi)$ and the $\alpha_j(\pi)$ are uniquely defined, so that $\alpha(\pi)$ is well defined for any partition $\pi \in S_n$ and so the same is true also for $a(\pi)$. By definition $a(\pi)$ is known if if know $\alpha(\pi)$. Also as $\pi \in S_n$, so that $\alpha(\pi) \vdash n$ and so all parts of are $\leq n$ we have that $\alpha(\pi) = (n^{a_n(\pi)}, \dots, 1^{a_1(\pi)})$ and so we can find $\alpha(\pi)$ if we know $a(\pi)$, from which we have that knowing $\alpha(\pi)$ is equivalent to knowing $a(\pi)$.

By the definitions of $\alpha(\pi)$ and $a(\pi)$ for any $\pi \in S_n$ and as two elements of S_n are conjugate if and only if they have the same cycle partition, we have that $\sigma, \rho \in S_n$ are conjugate if and only if $\alpha(\sigma) = \alpha(\rho)$ if and only if $a(\sigma) = a(\rho)$ (lemma 1.2.6 of [2]).

3 Young Subgroups of S_n

This section is based on section 1.3 of [2].

Let $\mathbf{n} = \{1, \dots, n\}$ and let λ be an improper partition of n. Let $\mathbf{n}_i^\lambda$ be pairwise disjoint subsets of $\mathbf{n}$ such that $|\mathbf{n}_i^\lambda| = \lambda_i$ and let S_i^λ be the subgroup of S_n consisting of the elements which fixes all the elements of $\mathbf{n} \setminus \mathbf{n}_i^\lambda$, that is S_i^λ is the symmetric group over $\mathbf{n}_i^\lambda$.

Definition 17 (Young subgroup)**.**

$$S_\lambda = S_1^\lambda \times S_2^\lambda \times \cdots$$

is called the Young subgroup *corresponding to* $\mathbf{n}^\lambda = (\mathbf{n}_1^\lambda, \mathbf{n}_2^\lambda, \dots)$.

It is easy to see that $S_\lambda \cong S_{\lambda_1} \times S_{\lambda_2} \times \cdots$ and that this is actually a finite product as $S_{\lambda_i} = \{1\}$ whenever $\lambda_i = 0$.

If we write λ as a diagram where in each row there are λ_i position filled with the numbers in $\mathbf{n}_i^\lambda$ (that is if we consider a λ-tableau), then S_λ is the subgroup of S_n which fixes the rows of the diagram.

If H is any subgroup of S_n there are two trivial representations of H of degree 1, which then need to be irreducible. These two representations might be equal, and in fact it is easy to see that they are the same if and only if $H \subset A_n$ when we are in characteristics $\neq 2$ and that they are always the same if we are in characteristics 2. The first one is the identity representation of H, which will be denoted by IH. The second one is the alternating representation of H, denoted by AH, that is the representation of H over V, where V is a one dimensional vector space, given by

$$\begin{array}{rccc} AH: & S_\lambda & \to & \mathrm{GL}(V) \\ & \pi & \mapsto & \mathrm{sign}(\pi)\cdot \mathrm{id}_V. \end{array}$$

We will now show how we can calculate $i\left(\mathrm{Ind}_{S\lambda}^{S_n}(IS\lambda), \mathrm{Ind}_{S_\mu}^{S_n}(IS_\mu)\right)$ and $i\left(\mathrm{Ind}_{S_\lambda}^{S_n}(IS\lambda), \mathrm{Ind}_{S_\mu}^{S_n}(AS_\mu)\right)$ where λ and μ are improper partitions of n.

Let $\{\pi\}$ be representatives of the double cosets $S_\lambda \rho S_\mu$ in S_n. Using Mackey's intertwining number, theorem 14, we have that

$$\begin{array}{rcl} i\left(\mathrm{Ind}_{S_\lambda}^{S_n}(IS_\lambda), \mathrm{Ind}_{S_\mu}^{S_n}(IS_\mu)\right) & = & \sum_\pi i\left(\mathrm{Res}_{S_\lambda\cap{}^\pi S_\mu}^{S_\lambda}(IS_\lambda), \mathrm{Res}_{S_\lambda\cap{}^\pi S_\mu}^{{}^\pi S_\mu}\left(IS_\mu^{(\pi)}\right)\right) \\ & = & \sum_\pi i(I(S_\lambda\cap{}^\pi S_\mu), I(S_\lambda\cap{}^\pi S_\mu)) \\ & = & \sum_\pi 1 \end{array}$$

as $I(S_\lambda\cap{}^\pi S_\mu)$ is an irreducible representation. So $i(\mathrm{Ind}_{S_\lambda}^{S_n}(IS_\lambda), \mathrm{Ind}_{S_\mu}^{S_n}(IS_n))$ is equal to the number of double cosets $S_\lambda\rho S_\mu$ in S_n. Also we have that

$$\begin{array}{rcl} i\left(\mathrm{Ind}_{S_\lambda}^{S_n}(IS_\lambda), \mathrm{Ind}_{S_\mu}^{S_n}(AS_n)\right) & = & \sum_\pi i\left(\mathrm{Res}_{S_\lambda\cap{}^\pi S_\mu}^{S_\lambda}(IS_\lambda), \mathrm{Res}_{S_\lambda\cap{}^\pi S_\mu}^{{}^\pi S_\mu}\left(AS_\mu^{(\pi)}\right)\right) \\ & = & \sum_\pi i(I(S_\lambda\cap{}^\pi S_\mu), A(S_\lambda\cap{}^\pi S_\mu)). \end{array}$$

As $I(S_\lambda\cap{}^\pi S_\mu)$ and $A(S_\lambda\cap{}^\pi S_\mu)$ are both irreducible representations we have that

$$i(I(S_\lambda\cap{}^\pi S_\mu), A(S_\lambda\cap{}^\pi S_\mu)) = \begin{cases} 1 & I(S_\lambda\cap{}^\pi S_\mu) = A(S_\lambda\cap{}^\pi S_\mu) \\ 0 & I(S_\lambda\cap{}^\pi S_\mu) \neq A(S_\lambda\cap{}^\pi S_\mu) \end{cases}$$

By the previous considerations we have that $I(S_\lambda\cap{}^\pi S_\mu) = A(S_\lambda\cap{}^\pi S_\mu)$ if and only if $S_\lambda\cap{}^\pi S_\mu \subset A_n$ when the characteristics of the field we are working with is $\neq 2$ (in particular when it is equal to 0). By definition of S_λ and S_μ we have that

$$S_\lambda\cap{}^\pi S_\mu = (\Pi_i S_{\mathbf{n}_i^\lambda}) \cap \pi(\Pi_j S_{\mathbf{n}_j^\mu})\pi^{-1} = \Pi_{i,j} S_{\mathbf{n}_i^\lambda \cap \pi(\mathbf{n}_j^\mu)} \qquad (1)$$

and so it is easy to see that $S_\lambda\cap{}^\pi S_\mu \subset A_n$ if and only if $S_\lambda\cap{}^\pi S_\mu = 1$, that is, if we are in characteristics $\neq 2$, $i\left(\mathrm{Ind}_{S_\lambda}^{S_n}(IS_\lambda), \mathrm{Ind}_{S_\mu}^{S_n}(AS_\mu)\right)$ is equal to the number of double cosets $S_\lambda \pi S_\mu$ for which $S_\lambda\cap{}^\pi S_\mu = 1$.

We will now give some methods for finding the number of double cosets $S_\lambda \rho S_\mu$ and the number of such cosets satisfying $S_\lambda \cap {}^\pi S_\mu = 1$.

This lemma is lemma 1.3.8 of [2].

Lemma 15. *$\rho \in S_\lambda \pi S_\mu$ if and only if $\left|\mathbf{n}_i^\lambda \cap \pi \mathbf{n}_k^\mu\right| = \left|\mathbf{n}_i^\lambda \cap \rho \mathbf{n}_k^\mu\right|$ for all i and k.*

Proof. $\Rightarrow$ Assume that $\rho \in S_\lambda \pi S_\mu$. Then $\rho = \sigma\pi\tau$ for some $\sigma \in S_\lambda$ and $\tau \in S_\mu$. Then we have

$$\mathbf{n}_i^\lambda \cap \rho \mathbf{n}_k^\mu = \mathbf{n}_i^\lambda \cap \sigma\pi\tau \mathbf{n}_k^\mu = \sigma \mathbf{n}_i^\lambda \cap \sigma\pi \mathbf{n}_k^\mu = \sigma(\mathbf{n}_i^\lambda \cap \pi \mathbf{n}_k^\mu)$$

for any i and k as σ fixes each set $\mathbf{n}_i^\lambda$ and τ fixes each set $\mathbf{n}_k^\mu$ by definition of S_λ and S_μ. In particular we have that $\left|\mathbf{n}_i^\lambda \cap \pi \mathbf{n}_k^\mu\right| = \left|\mathbf{n}_i^\lambda \cap \rho \mathbf{n}_k^\mu\right|$ for all i and k.

$\Leftarrow$ As for each i and k, $\mathbf{n}_i^\lambda \cap \pi \mathbf{n}_k^\mu$ and $\mathbf{n}_i^\lambda \cap \rho \mathbf{n}_k^\mu$ have the same number of elements and they are both contained in $\mathbf{n}_i^\lambda$, we can find $\sigma_i \in S_i^\lambda$ for each i such that $\sigma_i \left(\mathbf{n}_i^\lambda \cap \pi \mathbf{n}_k^\mu\right) = \mathbf{n}_i^\lambda \cap \rho \mathbf{n}_k^\mu$ for every k. As $\mathbf{n}_i^\lambda = \emptyset$ for all i such that $\lambda_i = 0$ and there are only finitely many j such that $\lambda_j \neq 0$, we have that only finitely many σ_i are different from 1, and so we can define $\sigma = \sigma_1\sigma_2\cdots$. It is easy to see that $\sigma\left(\mathbf{n}_i^\lambda \cap \pi \mathbf{n}_k^\mu\right) = \sigma_i\left(\mathbf{n}_i^\lambda \cap \pi \mathbf{n}_k^\mu\right)$ for all i and k and that by definition of S_λ we have that $\sigma \in S_\lambda$. So we have that for each i and k

$$\sigma\left(\mathbf{n}_i^\lambda \cap \pi \mathbf{n}_k^\mu\right) = \mathbf{n}_i^\lambda \cap \sigma\pi \mathbf{n}_k^\mu = \mathbf{n}_i^\lambda \cap \rho \mathbf{n}_k^\mu.$$

Taking the union over i we obtain $\sigma\pi\mathbf{n}_k^\mu = \rho\mathbf{n}_k^\mu$ and so by definition of S_μ we can find $\tau \in S_\mu$ such that $\sigma\pi\tau = \rho$. As $\sigma \in S_\lambda$ and $\tau \in S_\mu$ we then have that $\rho \in S_\lambda \pi S_\mu$. $\square$

The next theorem is theorem 1.3.10 of [2].

Theorem 16. *Let λ and μ be improper partitions of n and S_λ and S_μ be the corresponding Young subgroups. Then the map*

$$f : S_\lambda \pi S_\mu \mapsto \left(z_{i,k} := \left|\mathbf{n}_i^\lambda \cap \pi \mathbf{n}_k^\mu\right|\right)$$

is a bijection between the set of double cosets of S_λ and S_μ in S_n and the set of infinite matrices $(z_{i,k})$ over $\mathbb{N}$ satisfying

$$\sum_k z_{i,k} = \lambda_i, \qquad \sum_i z_{i,j} = \mu_k.$$

Even if in the theorem we are considering infinite matrices as the coefficients are non-negative we have that the i-th row (the k-th column) is 0 when λ_i (μ_k) is 0, so that if $\lambda_i = 0$ for $i > h$ and $\mu_k = 0$ for $k > l$ then the number of infinite matrices over $\mathbb{N}$ satisfying the conditions in the theorem is the same as the number of $h \times l$ matrices over $\mathbb{N}$ satisfying the same conditions, so that we only need to consider big enough finite matrices and not infinite matrices when we want to find the number of double cosets $S_\lambda \pi S_\mu$ in S_n.

Proof. We know by lemma 15 that f is well defined (it does not depend on the choice of the element of the double coset $S_\lambda \pi S_\mu$) and that f is injective. So we only need to show that it is surjective. Let $(z_{i,k})$ satisfy the properties in the theorem. As $\sum_i z_{i,k} = \mu_k$ for all k we can find a dissection $\mathbf{n}^\mu_{i,k}$ of $\mathbf{n}^\mu_k$ such that $\left|\mathbf{n}^\mu_{i,k}\right| = z_{i,k}$ for each i,k (that is $\mathbf{n}^\mu_k = \cup_i \mathbf{n}^\mu_{i,k}$ is a disjoint union and $\left|\mathbf{n}^\mu_{i,k}\right| = z_{i,k}$). Similarly for each i we can find a dissection $\mathbf{n}^\lambda_{i,k}$ of $\mathbf{n}^\lambda_i$ such that $\left|\mathbf{n}^\lambda_{i,k}\right| = z_{i,k}$. As $\left|\mathbf{n}^\lambda_{i,k}\right| = \left|\mathbf{n}^\mu_{i,k}\right|$ for each i and k we can find $\pi \in S_n$ such that $\pi\left(\mathbf{n}^\mu_{i,k}\right) = \mathbf{n}^\lambda_{i,k}$ for all i,k. It is easy to see that for such a π we have that $f(S_\lambda \pi S_\mu) = (z_{i,k})$ and so we have that f is surjective and then that f defines a bijection from the set of double cosets of S_λ and S_μ in S_n to the set of infinite matrices $(z_{i,k})$ over $\mathbb{N}$ satisfying $\sum_k z_{i,k} = \lambda_i$ and $\sum_i z_{i,j} = \mu_k$. $\square$

In particular we have the following

Corollary 17. *If λ and μ are improper partitions, $i\left(\mathrm{Ind}^{S_n}_{S_\lambda}(IS_\lambda), \mathrm{Ind}^{S_n}_{S_\mu}(IS_\mu)\right)$ is equal to the number of infinite matrices $(z_{i,k})$ over $\mathbb{N}$ with row sums λ_i and column sums μ_k.*

We will now show how to find the number of double cosets $S_\lambda \pi S_\mu$ satisfying $S_\lambda \cap {}^\pi S_\mu = 1$.

Lemma 18. $S_\lambda \cap {}^\pi S_\mu = 1$ *if and only if* $\left|\mathbf{n}^\lambda_i \cap \pi \mathbf{n}^\mu_k\right| = 0, 1$ *for all* i, k.

Proof. By equation (1) we have that $S_\lambda \cap {}^\pi S_\mu = \Pi_{i,k} S_{\mathbf{n}^\lambda_i \cap \pi(\mathbf{n}^\mu_k)}$ from which we have that $|S_\lambda \cap {}^\pi S_\mu| = \Pi_{i,k}\left|\mathbf{n}^\lambda_i \cap \pi(\mathbf{n}^\mu_k)\right|!$. In particular $|S_\lambda \cap {}^\pi S_\mu| = 1$ if and only if all the $\left|\mathbf{n}^\lambda_i \cap \pi(\mathbf{n}^\mu_k)\right|$ are either 0 or 1. $\square$

Form this lemma and theorem 16 we have the following corollary (corollary 1.3.13 of [2])

Corollary 19. *The number of double cosets $S_\lambda \pi S_\mu$ satisfying $S_\lambda \cap {}^\pi S_\mu = 1$ is equal to the number of infinite matrices with coefficients 0 and 1 with row sums λ_i and column sums μ_k.*

and so

Corollary 20. *If we are working in characteristics $\neq 2$ and λ and μ are improper partitions we have that $i\left(\mathrm{Ind}_{S_\lambda}^{S_n}(IS_\lambda), \mathrm{Ind}_{S_\mu}^{S_n}(AS_\mu)\right)$ is equal to the number of infinite matrices with coefficients 0 and 1 with row sums λ_i and column sums μ_k.*

Also in this case even if in the corollaries we are considering infinite matrices we could just consider some big enough finite matrix.

4 The Dominance Order

This section is based on section 1.4 of [2].

We know want to find for which partitions α, β of n we have that

$$i(Ind_{S_\alpha}^{S_n}(IS_\alpha), Ind_{S_\beta}^{S_n}(AS_\beta)) \neq 0.$$

To do this we will first define a partial order on the set of partitions of n.

Definition 18 (Dominance order)**.** *Let α and β be two partitions of n. We say that $\alpha \trianglelefteq \beta$ if for all i we have that*

$$\sum_{j=1}^{i} \alpha_j \leq \sum_{j=1}^{i} \beta_j.$$

The order defined by $\trianglelefteq$ on the set of partitions of n is called the dominance order*.*

It is easy to see that if $\alpha \trianglelefteq \beta$ then $\alpha \leq \beta$ where $\leq$ is the *lexicographic order*, that is we can find i such that $\alpha_j = \beta_j$ for $j < i$ and $\alpha_i < \beta_i$ (unless $\alpha = \beta$).

The next theorem is one direction of the Gale and Ryser's theorem (theorem 1.4.17 of [2]).

Theorem 21. *Let α and β be partitions of n. If there exists 0-1 matrices with rows sums α_i and column sums β'_k, where β' is the partition associated to β, then $\alpha \trianglelefteq \beta$.*

Proof. Assume that such a matrix $(z_{i,k})$ exists. Let $d_{j,h}$ be the number of columns for which $\sum_{i=1}^{h} z_{i,k} = j$ (that is $d_{j,h}$ is the number of columns of $(z_{i,h})$ which contain exactly j 1's in the first h rows) and let e_j be the number of parts of β' which are equal to j. Then for any h we have that

$$\sum_{j=1}^{h} \alpha_j = \sum_{j=1}^{h} j \cdot d_{j,h} \leq \sum_{j=1}^{h-1} j \cdot e_j + h \sum_{m=h}^{\infty} e_m = \sum_{j=1}^{h} \beta_j$$

as if the first h rows of column k of $(z_{i,k})$ contain exactly j 1's then $\beta'_k \geq j$, and so $\alpha \trianglelefteq \beta$. $\square$

From this theorem and corollary 20 we have the following, which is one direction of Ruch and Schönofer's theorem (theorem 1.4.18 of [2])

Theorem 22. *If α and β are partitions of n and S_α and $S_{\beta'}$ are Young subgroups for the partitions α and β' respectively and the characteristic of the ground-field is not 2, we have that if $i\left(\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha), \mathrm{Ind}_{S_{\beta'}}^{S_n}(AS_{\beta'})\right) \neq 0$ then $\alpha \trianglelefteq \beta$.*

5 The Ordinary Irreducible Representations of S_n

This section is based on section 2.1 of [2].

We will now be considering representations of S_n over $\mathbb{C}$ (the ordinary representations of S_n) and show that any irreducible representation over $\mathbb{C}$ can actually be realized over $\mathbb{Q}$. In order to do this we will first show that $i\left(Ind_{S_\alpha}^{S_n}(IS_\alpha), Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})\right) = 1$ for any α partition of n.

By section 3 we have that $i\left(Ind_{S_\alpha}^{S_n}(IS_\alpha), Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})\right)$ is equal to the number of 0-1 matrices with row sums α_i and column sums α'_k. In any such matrix $(z_{i,k})$ we need to have α_1 1's in the first row and as α' has exactly α_1 non-zero parts we can only fill up the first row in a unique way, that is by having $z_{1,k}$ equal to 1 if $k \leq \alpha_1$ and equal to 0 if $k > \alpha_1$. Assume that we have now filled up until row $i-1$ so that for any $j < i$ we have that $z_{j,k} = 1$ if $k \leq \alpha_j$ and $z_{j,k} = 0$ if $k > \alpha_j$. If now $k > \alpha_i$ we have that the number of 1's in the kth column in the first $i-1$ rows is equal to the number of α_j which are bigger or equal to k (as the parts of α are non-increasing), that is $\sum_{j=1}^{i-1} z_{j,k} = \alpha'_k$, and so as we also need to have that $\sum_{j\geq 1} z_{j,k} = \alpha'_k$ and all $z_{j,k} \in \{0,1\}$, we need to have that $z_{i,k} = 0$ for all $k > \alpha_i$. If now $k \leq \alpha_i$ we have that $z_{j,k} = 1$ for all $j < i$ and so $\sum_{j=1}^{i-1} z_{j,k} = i-1 < \alpha'_k$ and so we can set $z_{i,k}$ to be equal to 1 in this case. As we have exactly α_i columns for which $z_{i,k}$ can be equal to 1, $\sum_k z_{i,k}$ needs to be α_i and all $z_{i,k} = 0, 1$, we need to have that $z_{i,k} = 1$ if $k \leq \alpha_i$. So we have that the only 0-1 matrix with row sums α_i and column sums α'_k is given by $(z_{i,k})$ where

$$z_{i,k} = \begin{cases} 1 & k \leq \alpha_i \\ 0 & k > \alpha_i. \end{cases}$$

In particular we have that $i\left(Ind_{S_\alpha}^{S_n}(IS_\alpha), Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})\right) = 1$ for any $\alpha \vdash n$. As both IS_α and $AS_{\alpha'}$ can be defined over $\mathbb{Q}$ we have that the same is true for $Ind_{S_\alpha}^{S_n}(IS_\alpha)$ and $Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})$ and as

$$i\left(Ind_{S_\alpha}^{S_n}(IS_\alpha), Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})\right) = 1$$

we have that $Ind_{S_\alpha}^{S_n}(IS_\alpha)$ and $Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})$ contain a unique common representation which is irreducible over $\mathbb{Q}$, which appears exactly once in each one of them. Let $[\alpha]$ be this representation. This representation satisfies

$$1 \leq i([\alpha],[\alpha]) \leq i\left(Ind_{S_\alpha}^{S_n}(IS_\alpha), Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})\right) = 1$$

and so

$$i([\alpha],[\alpha]) = 1$$

and then we have that $[\alpha]$ is irreducible also over $\mathbb{C}$ and $[\alpha]$ needs to be the only irreducible representation over $\mathbb{C}$ which appears in both $Ind_{S_\alpha}^{S_n}(IS_\alpha)$ and $Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})$. So we have that the following theorem (theorem 2.1.3 of [2]) is true

Theorem 23. *If α is a partition on n and S_α and $S_{\alpha'}$ are Young subgroups corresponding to α and α' respectively we have that $Ind_{S_\alpha}^{S_n}(IS_\alpha)$ and $Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})$ have exactly one ordinary irreducible component $[\alpha]$ in common. We also have that $[\alpha]$ appears in each one of $Ind_{S_\alpha}^{S_n}(IS_\alpha)$ and $Ind_{S_{\alpha'}}^{S_n}(AS_{\alpha'})$ only once and that $[\alpha]$ can be realized over $\mathbb{Q}$.*

For example if we let $\alpha = (1^n)$ we have that $\alpha' = (n)$ and we have that $S_\alpha = 1$ and $S_{\alpha'} = S_n$ and so we have that

$$\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha) = \mathrm{Ind}_{S_\alpha}^{S_n}(AS_\alpha) = \mathrm{Ind}_{S_\alpha}^{S_n}(RS_\alpha) = RS_n$$

where RG denotes the regular representation on any group G, as IS_α, AS_α and RS_α are actually the same representation as $S_\alpha = 1$ and as $\mathrm{Ind}_H^G(RH) = RG$ for any $H \subset G$ (example 1 in section 3.3 of [8]), and that

$$\mathrm{Ind}_{S_{\alpha'}}^{S_n}(IS_{\alpha'}) = IS_n, \qquad \mathrm{Ind}_{S_{\alpha'}^{S_n}}(AS_{\alpha'}) = AS_n$$

as $S_{\alpha'} = S_n$. So as IS_n and AS_n are irreducible and they appear in the decomposition of RS_n into irreducible representations we have that

$$[(1^n)] = AS_n \qquad \text{and} \qquad [(n)] = IS_n.$$

The fact that $[(1^n)] = AS_n = AS_n \otimes IS_n = AS_n \otimes [(n)]$ is not casual. If now α is any partition of n we have by theorem 11 that

$$\mathrm{Ind}_{S_\alpha}^{S_n}(AS_\alpha) = \mathrm{Ind}_{S_\alpha}^{S_n}(AS_\alpha \otimes IS_\alpha) = AS_n \otimes \mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha).$$

Also as AS_n has degree 1 we have that any representation ρ is irreducible if and only if $AS_n \otimes \rho$ is irreducible (it can be easily seen by considering the characters). Then we have that $AS_n \otimes [\alpha]$ is irreducible and appears in the decomposition of both $\mathrm{Ind}_{S_{\alpha'}}^{S_n}(IS_{\alpha'})$ and $\mathrm{Ind}_{S_\alpha}^{S_n}(AS_\alpha)$ (as if $\rho = \oplus\rho_i$ is a decomposition of any representation ρ then $AS_n \otimes \rho = \oplus(AS_n \otimes \rho_i)$ is a decomposition of $AS_n \otimes \rho$) and so $AS_n \otimes [\alpha]$ needs to be the unique irreducible representation which appears in both $\mathrm{Ind}_{S_{\alpha'}}^{S_n}(IS_{\alpha'})$ and $\mathrm{Ind}_{S_\alpha}^{S_n}(AS_\alpha)$, and then for any $\alpha \vdash n$ we have that

$$[\alpha'] = AS_n \otimes [\alpha].$$

We now want to show that $\{[\alpha]\}$ is the complete set of ordinary irreducible representations of S_n. The next lemma is lemma 2.1.10 of [2].

Lemma 24. *If* $i\left(\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha), [\beta]\right) \neq 0$ *then* $\alpha \trianglelefteq \beta$.

As $[\beta]$ is an irreducible representation the lemma is saying that if $[\beta]$ appears in the decomposition of $\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha)$ in irreducible representations then $\alpha \trianglelefteq \beta$.

Proof. As $[\beta]$ is a subrepresentation of $\mathrm{Ind}_{S_{\beta'}}^{S_n}(AS_{\beta'})$ we have that whenever $i\left(\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha), [\beta]\right) \neq 0$ then also $i\left(\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha), \mathrm{Ind}_{S_{\beta'}}^{S_n}(AS_{\beta'})\right)$ is non-zero and so by theorem 22 we have that $\alpha \trianglelefteq \beta$. $\square$

The following theorem is theorem 2.1.11 of [2].

Theorem 25. $\{[\alpha] : \alpha \vdash n\}$ *is a complete set of equivalence classes of the ordinary irreducible representations of* S_n.

Proof. We will first prove that if $[\alpha] = [\beta]$ then $\alpha = \beta$. Assume that $[\alpha] = [\beta]$ (here equality means that $[\alpha]$ and $[\beta]$ are equivalent as representations of S_n). Then by definition of $[\alpha]$ and $[\beta]$ we have that

$$i\left(\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha), [\beta]\right) = i\left(\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha), [\alpha]\right) = 1$$

and

$$i\left(\mathrm{Ind}_{S_\beta}^{S_n}(IS_\beta), [\alpha]\right) = i\left(\mathrm{Ind}_{S_\beta}^{S_n}(IS_\beta), [\beta]\right) = 1.$$

Applying the previous lemma we then have that $\alpha \trianglelefteq \beta$ and $\beta \trianglelefteq \alpha$ and so we need to have that $\alpha = \beta$, as by definition of the dominance order we then get by induction that $\alpha_i = \beta_i$ for each i.

So we have that the representations $[\alpha]$, $\alpha \vdash n$, are pairwise non-equivalent, and so as the number of conjugacy classes of S_n and so from theorem 8 we have that the number of equivalence classes of irreducible representations of S_n is equal to the number of partitions of n, and so we have that $\{[\alpha] : \alpha \vdash n\}$ is a complete set of equivalence classes of the ordinary irreducible representations of S_n. □

6 The representations $\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha)$

This section follows section 2.2 of [2].

We have seen that the regular irreducible representations of S_n are exactly the representations $\{[\alpha] : \alpha \vdash n\}$. We still don't know however the representations $[\alpha]$ or their characters. We will now show how they can be found.

Let $p(n)$ be the number of partitions of n. We can put the partitions of n in order, so that

$$(1^n) = \alpha^1 < \alpha^2 < \ldots < \alpha^{p(n)} = (n)$$

where $<$ is given by the lexicographic order. We can now define a matrix

$$M_n := (m_{i,k}) = i\left(\mathrm{Ind}_{S_{\alpha^i}}^{S_n}(IS_{\alpha^i}), \left[\alpha^k\right]\right).$$

As whenever $\alpha \trianglelefteq \beta$ we also have that $\alpha \leq \beta$ we have by lemma 24 that if $i > k$ (that is if $\alpha^i > \alpha^k$) then $m_{i,k} = 0$. Also by definition of $[\alpha^i]$ we have that $m_{i,i} = 1$ for any i and so we have that M_n is an upper triangular matrix with 1's on the diagonal. If α and β are partitions of n, let ζ^α and ξ^α denote the characters of $[\alpha]$ and $\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha)$ respectively and ζ^α_β and ξ^α_β their value on the conjugacy class of S_n with cycle partition β. By theorem 25 and by definition of $m_{i,k}$ we then have that

$$\xi^{\alpha^i} = \sum_k m_{i,k}\zeta^{\alpha^k}.$$

If we now define two new matrices Z_n and Ξ_n by

$$Z_n := \left(\zeta^{\alpha^i}_{\alpha^k}\right), \qquad \Xi_n := \left(\xi^{\alpha^i}_{\alpha^k}\right)$$

we can easily seen that

$$\Xi_n = M_n Z_n.$$

As M_n is a matrix over $\mathbb{Z}$ and has determinant equal to 1 (as it is upper diagonal and all the diagonal entries are equal to 1) we have that M_n is invertible and the inverse is also a matrix over $\mathbb{Z}$. In particular we have that the characters of $[\alpha^i]$ can be obtained as a linear combination with coefficients in $\mathbb{Z}$ of the characters of $\mathrm{Ind}_{S_{\alpha^k}}^{S_n}(IS_{\alpha^k})$.

In particular we have the following theorem (theorem 2.2.10 of [2])

Theorem 26. *The ring* $\mathrm{char}(S_n) = \oplus\mathbb{Z}\zeta^{\alpha^i}$ *of generalized ordinary characters of* S_n *has also* $\{\xi^\alpha : \alpha \vdash n\}$ *as a* $\mathbb{Z}$*-basis.*

Even if we have that $Z_n = M_n^{-1}\Xi_n$ we still don't know the coefficients $m_{i,k}$, as these are defined in terms of $[\alpha^k]$ which we still don't know, and so we haven't found yet the characters of the irreducible representations of S_n. Though if we consider the scalar product of the i-th and the j-th rows of M_n we have by definition of $m_{i,k}$ and corollary 5 and the note after definition 3 that

$$\sum_k m_{i,k}m_{j,k} = \sum_k \left(\xi^{\alpha^i}, \zeta^{\alpha^k}\right) \cdot \left(\xi^{\alpha^j}, \zeta^{\alpha^k}\right) = \left(\xi^{\alpha^i}, \xi^{\alpha^j}\right).$$

So we have a way to evaluate $\sum_k m_{i,k}m_{j,k}$ as we can find the characters ξ^{α^k} by theorem 10. As we also know that M_n is upper triangular and has 1's on the diagonal this can be used to actually find M_n, as we already know the last row and as if we know the last $p(n) - i$ rows (M_n has size $p(n) \times p(n)$) we can find the coefficients $m_{i,k}$, $k > i$ inductively starting with $k = p(n)$ by

$$m_{i,k} = \left(\xi^{\alpha^i}, \xi^{\alpha^k}\right) - \sum_{j>k} m_{i,j}m_{k,j}.$$

We will now give an other method to find $\xi^\alpha(\pi)$ for any $\pi \in S_n$ and any $\alpha \vdash n$.

Definition 19 (Tableau)**.** *We say that t is a* tableau *of shape α, where α is a partition of n, if t is obtained by the Young diagram of α by placing the numbers from 1 to n in the nodes of α, so that each of these numbers appears exactly once in t.*

Definition 20 (Tabloid)**.** *A* tabloid *of shape α is an equivalence class of tableaux of shape α consisting of all the tableaux which contains the same elements on each row.*

For example if $\alpha = (3, 1)$ we have that

$$\begin{matrix} 1 & 2 & 3 \\ 4 & & \end{matrix}$$

is a tableau of shape α and the corresponding tabloid is given by

$$\left\{ \begin{matrix} 1 & 2 & 3 \\ 4 & & \end{matrix}, \begin{matrix} 1 & 3 & 2 \\ 4 & & \end{matrix}, \begin{matrix} 2 & 1 & 3 \\ 4 & & \end{matrix}, \begin{matrix} 2 & 3 & 1 \\ 4 & & \end{matrix}, \begin{matrix} 3 & 1 & 2 \\ 4 & & \end{matrix}, \begin{matrix} 3 & 2 & 1 \\ 4 & & \end{matrix} \right\}.$$

If t is a tableau we write $\{t\}$ for the tabloid containing t.

If t is a tableau let $\mathbf{n}_i$ be the set of elements which are in row i. Let $H(t) = S_{\mathbf{n}_1} \times S_{\mathbf{n}_2} \times \cdots$ be the row stabilizer of t (that is any element of $H(t)$ move any number on the i-th row of t to some number still on the i-th row of t for any i). Then we have that t and t' are in the same tabloid if and only if $t' = \pi t$ for some $\pi \in H(t)$. Also it clear that if t has shape α then $H(t)$ is a Young subgroup for α and any Young subgroup for α can be obtained this way. So if S_α is any Young subgroup for α we have that laterals of S_α correspond bijectively to the tabloids of shape α and that the action of S_n on the laterals of S_α is equivalent to the action of S_n on the α-tabloids, and so it can be seen that $\mathrm{Ind}_{S_\alpha}^{S_n}(IS_\alpha)$ is equivalent to the representation ρ of S_n on $V = \oplus_{\{t\}} K e_{\{t\}}$, where K is the ground-field, and ρ is given by extending by linearity $\rho(\pi)\left(e_{\{t\}}\right) = e_{\{\pi t\}}$. In particular it is easy to see (as ρ is a permutation representation) that we have that

$$\xi^\alpha(\pi) = \sum_{\{t\} : \{t\} = \{\pi t\}} 1,$$

that is $\xi^\alpha(\pi)$ is equal to the number of α-tabloids fixed by π.

7 The Ordinary Irreducible Characters as $\mathbb{Z}$-linear Combinations of Permutations Characters

This section follows section 2.3 of [2].

Even if in the previous section we found a method to compute the characters of the representations $[\alpha]$, the method we found requires many calculations, so we would like to find an easier one. This is what we will do in this section. If $\alpha^{(i)} \vdash n_i$ we have that $\left[\alpha^{(i)}\right]$ are representations of S_{n_i} and so we have that

$$\left[\alpha^{(1)}\right] \otimes \cdots \otimes \left[\alpha^{(k)}\right]$$

is a representation of

$$S_{n_1} \times \cdots \times S_{n_k} \subset S_{n_1+\ldots+n_k}.$$

Let

$$\left[\alpha^{(1)}\right] \cdots \left[\alpha^{(k)}\right] = \mathrm{Ind}_{S_{n_1}\times\cdots\times S_{n_k}}^{S_{n_1+\ldots+n_k}} \left(\left[\alpha^{(1)}\right] \otimes \cdots \otimes \left[\alpha^{(k)}\right]\right).$$

If $\alpha^{(i)} = (n_i)$ we know by section 5 that $\left[\alpha^{(i)}\right] = IS_{n_i}$ and so if we let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $S_\alpha = S_{n_1} \times \cdots \times S_{n_k}$ (S_α under the identification of $S_{n_1} \times \cdots \times S_{n_k}$ as a subgroup of $S_{n_1+\ldots+n_k}$ is actually a Young subgroup for α) we have that

$$[(n_1)] \otimes \cdots \otimes [(n_k)] = IS_{n_1} \otimes \cdots \otimes IS_{n_k} = IS_\alpha$$

and so

$$[(n_1)] \cdots [(n_k)] = \mathrm{Ind}_{S_\alpha}^{S_{n_1+\ldots+n_k}}(IS_\alpha).$$

Let now $\alpha = (\alpha_1, \ldots, \alpha_h)$ be a partition of n. If we let $[a] = 0$ whenever $a < 0$, $[0] = 1$ and $[a] = [(a)]$ if $a > 0$ we have that

$$|[\alpha_i + j - i]| = \begin{vmatrix} [\alpha_1] & [\alpha_1 + 1] & \ldots & [\alpha_1 + h - 1] \\ [\alpha_2 - 1] & [\alpha_2] & \ldots & [\alpha_2 + h - 2] \\ \vdots & \vdots & & \vdots \\ [\alpha_h + 1 - h] & [\alpha_h + 2 - h] & \ldots & [\alpha_h] \end{vmatrix} \tag{2}$$

is a generalized representation of S_n as any term of the determinant is of the form

$$\mathrm{sign}(\pi) \prod_i [\alpha_i + \pi(i) - i]$$

with $\pi \in S_h$ and as we have that $\prod[\alpha_i + \pi(i) - i]$ is either 0 if some term $\alpha_i + \pi(i) - i < 0$ or else it is a representation of S_n as if all $\alpha_i + \pi(i) - i$ are non-negative we have that $(\alpha_1 + \pi(1) - 1, \ldots, \alpha_h + \pi(h) - h)$ is an improper partition of

$$\sum_{i=1}^{h} \alpha_i + \pi(i) - i = \sum_{i=1}^{h} \alpha_i + \sum_{i=1}^{h} \pi(i) - \sum_{i=1}^{h} i = \sum_{i=1}^{h} \alpha_i = n. \tag{3}$$

Letting $[a] = 0$ for $a < 0$ and $[0] = 1$ makes sure that the determinant in equation (2) is independent of the choice of h such that $\alpha_i = 0$ for $i > h$, as we have that for any such h

$$\left\{\begin{matrix} [\alpha_1] & [\alpha_1+1] & \dots & [\alpha_1+(h+1)-1] \\ [\alpha_2-1] & [\alpha_2] & \dots & [\alpha_2+(+1)h-2] \\ \vdots & \vdots & & \vdots \\ [\alpha_{h+1}+1-(h+1)] & [\alpha_{h+1}+2-(h+1)] & \dots & [\alpha_{h+1}] \end{matrix}\right\} = \left\{\begin{matrix} [\alpha_1] & [\alpha_1+1] & \dots & [\alpha_1+h-1] & [\alpha_1+(h+1)-1] \\ [\alpha_2-1] & [\alpha_2] & \dots & [\alpha_2+h-2] & [\alpha_2+(h+1)-2] \\ \vdots & \vdots & & \vdots & \vdots \\ [\alpha_h+1-h] & [\alpha_h+2-h] & \dots & [\alpha_h] & [\alpha_h+1] \\ 0 & 0 & \dots & 0 & 1 \end{matrix}\right\}$$

as $\alpha_{h+1}=0$ and so $\alpha_{h+1}+k-(h+1)<0$ for all $k<h+1$ and so we have that $|[\alpha_i+j-i]|$ depends only on α. We want to show that

$$[\alpha]=|[\alpha_i+j-i]|\,.$$

In order to do this we will also need to consider compositions of n, that is sequences $\lambda=(\lambda_1,\lambda_2,\dots)$ over $\mathbb{Z}$ such that $\sum_i\lambda_i=n$. The fact that the sum of the λ_i is n forces λ_i to be equal to 0 for i big enough. A composition of n is an improper partition of n if and only if all terms are non-negative. We can extend the definition of ξ^α, $\alpha\vdash n$, and define ξ^λ for any composition of n as follows

$$\xi^\lambda=\begin{cases}\mathrm{Ind}_{S_\lambda}^{S_n}(IS_\lambda) & \lambda\models n\\ 0 & \lambda\not\models n.\end{cases}$$

Let $S_\mathbb{N}$ be the group of permutations over $\mathbb{N}$, that is the group of bijections from $\mathbb{N}$ into itself fixing all but finitely many points (that is $S_\mathbb{N}$ is the union of S_n, $n\in\mathbb{N}$). As if $k\geq 0$ we have that $A_n=A_{n+k}\cap S_n$ we can define a sign function on $S_\mathbb{N}$ in an obvious way. If for $\pi\in S_\mathbb{N}$ and λ a composition of n we let

$$\lambda\circ\pi=\left(\lambda_{\pi^{-1}(1)},\lambda_{\pi^{-1}(2)},\dots\right)$$

we have that $\xi^\lambda=\xi^{\lambda\circ\pi}$, as if λ is not an improper partition of n then λ has some negative parts and so also $\lambda\circ\pi$ must have some negative part, while if $\lambda\models n$ then also $\lambda\circ\pi$ is an improper partition of n and as clearly any Young subgroup S_λ of λ is also a Young subgroup $S_{\lambda\circ\pi}$ of $\lambda\circ\pi$ we have that $\mathrm{Ind}_{S_\lambda}^{S_n}(IS_\lambda)=\mathrm{Ind}_{S_\lambda\circ\pi}^{S_n}(IS_\lambda\circ\pi)$ and so also in this case $\xi^\lambda=\xi^{\lambda\circ\pi}$. If λ is again any composition of n and $\pi\in S_\mathbb{N}$, we can also define

$$\lambda-\mathrm{id}+\pi=(\lambda_1-1+\pi(1),\lambda_2-2+\pi(2),\dots).$$

As $\pi(i)\neq i$ for only finitely many i we have that $(\lambda-\mathrm{id}+\pi)_i\neq\lambda_i$ only on finitely many i. Also as $\lambda_i\neq 0$ for only finitely many i we have that the same is true for $(\lambda-\mathrm{id}+\pi)_i$ from which we have that $\sum_i(\lambda-\mathrm{id}+\pi)_i$ is well defined and then similarly to equation (3) we have that $\sum_i(\lambda-\mathrm{id}+\pi)_i = n$, so that $\lambda-\mathrm{id}+\pi$ is a composition of n for any λ is a composition of n and $\pi \in S_{\mathbb{N}}$.

If λ and μ are compositions of n and k respectively we can define $\lambda-\mu$ by $(\lambda-\mu)_i = \lambda_i - \mu_i$. It can be easily seen that $\lambda-\mu$ is a composition of $n-k$.

Assume now that $\lambda_i = 0$ for $i > h$ and that $\pi \in S_{\mathbb{N}} \setminus S_h$, that is π does not fix all $k > h$. Let $\overline{k}$ be the biggest k which is not fixed by π ($\overline{k}$ exists as π fixes all but finitely many k's). Then we need to have that $\pi(\overline{k}) < \overline{k}$ and as $\overline{k} > h$ we have that $\lambda_{\overline{k}} = 0$ and so $\lambda_{\overline{k}} - \overline{k} + \pi\left(\overline{k}\right) < 0$. So $\lambda-\mathrm{id}+\pi$ isn't an improper partition of n if $\lambda_i = 0$ for all $i > h$ and $\pi \notin S_h$. In particular $\lambda-\mathrm{id}+\pi \models n$ for only finitely many π from which $\xi^{\lambda-\mathrm{id}+\pi} \neq 0$ only for finitely many π, so that it makes sense to define

$$\chi^\lambda := \sum_{\pi \in S_{\mathbb{N}}} \mathrm{sign}(\pi)\xi^{\lambda-\mathrm{id}+\pi}.$$

It is easy to see by the previous remark and by the definition of the determinant that if $\alpha \vdash n$ then χ^α is the character of $|[\alpha_i + j - i]|$.

The next lemmas are lemmas 2.3.9 and 2.3.10 of [2].

Lemma 27. *If λ is a composition of n and*

$$\mu = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_i + 1, \lambda_{i+2}, \ldots)$$

for some $i \in \mathbb{N}$, we have that

$$\chi^\mu = -\chi^\lambda.$$

Proof. Let $\tau = (i, i+1) \in S_{\mathbb{N}}$. Then for any $\pi \in S_{\mathbb{N}}$ we have that if $j \neq i, i+1$

$$(\mu-\mathrm{id}+\pi)_j = \mu_j - j + \pi(j) = \lambda_j - j + \pi(\tau(j)) = (\lambda-\mathrm{id}+\pi\tau)_j.$$

Also we have that

$$(\mu-\mathrm{id}+\pi)_i = \mu_i - i + \pi(i) = \lambda_{i+1} - 1 - i + \pi(\tau(i+1)) = (\lambda-\mathrm{id}+\pi\tau)_{i+1},$$

$$(\mu-\mathrm{id}+\pi)_{i+1} = \mu_{i+1} - (i+1) + \pi(i+1) = \lambda_i + 1 - i - 1 + \pi(\tau(i)) = (\lambda-\mathrm{id}+\pi\tau)_i.$$

That is we have that

$$\mu-\mathrm{id}+\pi = (\lambda-\mathrm{id}+\pi\tau) \circ \tau$$

and so

$$\xi^{\mu-\mathrm{id}+\pi} = \xi^{(\lambda-\mathrm{id}+\pi\tau)\circ\tau}.$$

From this we have that

$$\begin{aligned}\chi^{\lambda} &= \textstyle\sum_{\pi\in S_{\mathbb{N}}} \mathrm{sign}(\pi)\xi^{\mu-\mathrm{id}+\pi} \\ &= \mathrm{sign}(\tau)\textstyle\sum_{\pi\in S_{\mathbb{N}}} \mathrm{sign}(\pi\tau)\xi^{(\lambda-\mathrm{id}+\pi\tau)\circ\tau} \\ &= \mathrm{sign}(\tau)\textstyle\sum_{\pi\in S_{\mathbb{N}}} \mathrm{sign}(\pi\tau)\xi^{\lambda-\mathrm{id}+\pi\tau} \\ &= -\chi^{\lambda}.\end{aligned}$$

□

Lemma 28. *Let λ be a composition of $m+k$. Then we have that*

$$\mathrm{Res}^{S_{m+k}}_{S_m\times S_k}\left(\xi^{\lambda}\right) = \sum_{\mu\models k} \xi^{\lambda-\mu}\otimes\xi^{\mu}$$

and that

$$\mathrm{Res}^{S_{m+k}}_{S_m\times S_k}\left(\chi^{\lambda}\right) = \sum_{\mu\models k} \chi^{\lambda-\mu}\otimes\xi^{\mu}.$$

Proof. If $\lambda \not\models m+k$ and $\mu \models k$ then $\lambda-\mu \not\models m$ as we can then find i such that $\lambda_i < 0$ and so, as $\mu_i \geq 0$, we have that $(\lambda-\mu)_i < 0$. In particular if λ isn't a generalized partition of $m+k$ then $\xi^{\lambda-\mu} = 0$ for all μ generalized partitions of k and so

$$\mathrm{Res}^{S_{m+k}}_{S_m\times S_k}\left(\xi^{\lambda}\right) = \mathrm{Res}^{S_{m+k}}_{S_m\times S_k}(0) = 0 = \sum_{\mu\models k} 0\otimes\xi^{\mu} = \sum_{\mu\models k}\xi^{\lambda-\mu}\otimes\xi^{\mu}.$$

So assume now that $\lambda \models m+k$. Then $\xi^{\lambda} = Ind^{S_{m+k}}_{S_\lambda}(IS_\lambda)$ and so from Mackey's subgroup theorem (theorem 13) we have that

$$\mathrm{Res}^{S_{m+k}}_{S_m\times S_k}\left(\xi^{\lambda}\right) = \sum_{(S_m\times S_k)\pi S_\lambda} \mathrm{Ind}^{S_m\times S_k}_{(S_m\times S_k)\cap\pi S_\lambda\pi^{-1}}\left(I\left((S_m\times S_k)\cap\pi S_\lambda\pi^{-1}\right)\right).$$

As $S_m\times S_k$ is a Young subgroup of (m,k) we have by lemma 15 that the double cosets of $S_m\times S_k$ and S_λ in S_{m+k} are completely determined by knowing the $\left|\mathbf{m}\cap\pi\mathbf{n}^{\lambda}_j\right|$ and $\left|((\mathbf{m}+\mathbf{k})\setminus\mathbf{k})\cap\pi\mathbf{n}^{\lambda}_j\right|$. Also as by definition of $\mathbf{n}_i$ we have that $\left|\mathbf{m}\cap\pi\mathbf{n}^{\lambda}_j\right| + \left|((\mathbf{m}+\mathbf{k})\setminus\mathbf{k})\cap\pi\mathbf{n}^{\lambda}_j\right| = \lambda_j$ we have that the double cosets $(S_m\times S_k)\pi S_\lambda$ are completely determined by knowing the $\left|((\mathbf{m}+\mathbf{k})\setminus\mathbf{k})\cap\pi\mathbf{n}^{\lambda}_j\right|$. Let

$$\mu = \left(\left|((\mathbf{m}+\mathbf{k})\setminus\mathbf{k})\cap\pi\mathbf{n}^{\lambda}_1\right|, \left|((\mathbf{m}+\mathbf{k})\setminus\mathbf{k})\cap\pi\mathbf{n}^{\lambda}_2\right|, \ldots\right).$$

Then we have that $\mu \models k$ and as $\lambda_i - \mu_i \geq 0$ for all i we have that $\lambda - \mu \models m$. By equation (1) we have that

$$(S_m \times S_k) \cap \pi S_\lambda \pi^{-1} = S_{\lambda-\mu} \times S_\mu$$

for some Young subgroups $S_{\lambda-\mu}$ and S_μ of $\lambda - \mu$ and μ. So we have that

$$\begin{aligned}
\mathrm{Ind}_{(S_m \times S_k)\cap \pi S_\lambda \pi^{-1}}^{S_m \times S_k} \left(I\left((S_m \times S_k) \cap \pi S_\lambda \pi^{-1}\right)\right) &= \mathrm{Ind}_{S_{\lambda-\mu}\times S_\mu}^{S_m \times S_k}(I(S_{\lambda-\mu} \times S_\mu)) \\
&= \mathrm{Ind}_{S_{\lambda-\mu}\times S_\mu}^{S_m \times S_k}(IS_{\lambda-\mu} \otimes IS_\mu) \\
&= \mathrm{Ind}_{S_{\lambda-\mu}}^{S_m}(IS_{\lambda-\mu}) \otimes \mathrm{Ind}_{S_\mu}^{S_k}(IS_\mu).
\end{aligned}$$

Let now $\mu \models m+k$. If $\lambda - \mu \models m$, we can find a double coset $(S_m \times S_k)\pi S_\lambda$ for which $\mu = \left(\left|((\mathbf{m}+\mathbf{k}) \setminus \mathbf{k}) \cap \pi \mathbf{n}_1^\lambda\right|, \left|((\mathbf{m}+\mathbf{k}) \setminus \mathbf{k}) \cap \pi \mathbf{n}_2^\lambda\right|, \ldots\right)$ (just choose π that sends λ_i of the elements $\mathbf{n}_i^\lambda$ to elements of $\mathbf{m}$ and sending the other elements of $\mathbf{n}_i^\lambda$ to elements of $(\mathbf{m}+\mathbf{k}) \setminus \mathbf{k}$) and if $\lambda - \mu \not\models m$ then $\xi^{\lambda-\mu} \otimes \xi^\mu = 0 \otimes \xi^\mu = 0$ we have again by lemma 15 that

$$\mathrm{Res}_{S_m \times S_k}^{S_{m+k}} \left(\xi^\lambda\right) = \sum_{\mu \models k} \xi^{\lambda-\mu} \otimes \xi^\mu$$

and so the first formula is proved also for the case that $\lambda \models m+k$. For the second formula we now have that

$$\begin{aligned}
\mathrm{Res}_{S_m \times S_k}^{S_{m+k}} \left(\chi^\lambda\right) &= \textstyle\sum_{\pi \in S_\mathbb{N}} \mathrm{sign}(\pi) \xi^{\lambda - \mathrm{id} + \pi} \\
&= \textstyle\sum_{\pi \in S_\mathbb{N}} \mathrm{sign}(\pi) \sum_{\mu \models k} \xi^{\lambda - \mathrm{id} + \pi - \mu} \otimes \xi^\mu \\
&= \textstyle\sum_{\mu \models k} \left(\sum_{\pi \in S_\mathbb{N}} \mathrm{sign}(\pi) \xi^{\lambda - \mu - \mathrm{id} + \pi}\right) \otimes \xi^\mu \\
&= \textstyle\sum_{\mu \models k} \chi^{\lambda-\mu} \otimes \xi^\mu.
\end{aligned}$$

□

Let λ be a composition of n, μ a composition of k and $\pi \in S_\mathbb{N}$. Let h be such that $\lambda_i = \mu_i = 0$ and $\pi(i) = i$ for all $i > h$. Then

$$(\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi)_i = \lambda_i - i - (\mu_{\pi^{-1}(i)} - \pi^{-1}(i)) = \lambda_i - \mu_i = 0$$

for all $i > h$. As

$$\begin{aligned}
\textstyle\sum_i (\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi)_i &= \textstyle\sum_i (\lambda - \mathrm{id})_i - \sum_i ((\mu - \mathrm{id}) \circ \pi)_i \\
&= \textstyle\sum_i (\lambda - \mathrm{id})_i - \sum_j (\mu - \mathrm{id})_j \\
&= \textstyle\sum_i \lambda_i - \sum_j \mu_j \\
&= n - k
\end{aligned}$$

we have that $\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi$ is a composition of $n-k$ for any λ composition of n, μ composition of k and $\pi \in S_\mathbb{N}$.

Also let h be such that $\lambda_i = \mu_i = 0$ for all $i > h$ and let $-h' = \min\{\mu_i\}$. $-h' \leq 0$ as 0 is a part of μ. Assume that $\pi \notin S_{h+h'}$. Let $\overline{k}$ be the maximum of the elements not fixed by π. Then as $\overline{k} > h + h' \geq h$ we have that

$$(\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi)_{\overline{k}} = \lambda_{\overline{k}} - \overline{k} - \mu_{\pi^{-1}(\overline{k})} + \pi^{-1}(\overline{k}) = \pi^{-1}(\overline{k}) - \overline{k} - \mu_{\pi^{-1}(\overline{k})}.$$

If $\pi^{-1}(\overline{k}) > h$ we have that $\mu_{\pi^{-1}(\overline{k})} = 0$ and so $(\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi)_{\overline{k}} < 0$ by the maximality of $\overline{k}$. If $\pi^{-1}(\overline{k}) \leq h$ we have that $\pi^{-1}(\overline{k}) - \overline{k} < -h'$ as $\overline{k} > h + h'$ and so as $\mu_{\pi^{-1}(\overline{k})} \geq -h'$ we again have that $(\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi)_{\overline{k}} < 0$. So in any case we have that $\xi^{\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi} = 0$ if $\pi \notin S_{h+h'}$ and so $\xi^{\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi} = 0$ for all but finitely many $\pi \in S_{\mathbb{N}}$ and then we can define

$$\chi^{\lambda/\mu} := \sum_{\pi \in S_{\mathbb{N}}} \mathrm{sign}(\pi) \xi^{\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi}.$$

The next lemma is lemma 2.3.12 of [2].

Lemma 29. *If λ is a composition of $m + k$ we have that*

$$\mathrm{Res}^{S_{m+k}}_{S_m \times S_k} \left(\chi^{\lambda}\right) = \sum_{\beta \vdash k} \chi^{\lambda/\beta} \otimes \chi^{\beta}.$$

Proof. From the last part of the proof of lemma 28 and as for a given $\pi \in S_{\mathbb{N}}$ and for any n, $\mu \mapsto \mu \circ \pi$ gives a bijection between the set of improper partitions of n, we get that

$$\mathrm{Res}^{S_{m+k}}_{S_m \times S_k} \left(\chi^{\lambda}\right) = \sum_{\pi \in S_{\mathbb{N}}} \mathrm{sign}(\pi) \sum_{\mu \circ \pi \models k} \xi^{\lambda - \mathrm{id} + \pi - \mu \circ \pi} \otimes \xi^{\mu \circ \pi}.$$

As $\xi^{\mu \circ \pi} = \xi^{\mu}$ this means that

$$\mathrm{Res}^{S_{m+k}}_{S_m \times S_k} \left(\chi^{\lambda}\right) = \sum_{\pi \in S_{\mathbb{N}}} \mathrm{sign}(\pi) \sum_{\mu \models k} \xi^{\lambda - \mathrm{id} - (\mu - \mathrm{id}) \circ \pi} \otimes \xi^{\mu}. \tag{4}$$

As $\lambda_i \geq 0$ as $\lambda \models k$ we have that $\lambda_i - i \geq -i$ and so for any j there are at least j's i for which $(\lambda - \mathrm{id})_i \geq -j$. Let $\sigma \in S_{\mathbb{N}}$ be such that the parts of $(\lambda - \mathrm{id}) \circ \sigma^{-1}$ are non-increasing. σ exists as if $\lambda_i = 0$ for $i \geq h$ then $\lambda_i - i < \lambda_k - k$ for all $i > h$ and all $k < i$, so any bijection σ for which the parts of $(\lambda - \mathrm{id}) \circ \sigma^{-1}$ are non-increasing fixes all $i > h$, and so any such $\sigma \in S_h \subset S_{\mathbb{N}}$. Even if such a σ is not uniquely defined $(\lambda - \mathrm{id}) \circ \sigma^{-1}$ is uniquely defined, so also $\beta = (\lambda - \mathrm{id}) \circ \sigma^{-1} + \mathrm{id}$ is uniquely defined. So we have that

$$(\beta - \mathrm{id}) \circ \sigma = \lambda - \mathrm{id}.$$

Now it is easy to see that the number of σ which satisfy this properties is equal to $c_\beta = \prod_{j\in\mathbb{Z}} a_j$, where a_j is the number of parts of $\beta - \mathrm{id}$ which are equal to j. As we have already seen that only the first h parts of $\lambda - \mathrm{id}$ can be repeated (h is such that $\lambda_i = 0$ for $i > h$) and the parts of $\beta - \mathrm{id}$ are the same and with the same multiplicity of the parts of $\lambda - \mathrm{id}$ (they are just in a different order) we have that c_β is finite. Also as for any j, $\lambda - \mathrm{id}$ has at least j parts which are bigger or equal to $-j$ the same is true for $\beta - \mathrm{id}$, and so as $\beta - \mathrm{id}$ is obtained by $\lambda - \mathrm{id}$ by reordering the parts so that they are in non-increasing order we have that $(\beta - \mathrm{id})_j \geq -j$ for all j and so $\beta_j \geq 0$ for all j. Also as it is clear that

$$\sum_i \beta_i = \sum_i ((\lambda - \mathrm{id}) \circ \sigma^{-1} + \mathrm{id})_i = \sum_i \lambda_i = k$$

we have that β is also an improper partition of k. Also as if $(\beta - \mathrm{id}) \circ \sigma + \mathrm{id}$ isn't an improper partition of k we have that $\xi^{(\beta-\mathrm{id})\circ\sigma+\mathrm{id}} = 0$ (so that we may add terms corresponding to such decompositions of k in the summation in equation (4)) we have that

$$\begin{aligned}
\mathrm{Res}^{S_{m+k}}_{S_m\times S_k}\left(\chi^\lambda\right) &= \textstyle\sum_{\pi\in S_\mathbb{N}} \mathrm{sign}(\pi) \sum_{\sigma\in S_\mathbb{N}} \sum_\beta 1/c_\beta\, \xi^{\lambda-\mathrm{id}-(\beta-\mathrm{id})\circ\sigma\circ\pi} \otimes \xi^{(\beta-\mathrm{id})\circ\sigma+\mathrm{id}} \\
&= \textstyle\sum_{\pi,\sigma} \mathrm{sign}(\pi) \sum_\beta 1/c_\beta\, \xi^{\lambda-\mathrm{id}-(\beta-\mathrm{id})\circ\sigma\circ\pi} \otimes \xi^{(\beta-\mathrm{id})\circ\sigma+\mathrm{id}} \\
&= \textstyle\sum_{\pi,\sigma} \mathrm{sign}(\pi) \sum_\beta 1/c_\beta\, \xi^{\lambda-\mathrm{id}-(\beta-\mathrm{id})\circ\sigma\circ\pi} \otimes \xi^{\beta-\mathrm{id}+\sigma^{-1}} \\
&= \textstyle\sum_\beta 1/c_\beta \sum_{\rho,\theta} \left(\mathrm{sign}(\rho)\, \xi^{\lambda-\mathrm{id}-(\beta-\mathrm{id})\circ\rho}\right) \otimes \left(\mathrm{sign}(\theta)\xi^{\beta-\mathrm{id}+\theta}\right) \\
&= \textstyle\sum_\beta 1/c_\beta \left(\sum_\rho \mathrm{sign}(\rho)\, \xi^{\lambda-\mathrm{id}-(\beta-\mathrm{id})\circ\rho}\right) \otimes \left(\sum_\theta \mathrm{sign}(\theta)\xi^{\beta-\mathrm{id}+\theta}\right) \\
&= \textstyle\sum_\beta 1/c_\beta\, \chi^{\lambda/\beta} \otimes \chi^\beta.
\end{aligned}$$

where β varies between those $\beta \models n$ satisfying $\beta_i - i \geq \beta_{i+1} - i - 1$ for any i and where $\rho = \sigma\pi$ and $\theta = \sigma^{-1}$, so that $\sum_{\rho,\theta} = \sum_{\pi,\sigma}$. Also even if we are rearranging terms this doesn't change the result, as in all summations there are only finitely many non-zero terms.

Also assume that for some β that appears in the summation we have that for some i, $\beta_i - i = \beta_{i+1} - i - 1$. Then $\beta_i = \beta_{i+1} - 1$ and so by lemma 27 we have that in that case $\chi^\beta = -\chi^\beta$ and so $\chi^\beta = 0$. So actually the only β which appear in the summations are those of the form $\beta_i - i > \beta_{i+1} - i - 1$ for all i. For these β the parts of $\beta - \mathrm{id}$ are all distinct and so $c_\beta = 1$. Also for any such β we have that $\beta_i > \beta_{i+1} - 1$ and so $\beta_i \geq \beta_{i+1}$ and as $\beta \models k$ we then have that β is actually a partition of k. Putting all of this together we have that

$$\mathrm{Res}^{S_{m+k}}_{S_m\times S_k}\left(\chi^\lambda\right) = \sum_{\beta\vdash k} \chi^{\lambda/\beta} \otimes \chi^\beta$$

which is what we wanted to prove. $\square$

The next theorem is theorem 2.3.12 of [2].

Theorem 30. *Let $\alpha \vdash n$ and $\beta \vdash k$, where $k \leq n$. Then we have that*

i) if $\chi^{\alpha/\beta} \neq 0$ we have that $\alpha_i \geq \beta_i$ for all i,

ii) $\left(\chi^{\alpha/\beta}, \xi^{(n-k)}\right) = \begin{cases} 1 & \alpha_i \geq \beta_i \geq \alpha_{i+1} \quad \forall i \\ 0 & \textit{otherwise.} \end{cases}$

Proof. As α and β are partitions of n and $k \leq n$ respectively, we have by the remarks before lemma 29 that $\xi^{\alpha-\mathrm{id}-(\beta-\mathrm{id})\circ\pi} = 0$ if $\pi \notin S_n$ and it is so easy to see by the definition of $\chi^{\alpha/\beta}$ that

$$\chi^{\alpha/\beta} = |[\alpha_i - i - (\beta_j - j)]|\,.$$

Assume now that $\beta_h > \alpha_h$ for some h. As $\alpha_m - m$ and $\beta_m - m$ are strictly decreasing we have that then if $i \geq h$ and $j \leq h$ we have that $\alpha_i - i \leq \alpha_h - h < \beta_h - h \leq \beta_j - j$ and so $\alpha_i - i - (\beta_j - j) < 0$ and so $[\alpha_i - i - (\beta_j - j)] = 0$ if $i \geq h$ and $j \leq h$. So as there is a block of size $(n-h+1) \times h$ which contains only 0's in a matrix of size $n \times n$, we need to have that the determinant of the matrix is 0, so $\chi^{\alpha/\beta} = 0$ if there is some h for which $\beta_h > \alpha_h$ and so *i)* holds.

By section 5 we have that $\xi^{(n-k)} = IS_{n-k}$. Let now λ be any composition of $n-k$. If $\lambda \not\models n-k$ then $\xi^\lambda = 0$ and so $\left(\xi^\lambda, \xi^{(n-k)}\right) = 0$ when $\lambda \not\models n-k$. If instead $\lambda \models n-k$ we have by theorem 12 and by the symmetry of $i(\cdot,\cdot)$ that

$$\begin{aligned}\left(\xi^\lambda, \xi^{(n-k)}\right) &= i\left(\mathrm{Ind}_{S_\lambda}^{S_{n-k}}(IS_\lambda), IS_{n-k}\right) \\ &= i\left(IS_\lambda, \mathrm{Res}_{S_\lambda}^{S_{n-k}}(IS_{n-k})\right) \\ &= i(IS_\lambda, IS_\lambda) \\ &= 1\end{aligned}$$

and so $\left(\xi^\lambda, \xi^{(n-k)}\right) = 1$ if $\lambda \models n-k$. So we have that $\left(\xi^\lambda, \xi^{(n-k)}\right)$ is either equal to 1 or 0 depending on whether all parts of λ are non-negative or not.

By definition of $\chi^{\alpha/\beta}$ we now have that

$$\begin{aligned}\left(\chi^{\alpha/\beta}, \xi^{(n-k)}\right) &= \left(\textstyle\sum_{\pi\in S_{\mathbb{N}}} \mathrm{sign}(\pi)\xi^{\alpha-\mathrm{id}-(\beta-\mathrm{id})\circ\pi}, \xi^{(n-k)}\right) \\ &= \textstyle\sum_{\pi\in S_{\mathbb{N}}} \mathrm{sign}(\pi)\left(\xi^{\alpha-\mathrm{id}-(\beta-\mathrm{id})\circ\pi}, \xi^{(n-k)}\right) \\ &= \textstyle\sum_{\pi\in S_{\mathbb{N}}} \mathrm{sign}(\pi)\prod_i \delta_{(\alpha-\mathrm{id}-(\beta-\mathrm{id})\circ\pi)_i \geq 0}.\end{aligned}$$

As α and β are partitions of n and $k \leq n$ respectively we have that if $\pi \notin S_n$ then $\alpha - \mathrm{id} - (\beta - \mathrm{id}) \circ \pi \not\models n-k$, so that if $\left(\xi^{\alpha-\mathrm{id}-(\beta-\mathrm{id})\circ\pi}, \xi^{(n-k)}\right) \neq 0$ we need to have that $\pi \in S_n$. Also as if $i > n$, so that $\alpha_i = \beta_i = 0$ and $\pi \in S_n$ we have that $(\alpha - \mathrm{id} - (\beta - \mathrm{id}) \circ \pi)_i = \alpha_i - i - (\beta_i - i) = 0$ we have that

$$\left(\chi^{\alpha/\beta}, \xi^{(n-k)}\right) = \sum_{\pi \in S_n} \mathrm{sign}(\pi) \prod_{i=1}^{n} \delta_{(\alpha - \mathrm{id} - (\beta - \mathrm{id}) \circ \pi)_i \geq 0} = \left|\delta_{\alpha_i - i - \beta_j + j \geq 0}\right|$$

where in the matrix i and j vary between 1 and n. Also as $\alpha_i - i$ is strictly decreasing and $-\beta_j + j$ is strictly increasing we have that if $\delta_{\alpha_i - i - \beta_j + j \geq 0} = 1$ then $\delta_{\alpha_{i'} - i' - \beta_{j'} + j' \geq 0} = 1$ for all $i' \leq i$ and $j' \geq j$, while if $\delta_{\alpha_i - i - \beta_j + j \geq 0} = 0$ then $\delta_{\alpha_{i'} - i' - \beta_{j'} + j' \geq 0} = 0$ for all $i' \geq i$ and $j' \leq j$. So we have that two rows i and i' are equal and non-zero if and only if $\delta_{\alpha_i - i - \beta_j + j \geq 0} = 0$, $\delta_{\alpha_i - i - \beta_{j+1} + j + 1 \geq 0} = 1$, $\delta_{\alpha_{i'} - i' - \beta_j + j \geq 0} = 0$ and $\delta_{\alpha_{i'} - i' - \beta_{j+1} + j + 1 \geq 0} = 1$, for some $1 \leq j < n$, or if $\delta_{\alpha_i - i - \beta_1 + 1 \geq 0} = 1$ and $\delta_{\alpha_{i'} - i' - \beta_1 + 1 \geq 0} = 1$, as for each $i \leq n$ we have that $\alpha_i \geq \beta_n$ and so $\alpha_i - i - \beta_n + n = \alpha_i - \beta_n + n - i \geq 0$. So there are exactly n different possible non-zero rows and so each one of them must appear in $\{\delta_{\alpha_i - i - \beta_j + j \geq 0}\}$ if the determinant must be non-zero. Also by the remarks we just made we need to have that $\{\delta_{\alpha_i - i - \beta_j + j \geq 0}\}$ has 0's under the diagonal and 1's above and on the diagonal if it has non-zero determinant in which case the determinant is 1. So we have that $\left|\delta_{\alpha_i - i - \beta_j + j \geq 0}\right| = 1$ if $\delta_{\alpha_i - i - \beta_j + j \geq 0} = 1$ if and only if $j \geq i$ and $\left|\delta_{\alpha_i - i - \beta_j + j \geq 0}\right| = 0$ otherwise. Also $\delta_{\alpha_i - i - \beta_j + j \geq 0} = 1$ if and only if $j \geq i$ if and only if $\delta_{\alpha_i - i - \beta_i + i \geq 0} = 1$ for $1 \leq i \leq n$ and $\delta_{\alpha_{i+1} - i - 1 - \beta_i + i \geq 0} = 0$ for $1 \leq i < n$, that is if $\alpha_i \geq \beta_i$ for $i \leq n$ and $\beta_i > \alpha_{i+1} - 1$ (that is $\beta_i \geq \alpha_{i+1}$) for $i < n$. As when $j > n$ we have that $\alpha_j = \beta_j = 0$ we then have that $\left|\delta_{\alpha_i - i - \beta_j + j \geq 0}\right| = 1$ if and only if $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \ldots$ and the determinant is zero otherwise. So as

$$\left(\chi^{\alpha/\beta}, \xi^{(n-k)}\right) = \left|\delta_{\alpha_i - i - \beta_j + j \geq 0}\right|$$

we have that

$$\left(\chi^{\alpha/\beta}, \xi^{(n-k)}\right) = \begin{cases} 1 & \alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \ldots \\ 0 & \text{otherwise} \end{cases}$$

as we wanted to prove. □

We will now prove theorem 2.3.14 of [2].

Theorem 31 (Young's rule (First version)). *Let $\lambda \models n$ be an improper partition of n such that $\lambda_i = 0$ for $i > n$ and let $\alpha \vdash n$. Then we have that $\left(\chi^{\alpha}, \xi^{\lambda}\right)$ equals the number of $(n-1)$-tuples $\left(\beta^{(1)}, \ldots, \beta^{(n-1)}\right)$ satisfying*

i) $\beta^{(i)} \vdash \sum_{j=1}^{i} \lambda_j$ *for* $1 \leq i \leq n - 1$,

ii) $\beta_j^{(1)} \leq \beta_j^{(2)} \leq \ldots \leq \beta_j^{(n-1)} \leq \alpha_j$ *for any* $j \geq 1$, *iii)* $\beta_j^{(i)} \leq \beta_{j-1}^{(i-1)}$ *for any* $j > 1$ *and any* $1 \leq i \leq n$, *where* $\beta^{(0)} = (0)$ *and* $\beta^{(n)} = \alpha$.

Proof. Let now $\beta^{(i)}$ be a partition of $\sum_j \lambda_j^{(i)}$, where $\lambda_j^{(i)} = 0$ if $j > i$. Then we have by lemma 29 and by Frobenius's reciprocity law that

$$\begin{aligned}
\left(\chi^{\beta^{(i)}}, \xi^{\lambda^{(i)}}\right) &= \left(\mathrm{Res}^{S_{\lambda_1^{(i)}+\ldots+\lambda_i^{(i)}}}_{S_{\lambda_1^{(i)}+\ldots+\lambda_{i-1}^{(i)}} \times S_{\lambda_i^{(i)}}} \chi^{\beta^{(i)}}, \xi^{\left(\lambda_1^{(i)},\ldots,\lambda_{i-1}^{(i)}\right)} \otimes \xi^{\left(\lambda_i^{(i)}\right)}\right) \\
&= \sum_{\beta^{(i-1)} \vdash \sum_{j=1}^{i-1} \lambda_j^{(i)}} \left(\chi^{\beta^{(i-1)}} \otimes \chi^{\beta^{(i)}/\beta^{(i-1)}}, \xi^{\left(\lambda_1^{(i)},\ldots,\lambda_{i-1}^{(i)}\right)} \otimes \xi^{\left(\lambda_i^{(i)}\right)}\right) \\
&= \sum_{\beta^{(i-1)} \vdash \sum_{j=1}^{i-1} \lambda_j^{(i)}} \left(\chi^{\beta^{(i-1)}}, \xi^{\left(\lambda_1^{(i)},\ldots,\lambda_{i-1}^{(i)}\right)}\right) \left(\chi^{\beta^{(i)}/\beta^{(i-1)}}, \xi^{\left(\lambda_i^{(i)}\right)}\right).
\end{aligned}$$

and by theorem 30 we have that

$$\left(\chi^{\beta^{(i)}}, \xi^{\lambda^{(i)}}\right) = \sum_{\substack{\beta^{(i-1)} \vdash \sum_{j=1}^{i-1} \lambda_j^{(i)} \\ \beta_j^{(i)} \geq \beta_j^{(i-1)} \geq \beta_{j+1}^{(i)} \forall j \geq 1}} \left(\chi^{\beta^{(i-1)}}, \xi^{\left(\lambda_1^{(i)},\ldots,\lambda_{i-1}^{(i)}\right)}\right).$$

Let now $\lambda^{(i)} = (\lambda_1, \ldots, \lambda_i, 0, \ldots)$ and $\beta^{(n)} = \alpha$. Then we have that

$$\begin{aligned}
\left(\chi^\alpha, \xi^\lambda\right) &= \sum_{\beta^{(n-1)}} \left(\chi^{\beta^{(n-1)}}, \xi^{(\lambda_1,\ldots,\lambda_{n-1})}\right) \\
&= \sum_{\beta^{(n-1)}} \sum_{\beta^{(n-2)}} \left(\chi^{\beta^{(n-2)}}, \xi^{(\lambda_1,\ldots,\lambda_{n-2})}\right) \\
&= \ldots \\
&= \sum_{\beta^{(n-1)}} \ldots \sum_{\beta^{(1)}} \left(\chi^{\beta^{(1)}}, \xi^{(\lambda_1)}\right) \\
&= \sum_{\beta^{(n-1)}} \ldots \sum_{\beta^{(1)}} \left(\chi^{\beta^{(0)}}, \xi^{(0)}\right) \\
&= \sum_{\beta^{(n-1)}} \ldots \sum_{\beta^{(1)}} 1
\end{aligned}$$

where the last equality follows by the fact that $\mathrm{Ind}_{S_0}^{S_0}(IS_0) = IS_0$, where $S_0 = \{1\}$ and as $\beta^{(0)} - \mathrm{id} + \pi \not\models 0$ if $\pi \neq 1$ and where the summations are taken with $\beta^{(n-1)} \vdash \sum_{j=1}^{n-1} \lambda_j$ such that $\alpha_j \geq \beta_j^{(n-1)} \geq \alpha_{j+1}$ for $j \geq 1$, $\beta^{(i)} \vdash \sum_{j=1}^{i} \lambda_j$ such that $\beta_j^{(i+1)} \geq \beta_j^{(i)} \geq \beta_{j+1}^{(i+1)}$ for $0 \leq i \leq n-2$ and where $\beta^{(0)} = (0, 0, \ldots)$ is the only partition of 0. It is now easy to see that $\left(\chi^\alpha, \xi^\lambda\right)$ is equal to the number of $(n-1)$-tuples $\left(\beta^{(1)}, \ldots, \beta^{(n-1)}\right)$ satisfying conditions i), ii) and iii). $\square$

Theorem 32. *If* α *is a partition of* n *we have that* $\chi^\alpha = \zeta^\alpha$.

Proof. We will first show that if $\left(\chi^\alpha, \xi^\lambda\right) \neq 0$, where λ is an improper partition of n for which $\lambda_i = 0$ for $i > n$ (which we can always assume up to reordering the parts of λ, which would not change ξ^λ) then we have that $\lambda \trianglelefteq \alpha$. By Young's rule we have that if $\left(\chi^\alpha, \xi^\lambda\right) \neq 0$ then there exist $\left(\beta^{(1)}, \ldots, \beta^{(n-1)}\right)$ satisfying conditions i), ii) and iii) in the text of the previous theorem. In particular by iii) we have that $\beta^{(i)}_{i+j} \leq \beta^{(0)}_j = 0$ for any $i, j \geq 1$ and so as $\beta^{(i)}$ is a partition we need to have that $\beta^{(i)}_k = 0$ for $k > i$ and so by properties i) and ii) we have that, if $h < n$

$$\sum_{i=1}^h \lambda_i = \sum_i \beta^{(h)}_i = \sum_{i=1}^h \beta^{(h)}_i \leq \sum_{i=1}^h \alpha_i.$$

Also as $\lambda_i = \alpha_i = 0$ if $i > n$ and as $\lambda \models n$ and $\alpha \vdash n$ we have that $\sum_{i=1}^h \lambda_i \leq \sum_{i=1}^h \alpha_i$ also for $h \geq n$ and so if $\left(\chi^\alpha, \xi^\lambda\right) \neq 0$ we need to have that $\lambda \trianglelefteq \alpha$.

Now again by Young's rule we have that $(\chi^\alpha, \xi^\alpha)$ is equal to the number of $(n-1)$-tuples $\left(\beta^{(1)}, \ldots, \beta^{(n-1)}\right)$ satisfying properties i), ii) and iii). Again we need to have that $\beta^{(i)}_k = 0$ if $k > i$ and so we have that $\beta^{(1)}_1 = \alpha_1$ and so by ii) $\beta^{(i)}_1 = \alpha_1$ for all i. Assume now that $j \geq 2$ and for all $k < j$ we have that $\beta^{(i)}_k$ is equal to α_k if $k \leq i$ and is 0 if $k > i$. Then by property ii) we also have that $\alpha_i = \beta^{(j-1)}_i \leq \beta^{(j)}_i \leq \alpha_i$ for any $i < j$ and so

$$\sum_{i=1}^{j-1} \alpha_i + \beta^{(j)}_j = \sum_{i=1}^{j} \beta^{(j)}_i = \sum_{i=1}^{j} \alpha_i = \sum_{i=1}^{j-1} \alpha_i + \alpha_j$$

and so we also need to have that $\beta^{(j)}_j = \alpha_j$. As we know that $\beta^{(j)}_i = 0$ if $i > j$ we then need to have that $\beta^{(h)}_j$ is equal to α_j if $h \geq j$ and is equal to 0 if $h < j$. So there is only one possible $(n-1)$-tuple satisfying the conditions, that is the one with $\beta^{(i)} = (\alpha_1, \ldots, \alpha_i, 0, \ldots)$ and it is easy to see that this $(n-1)$-tuple actually satisfies properties i), ii) and iii), so that $(\chi^\alpha, \xi^\alpha) = 1$.

We now want to show that χ^α is plus or minus an irreducible character. In order to do this it is enough to show that $(\chi^\alpha, \chi^\alpha) = 1$, as by definition of χ^α and theorem 26 we can write $\chi^\alpha = \sum n_\beta \zeta^\beta$ for some $n_\beta \in \mathbb{Z}$ and then we have that $(\chi^\alpha, \chi^\alpha) = \sum n_\beta^2$ and this is equal to 1 if and only if only one $n_\beta = 1$ and all the other $n_{\beta'} = 0$. By the previous part and by definition of χ^α it is enough to show that $\left(\chi^\alpha, \xi^{\alpha - \mathrm{id} + \pi}\right) = 0$ for $\pi \neq 1$. So assume that $\pi \neq 1$. Let i be the smallest element which is not fixed by π. Then $\pi(i) > i$ and so

$$\sum_{k=1}^i (\alpha - \mathrm{id} + \pi)_k = \sum_{k=1}^i \alpha_k - i + \pi(i) > \sum_{k=1}^i \alpha_k$$

and so $\alpha - \mathrm{id} + \pi \ntrianglelefteq \alpha$ and so by the first part we need to have that $\left(\chi^\alpha, \xi^{\alpha - \mathrm{id} + \pi}\right) = 0$ for $\pi \neq 1$ and so we need to have that $(\chi^\alpha, \chi^\alpha) = 1$.

If we can now show that $(\chi^\alpha, \zeta^\alpha) = 1$ we would then have that $\chi^\alpha = \zeta^\alpha$ as χ^α is plus or minus an irreducible character. As if $\pi \neq 1$ we have that $\alpha - \mathrm{id} + \pi \ntrianglelefteq \alpha$ we have by lemma 24 that $\left(\xi^{\alpha-\mathrm{id}+\pi}, \zeta^\alpha\right) = 0$ if $\pi \neq 1$. So we have that

$$(\chi^\alpha, \zeta^\alpha) = \left(\sum\nolimits_\pi \mathrm{sign}(\pi)\xi^{\alpha-\mathrm{id}+\pi}, \zeta^\alpha\right) = (\xi^\alpha, \zeta^\alpha) = 1$$

and so we have that

$$\chi^\alpha = \zeta^\alpha$$

and so as we also have that

$$|[\alpha_i + j - i]| = [\alpha].$$

$\square$

For example for $(n-1,1)$ we have that

$$[n-1,1] = \begin{vmatrix} [n-1] & [n] \\ [0] & [1] \end{vmatrix} = [n-1][1] - [n] = \mathrm{Ind}_{S_{n-1}}^{S_n}(IS_{n-1}) - IS_n.$$

As we saw in section 6 $\mathrm{Ind}_{S_{n-1}}^{S_n}(IS_{n-1})$ is equivalent to the representation ρ on V, where V has a basis $\left\{e_{\{t\}}\right\}$ indexed by the tabloids of shape $(n-1,1)$, and such that $\rho(\pi)(e_{\{t\}}) = e_{\{\pi t\}}$. Also it is easy to see that a tabloid of shape $(n-1,1)$ is completely determined by knowing the unique element which is on the second line of any tableau in the tabloid, so that actually ρ is equivalent to the permutation ϕ on W, where W has basis $\{e_i\}_{i=1,\dots,n}$, such that $\phi(\pi)(e_i) = e_{\pi(i)}$, which is called the *natural representation* of S_n. It is easy to see that $\xi^{(n-1,1)}(\pi)$ is equal to the number of elements of $\mathbf{n}$ fixed by π, that is that $\xi^{(n-1,1)}(\pi) = a_1(\pi)$, where $a_1(\pi)$ is the number of 1-cycles in the decomposition of π in disjoint cycles. So we also have that

$$\zeta^{(n-1,1)}(\pi) = a_1(\pi) - 1.$$

Theorem 33.

$$\zeta^\alpha((1\cdots n)) = \begin{cases} (-1)^r & \alpha = (n-r, 1^r), 0 \leq r < n \\ 0 & \textit{otherwise} \end{cases}$$

This theorem (theorem 2.3.17 of [2]) is actually a special case of the Murnaghan-Nakayama formula.

Proof. If $\alpha = (n) = (n-0, 1^0)$ we already know that ζ^α is the character of IS_n and so $\zeta^{(n)}((1\cdots n)) = 1$.

Assume now that $\alpha = (n-r, 1^r)$ for some $1 \le r \le n-1$. Then we have that

$$\zeta^\alpha = \begin{vmatrix} [n-r] & [n-r+1] & \dots & [n-r+r-1] & [n-r+r] \\ 1 & [1] & \dots & [r-1] & [r] \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & [1] \end{vmatrix}.$$

Also the only the only Young subgroup of S_n containing an n-cycle is S_n, as an n-cycle cannot be written as a product of disjoint cycles of length less than n. So if λ is any improper partition of n for which $S_\lambda \neq S_n$ we have that $[\lambda_1][\lambda_2]\cdots((1\cdots n)) = 0$, as then S_λ cannot contain any element conjugate to $(1\cdots n)$. So we get a contribution to $\zeta^\alpha((1\cdots n))$ only from those terms which appear in the determinant which contain $[n]$. As $[n]$ appears in the right-top corner of the matrix and as for any other coefficient $[a_{i,k}]$ of the matrix $a_{i,k} < n$ (as the $a_{i,k}$ are strictly increasing in k and strictly decreasing in i), we have that the sum of the terms of the determinant which contain $[n]$ is

$$(-1)^{r+1+1}[n] \cdot \begin{vmatrix} 1 & [1] & \dots & [r-1] \\ 0 & 1 & \dots & [r-2] \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix}.$$

As this smallest matrix is upper triangular and as 1's on the diagonal we have that the only term containing $[n]$ which appears in ζ^α is $(-1)^r[n]$ and so as $[n]((1\cdots n)) = 1$ we have that

$$\zeta^{(n-r,1^r)} = (-1)^r.$$

So we now only need to show that if $\alpha \neq (n-r, 1^r)$ then $\zeta^\alpha((1\cdots n)) = 0$. Let h minimal such that $\alpha_i = 0$ for $i > h$. Then $\alpha = (\alpha_1, \dots, \alpha_h)$ and $\alpha_j \neq 0$ for $j \le h$. As $\alpha \neq (n-r, 1^r)$ for any r we have that for some $2 \le j \le h$ we have that $\alpha_j > 1$ (that is we need to have that $\alpha_2 > 1$ as α is a permutation). Then $h - 1 < \sum_{j=2}^h \alpha_j$ and so $\alpha_1 + h - 1 < n$. As $[\alpha_1 + h - 1]$ is the upper right coefficient of $\{[\alpha_i + j - i]\}$, $i, j \le h$, and for all other coefficients $[a_{i,j}]$ we have that $a_{i,j} < \alpha_1 + h - 1$ we have that $[n]$ is not contained in any term of $|[\alpha_i + j - i]|$ and so $\zeta^\alpha((1\cdots n)) = 0$ if $\alpha \neq (n-r, 1^r)$. □

In the following we will indicate by χ^α, where $\alpha \vdash n$, the character of the irreducible representation labeled by α of S_n.

8 The Murnaghan-Nakayama Formula

In this section we will prove the Murnaghan-Nakayama formula. This section is based on the first part of section 2.4 of [2].

Let λ be a composition of n. For any i we can define

$$\lambda^{i-} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots),$$

$$\lambda^{i+} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \ldots).$$

For any i we have that λ^{i-} is a composition of $i-1$ and λ^{i+} is a composition of $i+1$. Also if $\lambda_i = 0$ we have that $(\lambda^{i-})_i < 0$ and so λ^{i-} is an improper partition of n for at most finitely many i, so that $\xi^{\lambda^{i-}} = 0$ for all but finitely many i.

Lemma 34.

$$\operatorname{Res}^{S_n}_{S_{n-1}}\left(\xi^{\lambda}\right) = \sum_i \xi^{\lambda^{i-}}.$$

Proof. By lemma 6 and as all improper partitions of 1 are of the form μ^i, where $\mu^i_j = \delta_{i,j}$ we have that

$$\operatorname{Res}^{S_n}_{S_{n-1}\times S_1}(\xi^{\lambda}) = \sum_i \xi^{\lambda-\mu^i} \otimes \xi^{\mu^i} = \sum_i \xi^{\lambda^{i-}} \otimes \xi^{\mu^i}$$

which is the formula in the lemma as $S_1 = 1$ and so $\xi^{\mu^i} = \xi^{(1)} = 1$. $\square$

Theorem 35 (Branching rule)**.** *Let $\alpha = (\alpha_1, \alpha_2, \ldots) \vdash n$. Then we have that*

$$\operatorname{Res}^{S_n}_{S_{n-1}}([\alpha]) = \sum_{i:\alpha_i > \alpha_{i+1}} \left[\alpha^{i-}\right]$$

and that

$$\operatorname{Ind}^{S_{n+1}}_{S_n}([\alpha]) = \sum_{i:\alpha_i < \alpha_{i-1}} \left[\alpha^{i+}\right]$$

where $a_0 = \infty$.

The first part of the branching rule is a particular case of the Murnaghan-Nakayama formula.

Proof. Applying the previous lemma and theorem 32 we have that

$$\begin{aligned}
\operatorname{Res}^{S_n}_{S_{n-1}}(\chi^{\alpha}) &= \textstyle\sum_\pi \operatorname{sign}(\pi)\xi^{\alpha-\mathrm{id}+\pi} \\
&= \textstyle\sum_\pi \operatorname{sign}(\pi) \sum_i \xi^{(\alpha-\mathrm{id}+\pi)^{i-}} \\
&= \textstyle\sum_i \sum_\pi \operatorname{sign}(\pi)\xi^{\alpha^{i-}-\mathrm{id}+\pi} \\
&= \textstyle\sum_i \chi^{\alpha^{i-}}.
\end{aligned}$$

Also by lemma 27 if $\alpha_i = \alpha_{i+1}$, that is if $\alpha_i^{i-} = \alpha_{i+1}^{i-} - 1$ we have that $\chi^{\alpha^{i-}} = 0$ and so the last summation can be considered only over those i for which $\alpha_i \neq \alpha_{i+1}$. As α is a partition, so that we always have that $\alpha_i \geq \alpha_{i+1}$ this means that

$$\mathrm{Res}_{S_{n-1}}^{S_n}(\chi^\alpha) = \sum_{i:\alpha_i > \alpha_{i+1}} \chi^{\alpha^{i-}}.$$

Also as α it is easily verified that if $\alpha_i > \alpha_{i+1}$ then α^{i-} is a partition and so we also have that

$$\mathrm{Res}_{S_{n-1}}^{S_n}([\alpha]) = \sum_{i:\alpha_i > \alpha_{i+1}} \left[\alpha^{i-}\right]$$

as χ^β is the character of $[\beta]$ for any partition β by theorem 32 and as two representations are equivalent if and only if they have the same character by corollary 4.

Using this part of the theorem and theorem 12 if $\beta \vdash n+1$ we now have that

$$\left(\mathrm{Ind}_{S_n}^{S_{n+1}}(\chi^\alpha), \chi^\beta\right) = \left(\chi^\alpha, \mathrm{Res}_{S_n}^{S_{n+1}}\left(\chi^\beta\right)\right) = \left(\chi^\alpha, \sum_{i:\beta_i > \beta_{i+1}} \chi^{\beta^{i-}}\right).$$

As $\left(\chi^\alpha, \chi^{\beta^{i-}}\right) = \delta_{\alpha,\beta^{i-}}$ and all the β^{i-} are distinct we need to have that $\left(\mathrm{Ind}_{S_n}^{S_{n+1}}(\chi^\alpha), \chi^\beta\right)$ is equal to 1 if there exists i such that $\beta^{i-} = \alpha$ and is 0 otherwise. Also $\alpha = \beta^{i-}$ if and only if $\beta = \alpha^{i+}$ and as β needs to be a partition we need to have that $\alpha_i < \alpha_{i-1}$. So the only possible β for which $[\beta]$ can appear in $\mathrm{Ind}_{S_n}^{S_{n+1}}([\alpha])$ are those of the form $\beta = \alpha^{i+}$ with $\alpha_i < \alpha_{i-1}$, and as in this case $\beta_i > \beta_{i+1}$ we have that $[\beta^{i-}]$ actually appears in $\mathrm{Res}_{S_n}^{S_{n+1}}([\beta])$ and so in this case $[\beta]$ does appear in $\mathrm{Ind}_{S_n}^{S_{n+1}}([\alpha])$. Putting all of this together we have that

$$\mathrm{Ind}_{S_n}^{S_{n+1}}([\alpha]) = \sum_{i:\alpha_i < \alpha_{i-1}} \left[\alpha^{i+}\right]$$

and so the theorem is proved. □

Let $\mu_k^i = ((\mu_k^i)_1, (\mu_k^i)_2, \ldots)$ be given by $(\mu_k^i)_j = k \cdot \delta_{i,j}$.

Lemma 36. *Let λ be a composition of $n = m + k$ and $\pi \in S_n$. Assume that π contains a k-cycle and has cycle type $(a_1(\pi), a_2(\pi), \ldots)$. Let $\rho \in S_m$ have cycle type*

$$(a_1(\pi), \ldots, a_{k-1}(\pi), a_k(\pi) - 1, a_{k+1}(\pi), \ldots).$$

Then we have that

$$\chi^\lambda(\pi) = \sum_i \chi^{\lambda - \mu_k^i}(\rho).$$

Proof. This follows easily from lemma 28 and as if $\mu \models k$ we have that $\xi^\mu(1\cdots k) = 0$ unless if $\mu = \mu_k^i$ for some i, as only in this case S_μ contains a conjugate of $(1\cdots k)$. The formula then follows as $\xi^{\mu_k^i} = 1$ as it is the character of IS_k. □

Theorem 37 (Murnaghan-Nakayama formula)**.** *Let α be a partition of n and let π and ρ be as in lemma 36. Then*

$$\chi^\alpha(\pi) = \sum_{i,j:h^\alpha_{i,j}=k} (-1)^{l^\alpha_{i,j}} \chi^{\alpha\setminus R^\alpha_{i,j}}(\rho).$$

Proof. In order to prove the Murnaghan-Nakayama formula it is enough to show that $\chi^{\alpha-\mu_k^i} = 0$ if there is no hook of length k in the i-th row and that if $h^\alpha_{i,j} = k$ then $\chi^{\alpha-\mu_k^i} = (-1)^{l^\alpha_{i,j}}\chi^{\alpha\setminus R^\alpha_{i,j}}$ as we then would get the formula by lemma 36 and by the fact that in each row there is at most one hook of length k.

In order to show that $\chi^{\alpha-\mu_k^i} = 0$ when there is no hook of length k in the i-th row we will show that if $\chi^{\alpha-\mu_k^i} \neq 0$ then we can find j such that $h^\alpha_{i,j} = k$. If $\alpha_i - k \geq \alpha_{i+1}$ we can easily see that $h^\alpha_{i,\alpha_i-k+1} = k$, so let's assume that $\alpha_i - k < \alpha_{i+1}$. As $\chi^{\alpha-\mu_k^i} \neq 0$ we then need to have that $\alpha_i - k \leq \alpha_{i+1} - 2$ as if $\alpha_i - k = \alpha_{i+1} - 1$ we would get a contradiction by lemma 27. In particular $\alpha_{i+1} - 1 \geq \alpha_i - k + 1$. Let $\alpha^{(1)} = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1} - 1, \alpha_i - k + 1, \alpha_{i+2}, \ldots)$. We will now construct $\alpha^{(h)}$ inductively until we would get an $\alpha^{(\overline{h})}$ which is a partition. As long as $\alpha^{(h)}_{i+h} < \alpha^{(h)}_{i+h+1}$ let

$$\begin{aligned}\alpha^{(h+1)} &= \left(\alpha^{(h)}_1, \ldots, \alpha^{(h)}_{i+h-1}, \alpha^{(h)}_{i+h+1} - 1, \alpha^{(h)}_{i+h} + 1, \alpha_{i+h+2}, \ldots\right)\\ &= (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1} - 1, \ldots, \alpha_{i+h+1} - 1, \alpha_i - k + h + 1, \alpha_{i+h+2}, \ldots).\end{aligned}$$

As $\alpha^{(h)}_{i+h}$ is strictly increasing and $\alpha^{(h)}_{i+h+1}$ are non-increasing we need to find some minimal $\overline{h}$ for which $\alpha^{(\overline{h})}_{i+\overline{h}} \geq \alpha^{(\overline{h})}_{i+\overline{h}+1}$. By the minimality pf $\overline{h}$ we need to have that $\alpha^{(\overline{h}-1)}_{i+\overline{h}-1} < \alpha^{(\overline{h}-1)}_{i+\overline{h}}$ and as again by lemma 27, $\chi^{\alpha^{(h)}} = (-1)^h\chi^\alpha \neq 0$ we need to have that $\alpha^{(\overline{h})}_{i+\overline{h}} = \alpha^{(\overline{h}-1)}_{i+\overline{h}-1} + 1 \leq \alpha^{(\overline{h}-1)}_{i+\overline{h}} - 1 = \alpha^{(\overline{h})}_{i+\overline{h}-1}$ and as it is easy to see that $\alpha^{(\overline{h})}_{j+1} \leq \alpha^{(\overline{h})}_j$ for $j \neq i + \overline{h}, i + \overline{h} - 1$ we have that $\alpha^{(\overline{h})}$ is a partition, as all of its terms must be non-negative as $\alpha^{(\overline{h})}_j = 0$ for j big enough as for $j > i + \overline{h}$, $\alpha^{(\overline{h})}_j = \alpha_j$ and α is a partition. We will now show that $h^\alpha_{i,\alpha_i-k+1+\overline{h}} = k$. First it is easy to see that $\overline{h} \leq k - 1$ as $\alpha_i - k + k - 1 + 1 = \alpha_i \geq \alpha_j$ for all $j > i$. So $(i, \alpha_i - k + 1 + \overline{h})$ is a node of α. Now the set of nodes of the form (i, j) for $j \geq \alpha_i - k + 1 + \overline{h}$ clearly contains

$k-\overline{h}$ nodes. So in order to show that $h^\alpha_{i,\alpha_i-k+1+\overline{h}}=k$ it is enough to show that the set of nodes $(h,\alpha_i-k+1+\overline{h})$ with $h>i$ contains exactly $\overline{h}$ nodes. As we have that $\alpha_{i+\overline{h}}=\alpha^{(\overline{h}-1)}_{i+\overline{h}}>\alpha^{(\overline{h}-1)}_{i+\overline{h}-1}=\alpha_i-k+\overline{h}-1$ and $\alpha^{(\overline{h}-1)}_{i+\overline{h}}>\alpha^{(\overline{h}-1)}_{i+\overline{h}-1}+1$ as $\chi^{\alpha^{(\overline{h}-1)}}\neq 0$ we get that $\alpha_{i+\overline{h}}\geq\alpha_i-k+\overline{h}+1$ and then we need to have that $(i+\overline{h},\alpha_i-k+1+\overline{h})$ is a node of α, while as $\alpha_{i+\overline{h}+1}=\alpha^{(\overline{h})}_{i+\overline{h}+1}\leq\alpha^{(\overline{h})}_{i+\overline{h}}=\alpha_i-k+\overline{h}$ we need to have that $(i+\overline{h}+1,\alpha_i-k+1+\overline{h})$ is not a node of α and so as α is a partition we need to have that the set of nodes $(h,\alpha_i-k+1+\overline{h})$ with $h>i$ contains exactly $\overline{h}$ points and so $h_{i,\alpha_i-k+1+\overline{h}}=k$ and so if $\chi^{\alpha-\mu^i_k}\neq 0$ we can find j such that $h^\alpha_{i,j}=k$.

Assume now that $h^\alpha_{i,j}=k$. By looking at the Young diagram of α it is easy to see that

$$(\alpha\setminus R^\alpha_{i,j})_h=\begin{cases}\alpha_h & h<i\text{ or }h>i+l^\alpha_{i,j}\\ \alpha_{h+1}-1 & i\leq h<i+l^\alpha_{i,j}\\ \alpha_i-k+l^\alpha_{i,j} & h=i+l^\alpha_{i,j}\end{cases}$$

so that

$$\alpha\setminus R^\alpha_{i,j}=(\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1}-1,\ldots,\alpha_{i+l^\alpha_{i,j}}-1,\alpha_1-k+l^\alpha_{i,j},\alpha_{i+l^\alpha_{i,j}+1},\ldots)$$

and so $\alpha\setminus R^\alpha_{i,j}$ can be obtained by $\alpha-\mu^i_k$ by applying lemma 27 recursively on $i,i+1,\ldots,i+l^\alpha_{i,j}-1$. As we need to apply the lemma $l^\alpha_{i,j}$ times we have that $\chi^{\alpha-\mu^i_k}=(-1)^{l^\alpha_{i,j}}\chi^{\alpha\setminus R^\alpha_{i,j}}=(-1)^{l^\alpha_{i,j}}\chi^{\alpha\setminus R^\alpha_{i,j}}$ where the last equality follows from theorem 32 and so we have that the theorem is proved. □

Let h maximal such that $\alpha_h\geq h$ (that is h is maximal such that (h,h) is in the Young diagram of α).

Corollary 38.
$$\chi^\alpha_{(h^\alpha_{1,1},\ldots,h^\alpha_{h,h})}=(-1)^{\sum_{i=1}^h\alpha'_i-i}.$$

Proof. We will first prove that for any $\beta\vdash n$ if k is maximal such that (k,k) is in the Young diagram and $\beta^1=\beta\setminus R^\beta_{1,1}$ then $h^{\beta^1}_{i,i}=h^\beta_{i+1,i+1}$ and that $(\beta^1)'_i=\beta'_{i+1}-1$ for any $1\leq i\leq k-1$.

If $i<k$ we have that $\beta_{i+1},\beta'_{i+1}>0$ and so it is easy to see that $(\beta^1)_i=\beta_{i+1}-1$ and $(\beta^1)'_i=\beta'_{i+1}-1$. Also it is easy to see that for any partition γ and any (i,j) node of γ we have that $h^\gamma_{i,j}=\gamma_i+\gamma'_j-i-j+1$ and so we have that

$$h^{\beta^1}_{i,i}=\left(\beta^1\right)_i+\left(\beta^1\right)'_i-2i+1=\beta_{i+1}-1+\beta'_{i+1}-1+2i+1=h^\beta_{i+1,i+1}.$$

If we define β^i inductively by $\beta^i = \beta^{i-1} \setminus R_{1,1}^{\beta^{i-1}}$ for $i < k$ we then have that $h_{1,1}^{\beta^i} = h_{i+1,i+1}^{\beta}$ and that $(\beta^i)'_1 = \beta'_{i+1} - i$.

Applying this to α and using the Murnaghan-Nakayama formula and as the only hook of length $h_{1,1}^{\alpha}$ is $H_{1,1}^{\alpha}$ and this has leg length $\alpha'_1 - 1$ we have that

$$\begin{aligned}
\chi^{\alpha}_{(h^{\alpha}_{1,1},\ldots,h^{\alpha}_{h,h})} &= (-1)^{\alpha'_1-1}\chi^{\alpha^1}_{(h^{\alpha}_{2,2},\ldots,h^{\alpha}_{h,h})} \\
&= (-1)^{\alpha'_1-1}(-1)^{\alpha'_2-2}\chi^{\alpha^2}_{(h^{\alpha}_{3,3},\ldots,h^{\alpha}_{h,h})} \\
&= \ldots \\
&= (-1)^{\alpha'_1-1}\cdots(-1)^{\alpha'_{h-1}-h+1}\chi^{\alpha^{h-1}}_{(h^{\alpha}_{h,h}} \\
&= (-1)^{\alpha'_1-1}\cdots(-1)^{\alpha'_h-h} \\
&= (-1)^{\sum_{i=1}^{h}\alpha'_i-i}.
\end{aligned}$$

□

Corollary 39. *If $\chi^{\alpha}_{\beta} \neq 0$ then $\beta \trianglelefteq (h^{\alpha}_{1,1},\ldots,h^{\alpha}_{d,d})$, where d is such that $(d,d) \in R^{\alpha}$, that is d is maximal such that $(d,d) \in \alpha$.*

Lemma 40. *Let $\alpha \vdash n$ and let (i,j) be a node of α which doesn't belong to the rim of α. If (h,l) is an other node of α we have that $h_{i,j}^{\alpha\setminus R^{\alpha}_{h,l}} \geq h^{\alpha}_{i+1,j+1}$.*

Proof. First as $(i,j) \in \alpha \setminus R^{\alpha}$ we have that $(i+1,j+1) \in \alpha$ and that $(i,j) \notin R^{\alpha}_{h,l}$, so $h^{\alpha}_{i+1,j+1}$ and $h_{i,j}^{\alpha\setminus R^{\alpha}_{h,l}}$ are defined. Assume that $i < h$ or $i > \alpha'_l$, where α' is the partition associated with α. Then it is easy to see that $(\alpha \setminus R^{\alpha}_{h,l})_i = \alpha_i > \alpha_{i+1} - 1$, as α is a partition. If $h \leq i < \alpha'_l$ then it can be seen by considering the Young diagram of α that $(\alpha \setminus R^{\alpha}_{h,l})_i = \alpha_{i+1} - 1$. Also when $i = \alpha'_l$ it can again be easily seen that $(\alpha \setminus R^{\alpha}_{h,l})_i \geq (\alpha \setminus R^{\alpha}_{h,l})_{i+1} = \alpha_{i+1}$. So we always have that $(\alpha \setminus R^{\alpha}_{h,l})_i \geq \alpha_{i+1} - 1$. Similarly it can be seen that $(\alpha \setminus R^{\alpha}_{h,l})'_j \geq \alpha'_{j+1} - 1$. Putting these two things together we get that

$$\begin{aligned}
h_{i,j}^{\alpha\setminus R^{\alpha}_{h,l}} &= \left(h_{i,j}^{\alpha\setminus R^{\alpha}_{h,l}}\right)_i + \left(h_{i,j}^{\alpha\setminus R^{\alpha}_{h,l}}\right)'_j - i - j + 1 \\
&\geq \alpha_{i+1} - 1 + \alpha'_{j+1} - 1 - i - j + 1 \\
&= \alpha_{i+1} + \alpha'_{j+1} - (i+1) - (j+1) + 1 \\
&= h^{\alpha}_{i+1,j+1}
\end{aligned}$$

and so the lemma is proved. □

We will now prove corollary 39.

Proof. Assume that for some i we have that

$$\sum_{j=1}^{i}\beta_j > \sum_{j=1}^{i}h^{\alpha}_{j,j}.$$

If $i = 1$ we can conclude by the Murnaghan-Nakayama formula as then $\beta_1 > h^\alpha_{1,1}$ and so α doesn't have any hook of length β_1. Assume now that $i > 1$ and that $\chi^\gamma_\delta = 0$ if $\sum_{j\leq i-1}\delta_j > \sum_{j\leq i-1} h^\gamma_{j,j}$. Then we have that $h^\alpha_{1,1} + \cdots + h^\alpha_{d,d} = n = \sum \beta_j$ and as the β_j are non-negative we need to have that $i < d$ and

$$\sum_{j>i}\beta_j < \sum_{j>i} h^\alpha_{j,j} = h^\alpha_{i+1,i+1} + \cdots + h^\alpha_{d,d}.$$

From the previous lemma if $i \leq j < d$ and $h^\alpha_{h,l} = \beta_1$ we have that $h^{\alpha\backslash R^\alpha_{h,l}}_{j,j} \geq h^\alpha_{j+1,j+1}$ and so we have that

$$\sum_{j>i}\beta_j < h^{\alpha\backslash R^\alpha_{h,l}}_{i,i} + \ldots + h^{\alpha\backslash R^\alpha_{h,l}}_{d-1,d-1} \leq \sum_{j>i-1} h^{\alpha\backslash R^\alpha_{h,l}}_{j,j}$$

and so the sum of the first $i-1$ terms of $(\beta_2, \beta_3, \ldots)$ needs to be bigger than $h^{\alpha\backslash R^\alpha_{h,l}}_{1,1} + \ldots + h^{\alpha\backslash R^\alpha_{h,l}}_{i-1,i-1}$ and so we have that $\chi^{\alpha\backslash R^\alpha_{h,l}}_{(\beta_2,\beta_3,\ldots)} = 0$ for any (h,l) such that $h^\alpha_{h,l} = \beta_1$ and then by induction and by applying the Murnaghan-Nakayama formula we have that $\chi^\alpha_\beta = 0$ whenever $\beta \not\trianglelefteq (h^\alpha_{1,1}, \ldots, h^\alpha_{d,d})$ and so the theorem is proved. $\square$

9 β-sets, cores and quotients

In this section we will define β-sets, cores and quotient and show how we can them to determine informations about hooks of a partition.

Definition 21 (β-set)**.** *A β-set is a finite set of $\mathbb{N}$, that is a β-set is a finite set of non-negative integers.*

Assume now that $X = \{y_1, \ldots, y_k\}$ is a β-set and that $y_i > y_{i+1}$ for i from 1 to $k-1$ (we can always assume this up to reordering the y_i). Then we have that

$$y_{i+1} - k + i + 1 \leq y_i - k + i$$

for $0 \leq i < k$ and so we have that

$$(y_1 - k + 1, y_2 - k + 2, \ldots, y_k)$$

is a partition (by what we just saw we have that $y_i - k + i \geq y_k \geq 0$ for all i).

Definition 22. *If $X = \{y_1, \ldots, y_k\}$ is a β-set and the y_i are decreasing we say that*

$$P^*(X) = (y_1 - k + 1, y_2 - k + 2, \ldots, y_k)$$

is the partition associated to X.

If $X = \{y_1, \ldots, y_k\}$ is a β-set and $s \geq 0$ is an integer we can define a new β-set by

$$X^{+s} = \{y_1, \ldots, y_k, s-1, s-2, \ldots, 0\}.$$

It is easy to see that (just use the definition)

Theorem 41. *If X is a β-set and $s \geq 0$ is an integer we have that*

$$P^*(X) = P^*(X^{+s}).$$

Also if $\alpha = (\alpha_1, \ldots, \alpha_h)$ is a partition with $\alpha_h > 0$ and

$$X_\alpha = \{h^\alpha_{1,1}, \ldots, h^\alpha_{h,1}\}$$

it can be easily seen that X_α, the set of first column hook-lengths of α, is a β-set and that $P^*(X_\alpha) = \alpha$. So for any partition α we can find some β-set X for which $P^*(X) = \alpha$.

It is easy to see that

Theorem 42. *If X is a β-set and α is a partition we have that $P^*(X) = \alpha$ if and only if $X = X^{+s}_\alpha$ for some $s \geq 0$.*

Lemma 43. *If $\alpha = (\alpha_1, \ldots, \alpha_h)$ is a partition and $\alpha_h > 0$, we have that for any $1 \leq i \leq h$ and any $1 \leq k \leq \alpha_i$,*

$$\prod_{j=i+1}^{h} \left(h^\alpha_{i,k} - h^\alpha_{j,k}\right) \prod_{v=k}^{\alpha_i} h^\alpha_{i,v} = h^\alpha_{i,1}!$$

Proof. As the $h^\alpha_{l,m}$ are strictly decreasing in both l and m we have that all terms in the products in the left hand side are between 1 and $h^\alpha_{i,k}$. Also by choice h we have that $l^\alpha_{i,k} = h - i$ and so the number of terms in the first product is equal to the leg-length of (i, k). As in the second product there are $a^\alpha_{i,k} + 1 = \alpha_i - k + 1$ terms, we have that the left hand side of the equation we want to prove consists of exactly $h^\alpha_{i,k}$ terms, just like the right hand side. So in order to show that the two sides are equal it is enough to show that all terms in the left hand side are different. As the $\left(h^\alpha_{i,k} - h^\alpha_{j,k}\right)$ are pairwise different and the same is true for the $h^\alpha_{i,v}$, we only need to show that $\left(h^\alpha_{i,k} - h^\alpha_{j,k}\right) \neq h^\alpha_{i,v}$ for each $i+1 \leq j \leq h$ and each $1 \leq v \leq \alpha_i$. As the $\left(h^\alpha_{i,k} - h^\alpha_{j,k}\right)$ are increasing and the $h^\alpha_{i,v}$ are decreasing it is enough to show that one of the following must hold

i) $h^\alpha_{i,v} < h^\alpha_{i,k} - h^\alpha_{i+1,k}$,

ii) $h^\alpha_{i,v} > h^\alpha_{i,k} - h^\alpha_{h,k}$,

iii) $h^\alpha_{i,k} - h^\alpha_{j,k} < h^\alpha_{i,v} < h^\alpha_{i,k} - h^\alpha_{j,k}$ for some $i < j < h$.

Let $j = \alpha'_v$, where α' is the partition conjugated to α. We need to have that $i \leq j \leq h$ as $(i,v) \in \alpha$ and as by definition of h we have that $\alpha'_l \leq h$ for any l. Assume that $j = i$. Then $v > \alpha_{i+1,1}$ and so we have that

$$h^\alpha_{i,v} = \alpha_i + \alpha'_v - i - v + 1 < \alpha_i + h - i - (k-1) - (\alpha_{i+1} + h - j - 1 - (k-1)) = h^\alpha_{i,k} - h^\alpha_{i+1,k}$$

and so i) holds in this case.

Assume now that $j = \alpha'_v = h$. Then $v \leq \alpha_h$ and so we have that

$$h^\alpha_{i,v} = \alpha_i + \alpha'_v - i - v + 1 > \alpha_i + h - i - (k-1) - (\alpha_h - (k-1)) = h^\alpha_{i,k} - h^\alpha_{h,k}$$

and then ii) holds in this case.

So assume now that $i < j = \alpha'_v < h$. We want to show that iii) holds in this case. By definition of j we have that $\alpha_j \geq v$ and $\alpha_{j+1} < v$. So

$$h^\alpha_{i,v} = \alpha_i + \alpha'_v - i - v + 1 < \alpha_i + h - i - (k-1) - (\alpha_{j+1} + h - j - 1 - (k-1)) = h^\alpha_{i,k} - h^\alpha_{j+1,k}$$

and

$$h^\alpha_{i,v} = \alpha_i + \alpha'_v - i - v + 1 > \alpha_i + h - i - (k-1) - (\alpha_j + h - j - (k-1)) = h^\alpha_{i,k} - h^\alpha_{j,k}$$

and then we have that $h^\alpha_{i,k} - h^\alpha_{j,k} < h^\alpha_{i,v} < h^\alpha_{i,k} - h^\alpha_{j+1,k}$, that is iii) holds.

So we have that the $\left(h^\alpha_{i,k} - h^\alpha_{j,k}\right)$ and the $h^\alpha_{i,v}$ are pairwise different and then the lemma is proved. □

Lemma 44. *If $\alpha = (\alpha_1, \ldots, \alpha_h)$ is a partition and $\alpha_h > 0$, we have that for any $1 \leq i \leq h$ and any $1 \leq k \leq \alpha_i$,*

$$\prod_{j=i}^{h} h^\alpha_{j,k} \prod_{v=k+1}^{\alpha_i} \left(h^\alpha_{i,k} - h^\alpha_{i,v}\right) = h^\alpha_{i,1}!$$

Proof. It follows by the previous lemma by considering α', as for any (j,v) node of α we have that $h^\alpha_{j,v} = h^{\alpha'}_{v.j}$. □

We will now use this lemma to prove the following theorem, which shows how we can remove hooks from a partition by simply considering one of its β-sets.

Theorem 45. *Let $X = \{y_1, \ldots y_k\}$ be a β-set and let $\alpha = P^*(X)$. Assume that the y_i decreasing. Then we have that for fixed i and $h > 0$ we can find j such that $h^\alpha_{i,j} = h$ if and only if $y_i - h \notin X$ and $y_i - h \geq 0$. In this case we have that if*

$$X' = \{y_1, \ldots y_{i-1}, y_i - h, y_{i+1}, \ldots, y_k\}$$

then $P^*(X') = \alpha \setminus R^\alpha_{i,j}$ and we have that $l^\alpha_{i,j}$ is equal to the number of l for which $y_i - h < y_l < y_i$.

Proof. Assume that $h^\alpha_{i,j} = h$ and that $Y = X^{+s}$, $s \in \mathbb{N}$. By definition of X^{+s} and as y_i is the i-th biggest element of X, we have that if $y_i - h \notin X$ and $y_i - h \geq 0$ then we have that $y_i + s$ is the i-th biggest element of Y, $(y_i+s)-h \notin Y$ and $(y_i+s)-h \geq 0$. If instead $X = Y^{+s}$ we need to have that as $P^*(Y) = \alpha$ and (i,j) is a node of α, Y needs to contain at least i elements. As $y_i - h \notin X$ we need to have that $(y_i - s) - h \notin Y$. Also y_i is the i-th biggest element of X, so $y_i - s$ needs to be the i-th biggest element of Y (as Y has at least i elements) and as $y_i - h \notin X$, $y_i - h \geq 0$ and $1, 2, \ldots s-1 \in X$ we need to have that $y_i - h \geq s$ and so $(y_i - s) - h \geq 0$. So if we can prove that $y_i - h \notin X$ and $y_i - h \geq 0$ for some β-set X such that $P^*(X) = \alpha$, where y_i is the i-th biggest element of X, then we have that for any β-set Y such that $P^*(Y) = \alpha$ we have that if z_i is the i-th biggest element of Y then $z_i - h \notin Y$ and $z_i - h \geq 0$, so it is enough to show that this property holds for one β-set for α. Let $X = X_\alpha$ be the set of first column hook-lengths of α. Then we have that $y_i = h^\alpha_{i,1}$.

By the last lemma we have that if we can find j such that $h^\alpha_{i,j} = h$ then $h^\alpha_{i,1} - h \neq h^\alpha_{i',1}$ for any $i' > i$ (this is trivial if $j = 1$ as then $h^\alpha_{i,1} - h = 0$) and as $h^\alpha_{i,1} - h < h^\alpha_{i,1} \leq h^\alpha_{i'',1}$ for any $i'' \leq i$, and so if for some j we have that $h^\alpha_{i,j} = h$ then $h^\alpha_{i,1} - h \notin X_\alpha$. Also it is clear that in this case we have that $h \leq h^\alpha_{i,1}$, so that we also have that $h^\alpha_{i,1} - h \geq 0$.

Assume now that $h^\alpha_{i,1} - h \geq 0$ and $h^\alpha_{i,1} - h \notin X_\alpha$. As $h \geq 1$ we in particular need to have that $1 \leq h \leq h^\alpha_{i,1}$ and $h^\alpha_{i,1} - h \neq h^\alpha_{i',1}$ for any $i' > i$, that is $h \neq h^\alpha_{i,1} - h^\alpha_{i',1}$ for any $i' > i$ and so again by the lemma we need to have that $h = h^\alpha_{i,j}$ for some j, (we can apply the lemma if $h \neq h^\alpha_{i,1}$ and if $h = h^\alpha_{i,1}$ we can just take $j = 1$) and so we have that the first part of the theorem is true.

As for any $s \geq 0$, if $X^{+s}_\alpha = \{y_1, \ldots, y_k\}$, where the y_i are decreasing, we have that $y_i - h < y_j < y_i$ if and only if $h^\alpha_{i,1} - h < h^\alpha_{j,1} < h^\alpha_{i,1}$ when $y_i - h \geq 0$ and $y_i - h \notin X^{+s}_\alpha$ and as by definition of X^{+s}_α we then need to have that $y_i - h \geq s$, it is easy to see that in order to show that $l^\alpha_{i,j}$ is equal to the number of l for which $y_i - h^\alpha_{i,j} < y_l < y_i$ for any β-set $X = \{y_1, \ldots y_k\}$ for α, it is again enough to show it for X_α.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$, with $\alpha_m > 0$. Then we have that $h^\alpha_{l,1} = \alpha_l + m - l$ for any $1 \leq l \leq m$. As the arm-length of (i,j) is equal to $\alpha_i - j$, we have that the leg-length of (i,j) is equal to $h^\alpha_{i,j} - 1 - \alpha_i + j$, so, as the $h^\alpha_{l,1}$ are decreasing, in order to prove this part of the theorem it is enough to prove that

$$h^\alpha_{h^\alpha_{i,j}-1-\alpha_i+j+i,1} > h^\alpha_{i,1} - h^\alpha_{i,1}$$

and that

$$h^\alpha_{h^\alpha_{i,j}-\alpha_i+j+i,1} < h^\alpha_{i,1} - h^\alpha_{i,1},$$

if $l^\alpha_{i,j}+i=h^\alpha_{i,j}-1-\alpha_i+j+i<m$. When $h^\alpha_{i,j}-1-\alpha_i+j+i=m$ (it can never be bigger than m) we have that $l^\alpha_{i,j}=m-i$ and in this case we would have proven by the previous point and by the fact that the $h^\alpha_{i',1}$ are decreasing that $h^\alpha_{i,1}-h^\alpha_{i,1}<h_{i',1}$ if and only if $i'>i$, and so also in this case we would have that if $h^\alpha_{h^\alpha_{i,j}-1-\alpha_i+j+i,1}>h^\alpha_{i,1}-h^\alpha_{i,1}$ then the leg-length of (i,j) is equal to the number of l for which $h^\alpha_{i,1}-h^\alpha_{i,j}<h^\alpha_{l,1}<h^\alpha_{i,1}$.

By definition of leg-length we need to have that $\alpha_{h^\alpha_{i,j}-1-\alpha_i+j+i}\geq j$, so

$$\begin{aligned} h^\alpha_{h^\alpha_{i,j}-1-\alpha_i+j+i+1,1} &= \alpha_{h^\alpha_{i,j}-1-\alpha_i+j+i}+m-(h^\alpha_{i,j}-1-\alpha_i+j+i)\\ &> \alpha_i+m-i-h^\alpha_{i,j}\\ &= h^\alpha_{i,1}-h^\alpha_{i,j}\end{aligned}$$

and as again by definition of leg-length we need to have that $\alpha_{h^\alpha_{i,j}-\alpha_i+j+i}<j$, whenever $h^\alpha_{i,j}-\alpha_i+j+i\leq m$, we have that in this case

$$\begin{aligned} h^\alpha_{h^\alpha_{i,j}-\alpha_i+j+i+1,1} &= \alpha_{h^\alpha_{i,j}-\alpha_i+j+i}+m-(h^\alpha_{i,j}-1-\alpha_i+j+i)\\ &< \alpha_i+m-i-h^\alpha_{i,j}\\ &= h^\alpha_{i,1}-h^\alpha_{i,j}\end{aligned}$$

and so we always have that the leg-length of (i,j) is equal to the number of l for which $y_i-h<y_l<y_i$, where $X=\{y_1,\ldots,y_k\}$ is any β-set of α.

Assume that $h^\alpha_{i,j}=h$ and that $l^\alpha_{i,j}=s$. Looking at the Young diagrams of α and of $\alpha\setminus R^\alpha_{i,j}$ and as $\alpha\setminus R^\alpha_{i,j}\vdash n-h$, it is easy to see that

$$\left(\alpha\setminus R^\alpha_{i,j}\right)_l=\begin{cases}\alpha_l & l<i\text{ or }l>i+s\\ \alpha_{l+1}-1 & i\leq l<i+s\\ \alpha_i-h+s & l=i+s.\end{cases}$$

As $\alpha\setminus R^\alpha_{i,j}$ has at most k parts different from 0 as the same is true for α as $|X|=k$ we can write

$$\alpha\setminus R^\alpha_{i,j}=\left(\left(\alpha\setminus R^\alpha_{i,j}\right)_1,\ldots,\left(\alpha\setminus R^\alpha_{i,j}\right)_k\right)$$

and so we have that

$$Y=\left\{\left(\alpha\setminus R^\alpha_{i,j}\right)_1+k-1,\left(\alpha\setminus R^\alpha_{i,j}\right)_2+k-2,\ldots,\left(\alpha\setminus R^\alpha_{i,j}\right)_k\right\}$$

is a β-set for $\alpha\setminus R^\alpha_{i,j}$. We will show that $Y=X'$.

As $X=\{\alpha_1+k-1,\alpha_2+k-2,\ldots,\alpha_k\}=\{y_1,\ldots,y_k\}$ we have that $y_l=\alpha_l+k-l$. Then we have that

$$\left(\alpha\setminus R^\alpha_{i,j}\right)_l+k-l=\begin{cases}\alpha_l+k-l=y_l & l<i\text{ or }l>i+s\\ \alpha_{l+1}-1+k-l=y_{l+1} & i\leq l<i+s\\ \alpha_i-h+k-i=y_i-h & l=i+s.\end{cases}$$

and so we have that $Y = \{y_l : l \neq i\} \cup \{y_i - h\} = X'$ and then X' is a β-set for $\alpha \setminus R^{\alpha}_{i,j}$ and so the theorem is proved. $\square$

Corollary 46. *Removing a kq-hooks is equivalent to removing a certain sequence of k hooks all of length q.*

Proof. Suppose that we are removing a kq-hook from a partition with β-set $X = \{y_1, \ldots, y_k\}$ by changing y_h to $y_h - kq$. Assume that the y_i are decreasing and let $h = i_1 > \ldots > i_l$ be those indexes j for which $y_h - kq < y_j \leq y_h$ and $y_h - y_j$ is divisible by q. We can recursively change in the following way the elements of X

$$\begin{aligned}
&y_{i_l} \to y_{i_l} - q, y_{i_l} - q \to y_{i_l} - 2q, \ldots, y_h - kq + q \to y_h - kq,\\
&y_{i_{l-1}} \to y_{i_{l-1}} - q, \ldots, y_{i_l} + q \to y_{i_l},\\
&\ldots,\\
&y_h \to y_h - q, \ldots, y_{i_2} + q \to y_{i_2}.
\end{aligned}$$

As the y_{i_m} are decreasing, $y_h - kq \geq 0$, $y_h - kq \notin X$ and by definition of the indexes i_m, it can be easily seen that each of these steps corresponds to removing a q-hook, that the β-set we obtain at the end is given by $\{y_1, \ldots, y_{h-1}, y_h - kq, y_{h+1}, \ldots, y_k\}$ and that the number of hooks we remove this way is exactly k, and so the corollary is proved. $\square$

In order to give the definitions of the q-core and the q-quotient of a partition for any positive integer q we will first introduce the q-abacus.

Definition 23 (q-abacus)**.** *The q-abacus consists of q vertical runners index starting from the left with the numbers $0, 1, \ldots, q-1$. The i-th runner contains positions $i, q+i, 2q+i, \ldots$ starting from the top and moving downwards.*

For example the 4-abacus is given by

$$\begin{array}{cccc}
0 & 1 & 2 & 3\\
4 & 5 & 6 & 7\\
8 & 9 & 10 & 11\\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

If X is a β-set we can place a bead on the q-abacus on the numbers which are contained in X. For example if $X = \{1, 3, 9\}$ we have that the 4-abacus for X is given by

$$\begin{array}{cccc}
0 & ① & 2 & ③\\
4 & 5 & 6 & 7\\
8 & ⑨ & 10 & 11\\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Definition 24 (q-core)**.** *Let α be a partition. The q-core of α is obtained by recursively removing q-hooks from α until we obtain a partition which doesn't contain any q-hook. The q-core of α is indicated by $\alpha_{(q)}$.*

By theorem 45 we can easily see that if X is a β-set for α then we can obtain a β-set for $\alpha_{(q)}$ by moving each bead on the q-abacus of X as high as possible, leaving each bead in its runner and without overlapping beads. This way it can also be seen that the q-core of a partition is unique, that is it doesn't depend on which sequence of q-hooks we recursively remove from α.

If we look again at the previous example we have $X = \{1, 3, 9\}$, so $\alpha = (7, 2, 1)$, and the 4-abacus for $\alpha_{(4)}$ is

$$\begin{array}{cccc} 0 & \textcircled{1} & 2 & \textcircled{3} \\ 4 & \textcircled{5} & 6 & 7 \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

so we have that $\{1, 3, 5\}$ is a β-set for $\alpha_{(4)}$ and then $\alpha_{(4)} = (3, 2, 1)$.

Definition 25 (q-quotient)**.** *Let α be a partition and let X be a β-set for α such that q divides the cardinality of X. The q-quotient of α is given by a q-tuple of partitions*

$$\alpha^{(q)} = (\alpha_0, \ldots, \alpha_{q-1}),$$

where $\alpha_i = P^*(X_i)$ and $j \in X_i$ if and only if $qj + i \in X$.

It is easy to see that for any α we can find a β-set X such that $q||X|$ and $P^*(X) = \alpha$ and that $\alpha^{(q)}$ doesn't depend on the choice of such an X, as if Y satisfies the same properties of X we have that $Y = X^{+kq}$ or $X = Y^{+kq}$ for some $k \geq 0$ (we can assume that $Y = X^{+kq}$) and then it is easy to see that $Y_i = X_i^{+k}$ and so the α_i are well defined. If we labeled positions on each runners of the q-abacus starting with 0, it can be easily seen that each X_i consists of the positions of the i-th runner of the q-abacus which correspond elements of X.

Definition 26 (q-weight)**.** *The q-weight of a partition α is the number of q-hooks that we need to recursively remove from α in order to obtain $\alpha_{(q)}$. The q-weight of α is indicated by $w_q(\alpha)$.*

As if $\gamma \vdash n$ and δ is obtained by γ by removing a q-hook we have that $\delta \vdash n - q$ we also have that as $\alpha_{(q)}$ is unique also $w_q(\alpha)$ is well defined, that is it doesn't depend on which hooks we are removing. More precisely if $\alpha \vdash n$ and $\alpha_{(q)} \vdash m$ we have that $w_q(\alpha) = (n - m)/q$.

Definition 27. *If α is a partition of n we define $|\alpha| = n$.*

Theorem 47. *If α is a partition and $\alpha^{(q)} = (\alpha_1, \ldots, \alpha_{q-1})$ we have that*

$$\sum_i |\alpha_i| = w_q(\alpha)$$

and for any positive integer k, there is a bijective correspondence between kq-hooks of α and k-hooks of $\alpha^{(q)}$. Also this correspondence is preserved after removing any kq-hook from α and the corresponding k-hook from $\alpha^{(q)}$.

In the theorem a hook of $\alpha^{(q)}$ is just a hook of one of the α_i.

Proof. We will start by proving the second part of the theorem. Let X be a β-set for α such that $q||X|$ and let $X_j = \{i : iq + j \in X\}$ for $0 \leq j < q$. Then by definition we have that X_j is a β-set for α_j for all j. So let now map the hook of α which corresponds to changing $iq + j$ to $(i - k)q + j$ in X to the hook of α_j which corresponds to changing i to $i - k$ in X_j. It is easy to see that by theorem 45 this gives a bijection between the kq-hooks of α and the k-hooks of $\alpha^{(q)}$, as $(i - k)q + j \notin X$ if and only if $i - k \notin X_j$ and $(i - k)q + j \geq 0$ if and only if $i - k \geq 0$ as $0 \leq j < q$. Also it is easy to see that this bijection is preserved by removing a kq-hook of α and the corresponding k-hook of $\alpha^{(q)}$.

In order to prove now the first part of the theorem we can notice that by the second part of the theorem, removing a sequence of m hooks of length q from α corresponds to removing a sequence of m hooks of length 1 from $\alpha^{(q)}$. By definition of $w_q(\alpha)$ we know that we can remove a sequence of $w_q(\alpha)$ hooks from α, so in particular $\alpha^{(q)}$ must contain at least $w_q(\alpha)$ nodes, as removing a 1-hook means that we are removing a node from $\alpha^{(q)}$. Also if $\alpha^{(q)}$ would contain more than $w_q(\alpha)$ nodes we could, after having removed the other $w_q(\alpha)$ 1-hooks, remove at least one more 1-hook, as then at least one of the partition obtained from the α_j after removing the $w_q(\alpha)$ 1-hooks from $\alpha^{(q)}$ would need to have some part different from 0, and so we could remove at least one1-hooks from that partition. But this would mean that we could actually remove a q-hook from $\alpha_{(q)}$, as after removing $w_q(\alpha)$ hooks of length q from α we always obtain $\alpha_{(q)}$. This gives though a contradiction as by definition $\alpha_{(q)}$ doesn't have any hook of length q. So we need to have that $\alpha^{(q)}$ contains exactly $w_q(\alpha)$ nodes and so, as the nodes of $\alpha^{(q)}$ are the nodes of the partitions α_j, we need to have that

$$\sum_i |\alpha_i| = w_q(\alpha.)$$

□

Corollary 48. *If β is obtained by α by removing l hooks of length q we have that $w_q(\beta) = w_q(\alpha) - l$.*

Proof. This is an easy application of 47 as then $\beta^{(q)}$ is obtained by $\alpha^{(q)}$ by removing l nodes. □

It can also be proved that there exists a unique partition which has a given q-core and a given q-quotient. This can be seen by considering a β-set of the q-core with sufficiently with cardinality sufficiently big and divisible by q.

We will now show how we can define a q-sign for a partition.

Definition 28 (Natural numbering)**.** *Let X be a β-set. The natural numbering on X is given by numbering the elements of X in increasing order.*

Definition 29 (q-numbering)**.** *Let X be a β-set. Let $x, y \in X$ be the j-th bead (we start count them in increasing order) on the i-th runner and the j_1-th bead on the i_1-bead of the q-abacus of X respectively. The q-numbering of X is the one for which x is indexed by a smaller number then y if and only if $j < j_1$ or $j = j_1$ and $i < i_1$.*

So for example if $X = \{1, 3, 6, 9\}$ we have that the natural numbering is

$$\begin{array}{cccc} 0 & 1_1 & 2 & 3_2 \\ 4 & 5 & 6_3 & 7 \\ 8 & 9_4 & 10 & 11 \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

and the 4-numbering of X is

$$\begin{array}{cccc} 0 & 1_1 & 2 & 3_3 \\ 4 & 5 & 6_2 & 7 \\ 8 & 9_4 & 10 & 11 \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Definition 30 (q-sign)**.** *Let X is a β-set and $\alpha = P^*(X)$. Let π be the permutation which sends the natural numbering of X in the q-numbering of X. The q-sign of X and α is given by*

$$\delta_q(X) = \delta_q(\alpha) = \operatorname{sign}(\pi).$$

In the previous example we have that

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

and so $\pi = (2\ 3)$ and so $\delta_4(X) = \delta_4(4, 2, 1) = -1$.

We will now show that the q-sign of a partition α is well defined.

Theorem 49. *Let $X = \{x_1, \ldots, x_k\}$ be β-set ordered with the natural numbering. Assume that $x_i - h \notin X$ and $x_i - h \geq 0$. Let $Y = \{x_a, \ldots, x_{i-1}, x_i - h, x_{i+1}, \ldots, x_k\}$. Let*

$$z_j = \begin{cases} x_j & j \neq i \\ x_i - h & j = i \end{cases}$$

and let $Y = \{y_1, \ldots, y_k\}$ be the natural numbering on Y. Let $\alpha = P^*(X)$ and let (l, m) be the node of α whose removal corresponds to changing X to Y. If π is the permutation for which $z_{\pi(j)} = y_j$ for all j we have that $\mathrm{sign}(\pi) = (-1)^{l^\alpha_{l,m}}$.

Proof. Notice that by theorem 45 we have that $l = k - i + 1$. By the same theorem we also have that $\pi = (k - i - l^\alpha_{l,m} + 1, k - i - l^\alpha_{l,m} + 2, \ldots, k - i + 1)$ and so π is a cycle of length $l^\alpha_{l,m} + 1$ and then $\mathrm{sign}(\pi) = (-1)^{l^\alpha_{l,m}}$. □

Corollary 50. *Let X and Y be β-sets and let Y be obtained by recursively removing a series of hooks from X (that is by removing hooks from the corresponding partitions). That is $X = \{x_1^{(1)}, \ldots, x_k^{(1)}\}$ with the natural numbering on X and $Y = \{x_1^{(h+1)}, \ldots, x_k^{(h+1)}\}$ for some $h+1$, where the $x_j^{(l)}$ are obtained recursively by*

$$x_j^{(l)} = \begin{cases} x_j^{(l-1)} & j \neq i_{l-1} \\ x_{i_{l-1}}^{(l-1)} - h_{l-1} & j = i_{l-1} \end{cases}$$

for some i_l and h_l such that $x_{i_l}^{(l)} - h_l \neq x_j^{(l)}$ for any j and $x_{i_l}^{(l)} - h_l \geq 0$.

Let l_j be the leg-length of the hook removed at step j. Assume that $x_i^{(1)}$ is the natural numbering of X and $Y = \{y_1, \ldots, y_k\}$ is the natural numbering on Y. Let π be such that $x_{\pi(i)}^{(h+1)} = y_i$. Then

$$\mathrm{sign}(\pi) = (-1)^{\sum_{j=1}^{h} l_j}.$$

Proof. For each $1 \leq j \leq h$ let $\pi_j \in S_k$ be such that $x_{\pi_j(i_1)}^{(j+1)} > x_{\pi_j(i_2)}^{(j+1)}$ if and only if $x_{i_1}^{(j)} > x_{i_2}^{(j)}$. It isn't hard to see that each π_j is given by the theorem when we consider the ordering on $\{1, \ldots, k\}$ given by $i_1 >' i_2$ if and only if $x_{i_1}^{(j)} > x_{i_2}^{(j)}$, instead of the ordering $1 < 2 < \ldots < k$. Even if we consider a new order on the set $\{1, \ldots, k\}$ we still have by theorem 45 that π_j is a cycle of length $l_j + 1$.

We want to show that $\pi = \pi_h \ldots \pi_1$, from which the corollary follows easily. Using the theorem we have that this is clearly satisfied when $h = 1$. So assume that the theorem is true when Y is obtained by X by removing a sequence of $h - 1$ hooks. As $\{x_1^{(h)}, \ldots, x_k^{(h)}\}$ is obtained from X by removing $h - 1$ hooks we have that $x_{\pi_{h-1} \ldots \pi_1(i_1)}^{(h)} > x_{\pi_{h-1} \ldots \pi_1(i_2)}^{(h)}$ if and only if $i_1 > i_2$.

Now using the definition of π_h we have $x^{(h+1)}_{\pi_h\pi_{h-1}\ldots\pi_1(i_1)} > x^{(h+1)}_{\pi_h\pi_{h-1}\ldots\pi_1(i_1)}$ if and only if $x^{(h)}_{\pi_{h-1}\ldots\pi_1(i_1)} > x^{(h)}_{\pi_{h-1}\ldots\pi_1(i_2)}$ if and only if $i_1 > i_2$ and as by definition of π we also have that $x^{(h+1)}_{\pi(i_1)} > x^{(h+1)}_{\pi(i_2)}$ if and only if $i_1 > i_2$ we then need to have that $\pi = \pi_h \ldots \pi_1$ and so the corollary is proved. □

In particular when we apply this corollary to the case where $P^*(X) = \alpha$ and $P^*(Y) = \alpha_{(q)}$ and all hooks removed have length q it can be seen that $x^{(1)}_{\pi(i)}$ is the q-numbering of X (as beads on the same runner of the q-abacus cannot jump each other while removing q-hooks) and so π sends the natural numbering of X in the q-numbering of X, and so $\delta_q(X) = \delta_q(\alpha) = \text{sign}(\pi)$. So as $\delta_q(\alpha)$ by definition doesn't depend on the choice of hooks which are recursively removed from α we have that $(-1)^{\sum l_j}$ is also constant and so the sum of the leg-lengths of the hooks which are recursively removed from α to obtain $\alpha_{(q)}$ is also well defined up to a multiple of 2.

10 Weights and characters values

In this section we will define the q-weight of a permutation and show that if α is a partition of n, $\sigma \in S_n$ and $\chi^\alpha(\sigma) \neq 0$ then we need to have that $w_q(\sigma) \leq w_q(\alpha)$ for any positive integer q. After having done this we will give a formula that can be used to find $\chi^\alpha(\sigma)$ when $w_q(\sigma) = w_q(\alpha)$. This formula is a generalization of formula 2.7.25 from [2].

Definition 31 (q-weight of a permutation). *Let σ be a permutation and let b_i be the lengths of the cycles of σ which are divisible by q, counted with multiplicity. The q-weight of σ is given by*

$$w_q(\sigma) = \sum b_i/q.$$

So for example we have that $w_2((1,2)(3,4,5,6)(7,8,9)) = 1 + 2 = 3$.

Theorem 51. *Let $\alpha \vdash n$ and $\sigma \in S_n$ be such that for some q, $w_q(\sigma) > w_q(\alpha)$. Then $\chi^\alpha(\sigma) = 0$.*

Proof. Let $b_1, \ldots, b_h$ be the lengths of the cycles of σ which are divisible by q. We can write $b_i = q\gamma_i$, for some positive integers γ_i. By assumption we have that $\sum \gamma_i > w_q(\alpha)$. By recursively applying the Murnaghan-Nakayama formula if $\chi^\alpha(\sigma)$ was different from 0, we could find a partition β, which is obtained by α by recursively removing hooks of length $q\gamma_i$. By corollary 46 we would then have that β could be obtained by α by removing a sequence of $\sum \gamma_i$ hooks all of length q. As $\sum \gamma_i > w_q(\alpha)$ and we cannot recursively remove more then $w_q(\alpha)$ hooks of length q from α we have a contradiction and so we have that $\chi^\alpha(\sigma) = 0$ when $w_q(\sigma) > w_q(\alpha)$ for some q. □

We will now apply the results of the previous section to obtain a formula for $\chi^\alpha(\rho\pi)$, where $\alpha \vdash n$ has q-weight $w_q(\alpha) = w$, ρ has cycle partition $q\gamma$ where $\gamma \vdash w$ and ρ and π act on distinct elements of $\mathbf{n}$. We want to show that

$$\chi^\alpha(\rho\pi) = \delta_q(\alpha) f^{\alpha^{(q)}}(\gamma)\chi^{\alpha_{(q)}}(\pi) \tag{5}$$

where $f^{\alpha^{(q)}}(\gamma)$ only depends on γ and the q-quotient of α. In particular, as in the following we will also find a formula for $f^{\alpha^{(q)}}(\gamma)$, this gives a formula for $\chi^\alpha(\sigma)$, when $w_q(\sigma) = w_q(\alpha)$ as in this case we can write $\sigma = \rho\pi$, where ρ is the products of the cycles of σ with length divisible by q and π is the product of the other cycles of σ and we then have that ρ and π satisfy the assumption we have just defined for them.

Let β be obtained from α by removing recursively hooks of length $q\gamma_i$. As by corollary 46 we have that β can also be obtained by α by removing $\sum \gamma_i$ hooks all of length q and as $\sum \gamma_i = w = w_q(\alpha)$ we need to have that $\beta = \alpha_{(q)}$.

So if (i_h, j_h) are the nodes corresponding to the hooks removed at each step and $\gamma = (\gamma_1, \ldots, \gamma_k)$ we have by recursively applying the Murnaghan-Nakayama formula that

$$\chi^\alpha(\rho\pi) = \sum_{((i_1,j_1),\ldots(i_k,l_k))} (-1)^{l_{i_1,j_1}+\cdots+l_{i_k,j_k}} \chi^{\alpha_{(q)}}(\pi), \tag{6}$$

where l_{i_h,j_h} is the leg-length of the hook removed at step h. So in order to prove formula (5) it is enough to show that

$$\sum_{((i_1,j_1),\ldots(i_k,l_k))} (-1)^{l_{i_1,j_1}+\cdots+l_{i_k,j_k}} = \delta_q(\alpha) f^{\alpha^{(q)}}(\gamma) \tag{7}$$

for some $f^{\alpha^{(q)}}(\gamma)$ depending on $\alpha^{(q)}$ and γ only.

Using theorem 47 we have that the removal of any sequence of $k_i q$-hooks from α correspond bijectively to the removal of a sequence of k_i-hooks from $\alpha^{(q)}$. Let $((i_l, j_l))$ be a sequence of nodes of α and $((i'_l, j'_l))$ the corresponding sequence of nodes of $\alpha^{(q)}$, such that $h_{i_l,j_l} = q\gamma_i$ and $h_{i'_l,j'_l} = \gamma_i$ in the partition from which they are removed. If l_{i_l,j_l} and $l_{i'_l,j'_l}$ are the leg-lengths of these hooks (again in the partition from which they are removed), in order to prove equation (7) it is enough to prove that

$$(-1)^{l_{i_1,j_1}+\cdots+l_{i_k,j_k}} = \delta_q(\alpha)(-1)^{l_{i'_1,j'_1}+\cdots+l_{i'_k,j'_k}} \tag{8}$$

as in this case we would have that

$$\sum_{((i_1,j_1),\dots(i_k,l_k))} (-1)^{l_{i_1,j_1}+\dots+l_{i_k,j_k}} = \delta_q(\alpha) \sum_{((i'_1,j'_1),\dots(i'_k,l'_k))} (-1)^{l_{i'_1,j'_1}+\dots+l_{i'_k,j'_k}}$$

and it is clear that

$$f^{\alpha^{(q)}}(\gamma) = \sum_{((i'_1,j'_1),\dots(i'_k,l'_k))} (-1)^{l_{i'_1,j'_1}+\dots+l_{i'_k,j'_k}}$$

only depends on $\alpha^{(q)}$ and γ.

Theorem 52. *If (i_l, j_l) and (i'_l, j'_l) are defined as before we have that*

$$(-1)^{l_{i_1,j_1}+\dots+l_{i_k,j_k}} = \delta_q(\alpha)(-1)^{l_{i'_1,j'_1}+\dots+l_{i'_k,j'_k}}.$$

Proof. Let $X = \{x_1, \dots, x_m\}$ be a β-set for α such that $q||X|$. We can assume that the x_i are increasing.

In order to prove the theorem we will use corollary 50. Let $\alpha^{(q)} = (\alpha_0, \dots, \alpha_{q-1})$. Let π be given by the corollary for recursively removing the hooks R_{i_l,j_l} from X and let π' be the permutation that sends the natural numbering of X in the q-numbering of X. By the notes after corollary 50 we have that π' is the permutation given by the same corollary for removing recursively any maximal sequence of q-hooks from X. For each $0 \le i < q$ let J_i be the set indexes s for which $x_s = i + qt$ for some $t \in \mathbb{N}$. Let $X_i = \{(x_s - i)/q : s \in J_i\}$ and let π_i be given by corollary 50 by removing from X_i those hooks of the sequence $(R_{i'_l,j'_j})$ of hooks of $\alpha^{(q)}$ which belong to α_i, where the elements of X_i are labeled by the indexes which belong to J_i (just let $X_i = \{x'_s : s \in J_i\}$ with $x'_s = (x_s - i)/q$).

We want to prove that

$$\pi = \pi'\pi_0 \cdots \pi_{q-1}$$

as then we would easily have by corollary 50 that

$$(-1)^{l_{i_1,j_1}+\dots+l_{i_k,j_k}} = \delta_q(\alpha)(-1)^{l_{i'_1,j'_1}+\dots+l_{i'_k,j'_k}},$$

as each hook $R_{i'_l,j'_l}$ of $\alpha^{(q)}$ belongs to exactly one of the α_i.

Notice that as the π_i act on distinct numbers they all commute with each other, so it doesn't matter which order we multiply them in. To show that $\pi = = \pi'\pi_0 \cdots \pi_{q-1}$ we need to show that if we write $X = \{x_1^{(1)}, \dots, x_m^{(1)}\}$ with the natural numbering (we just have that $x_s^{(1)} = x_s$ for each s) and

$Y = \{x_1^{(k+1)}, \ldots, x_m^{(k+1)}\}$ is obtained by X as in the text of corollary 50 for the sequence of hooks R_{i_l,j_l} we have that

$$x^{(k+1)}_{\pi'\pi_0\cdots\pi_{q-1}(i)} \leq x^{(k+1)}_{\pi'\pi_0\cdots\pi_{q-1}(j)} \text{ if and only if } i \leq j.$$

First as we have already noticed we have that $P^*(Y) = \alpha_{(q)}$ as we are removing from α hooks of length $\gamma_l q$ and $\sum \gamma_l = w_q(\alpha)$. Also as all hooks that we are removing have lengths divisible by q we have that if $x_i^{(1)}$ is on the h-th runner of the q-abacus of X then $x_i^{(k+1)}$ is on the h-th runner of the q-abacus of Y. So we will first show that if $x_i^{(1)}$ and $x_j^{(1)}$ are on the same runner of the q-abacus of X then $x^{(k+1)}_{\pi_0\cdots\pi_{q-1}(i)} \leq x^{(k+1)}_{\pi_0\cdots\pi_{q-1}(j)}$ if and only if $i \leq j$. But as when removing a hook from α a bead jumps an other bead on the same runner if and only if in when removing the corresponding hook in $\alpha^{(q)}$ the bead corresponding to the first one jumps the one corresponding to the second one, we have that if $x_i^{(1)}$ and $x_j^{(1)}$ are on the h-th runner then $x^{(k+1)}_{\pi_h(i)} \leq x^{(k+1)}_{\pi_h(j)}$ if and only if $i \leq j$. Also as if $s \neq h$ we have that π_s fixes all indexes of the beads on the h-th runner and π_h acts only on these indexes we need to have that

$$x^{(k+1)}_{\pi_0\cdots\pi_{q-1}(i)} \leq x^{(k+1)}_{\pi_0\cdots\pi_{q-1}(j)} \text{ if and only if } i \leq j$$

when $x_i^{(1)}$ and $x_j^{(1)}$ are on the same runner of the q-abacus of X and so it follows by the definition of π' that

$$x^{(k+1)}_{\pi'\pi_0\cdots\pi_{q-1}(i)} \leq x^{(k+1)}_{\pi'\pi_0\cdots\pi_{q-1}(j)} \text{ if and only if } i \leq j$$

and then the theorem is proved.

□

Lemma 53. *If $\delta = (\delta_1, \ldots, \delta_k)$ is a partition of $w_q(\alpha)$, there is a bijection between the set of sequences $((i'_1, j'_1), \ldots, (i'_k, j'_k))$ of nodes of $\alpha^{(q)}$ corresponding to hooks of lengths $(\delta_1, \ldots, \delta_k)$ which are recursively from δ and q-tuples of nodes $\left((i_{s_1}, j_{s_1}), \ldots, \left(i_{s_{k_s}}, j_{s_{k_s}}\right)\right)$ corresponding to hooks of lengths $\left(\delta_{s_1}, \ldots, \delta_{s_{k_s}}\right)$ which are recursively removed from α_s, the s-th component of $\alpha^{(q)}$ such that*

$$\left\{\delta_{0_1}, \ldots, \delta_{0_{k_0}}\right\} \cup \ldots \cup \left\{\delta_{q-1_1}, \ldots, \delta_{q-1_{k_{q-1}}}\right\} = \{1, \ldots, k\}$$

is a disjoint union and for each $0 \leq s \leq q-1$ we have that $\delta_{s_1} + \ldots + \delta_{s_{k_s}} = |\alpha_s|$.

In particular we have that this bijection satisfies that $(i'_l, j'_l) \in \alpha_s$ if and only if $l = s_t$ for some t and in this case we have that $(i'_l, j'_l) = (i_{s_t}, j_{s_t})$. In particular if $l_{i'_l,j'_l}$ and $l_{i_{s_t},j_{s_t}}$ are the leg-length of the corresponding hooks in the partition from which they are removed, we have that $l_{i'_l,j'_l} = l_{i_{s_t},j_{s_t}}$.

Proof. This bijection is given by the second part of the lemma. The fact that $\delta_{s_1}+\ldots+\delta_{s_{k_s}}=|\alpha_s|$ is due to the fact that $\delta\vdash w_q(\alpha)$, so that $\delta_1+\ldots+\delta_k=|\alpha_0|+\ldots+|\alpha_{q-1}|$ and so whenever we recursively remove a sequence of hooks of lengths $(\delta_1,\ldots,\delta_k)$ the sum of the hook-lengths of the hooks which are removed from α_s needs to be $|\alpha_s|$. □

Lemma 54. *If $\alpha,\beta\vdash n$ we have that*

$$\chi^\alpha_\beta=\sum_{((i_1,j_1),\ldots,(i_k,j_k))}(-1)^{\sum_l l_{i_l,j_l}}$$

where k is such that $\beta_k>0$ and $\beta_{k+1}=0$, the nodes (i_l,j_l) are such that $h_{i_l,j_l}=\beta_l$ in the partition obtained by recursively removing from α the rim-hooks corresponding to the first $l-1$ nodes and l_{i_l,j_l} is the leg-length of the node (i_l,j_l) in the same partition.

Proof. This lemma is actually an easy corollary of the Murnaghan-Nakayama formula. □

Theorem 55. *If α, ρ, π and γ are defined as at the beginning of this section we have that*

$$\chi^\alpha(\rho\pi)=\delta_q(\alpha)\left(\sum_{\left(0_1,\ldots,0_{k_0},\ldots,q-1_1,\ldots,q-1_{k_{q-1}}\right)}\prod_{i=0}^{q-1}\chi^{\alpha_i}_{\left(\gamma_{i_1},\ldots,\gamma_{i_{k_i}}\right)}\right)\chi^{\alpha^{(q)}}(\pi)$$

where the sequences $(s_1,\ldots,s_{k_s})$, $0\le s<q$, respect the same properties as in lemma 53 and $\alpha^{(q)}=(\alpha_0,\ldots,\alpha_{q-1})$.

Proof. By the equation (6) and theorem 52 we have that

$$\chi^\alpha(\rho\pi)=\delta_q(\alpha)f^{\alpha^{(q)}}(\gamma)\chi^{\alpha^{(q)}}(\pi)$$

where

$$f^{\alpha^{(q)}}(\gamma)=\sum_{((i'_1,j'_1),\ldots(i'_k,l'_k))}(-1)^{l_{i'_1,j'_1}+\cdots+l_{i'_k,j'_k}}$$

where the sequences $((i'_1,j'_1),\ldots(i'_k,l'_k))$ satisfy the same properties as in the previous part of this section. Also by applying lemma 53 we have that

$$f^{\alpha^{(q)}}(\gamma)=\sum_{\left(0_1,\ldots,q-1_{k_{q-1}}\right)}\sum_{\left(\left(i_{0_1},j_{0_1}\right),\ldots,\left(i_{(q-1)_{k_{q_1}}},j_{(q-1)_{k_{q_1}}}\right)\right)}(-1)^{\sum l_{i_{l_m},j_{l_m}}}$$

and by lemma 54 we have that

$$f^{\alpha^{(q)}}(\gamma) = \sum_{\left(0_1,\dots,(q-1)_{k_{q-1}}\right)} \prod \chi^{\alpha_j}_{\left(\gamma_{i_1},\dots,\gamma_{i_{h_i}}\right)}$$

from which the theorem follows. □

11 The hook-formula

Theorem 56 (Hook formula). *If $\alpha \vdash n$ we have that the degree of $[\alpha]$, the irreducible representation of S_n indexed by α, is*

$$f^\alpha = \frac{n!}{\prod_{(i,j)\in\alpha} h^\alpha_{i,j}}.$$

Proof. By definition of f^α we have that $f^\alpha = \chi^\alpha(1)$. Using the branching rule and as α has a 1-hook on row i if and only if $\alpha_i > \alpha_{i+1}$, that is if and only if α^{i-}, as defined at the beginning of section 8, is a partition, we have that

$$f^\alpha = \chi^\alpha(1) = \sum_i \chi^{\alpha^{i-}}(1) = \sum_i f^{\alpha^{i-}},$$

where the summation is taken over those i such that $\alpha_i > \alpha_{i+1}$.

By lemma 43 we have that

$$\prod_{(i,j)\in\alpha} h^\alpha_{i,j} = \prod_{i=1}^h \frac{h^\alpha_{i,1}!}{\prod_{i'>i}(h^\alpha_{i,1} - h^\alpha_{i',1})}.$$

As when $\alpha_i > \alpha_{i+1}$ we have that

$$h^{\alpha^{i-}}_{j,1} = \begin{cases} h^\alpha_{j,1} & j \neq i \\ h^\alpha_{i,1} - 1 & j = i \end{cases}$$

we need to have that have that

$$\begin{aligned}
\frac{1}{\prod_{(k,j)\in\alpha^{i-}} h^{\alpha^{i-}}_{k,j}} &= \frac{\prod_{1\le k<k'\le h}(h^{\alpha^{i-}}_{k,1} - h^{\alpha^{i-}}_{k',1})}{\prod_{k=1}^h h^{\alpha^{i-}}_{k,1}!} \\
&= \frac{\prod_{1\le k<k'\le h, k,k'\neq i}(h^\alpha_{k,1} - h^\alpha_{k',1})}{\prod_{k\neq i} h^\alpha_{k,1}!} \frac{\prod_{k<i}(h^\alpha_{k,1} - h^\alpha_{i,1}+1)\prod_{k>i}(h^\alpha_{i,1} - h^\alpha_{k,1}-1)}{(h^\alpha_{i,1}-1)!} \\
&= h^\alpha_{i,1} \frac{\prod_{k'>k}(h^\alpha_{k,1} - h^\alpha_{k',1})}{\prod_{k=1}^h h^\alpha_{k,1}!} \prod_{k<i} \frac{h^\alpha_{k,1} - h^\alpha_{i,1}+1}{h^\alpha_{k,1} - h^\alpha_{i,1}} \prod_{k>i} \frac{h^\alpha_{i,1} - h^\alpha_{k,1}-1}{h^\alpha_{i,1} - h^\alpha_{k,1}} \\
&= h^\alpha_{i,1} \frac{\prod_{k'>k}(h^\alpha_{k,1} - h^\alpha_{k',1})}{\prod_{k=1}^h h^\alpha_{k,1}!} \prod_{k\neq i} \frac{h^\alpha_{i,1} - h^\alpha_{k,1}-1}{h^\alpha_{i,1} - h^\alpha_{k,1}}.
\end{aligned}$$

If for some $i \leq h$ we have that $\alpha_i = \alpha_{i+1}$ then $h^\alpha_{i,1} = h^\alpha_{i+1,1} + 1$, so that $h^\alpha_{i,1} - h^\alpha_{i+1,1} - 1 = 0$ and as in this case we need to have that $i < h$, as $\alpha_h > 0$ and $\alpha_{h+1} = 0$, we have that $h^\alpha_{i,1} - h^\alpha_{i+1,1} - 1$ appears in $\prod_{k\neq i}(h^\alpha_{i,1} - h^\alpha_{k,1} - 1)$, so that

$$h^\alpha_{i,1}\frac{\prod_{k'>k}(h^\alpha_{k,1} - h^\alpha_{k',1})}{\prod_{k=1}^h h^\alpha_{k,1}!}\prod_{k\neq i}\frac{h^\alpha_{i,1} - h^\alpha_{k,1} - 1}{h^\alpha_{i,1} - h^\alpha_{k,1}} = 0$$

in this case.

As the hook-formula clearly holds when $n = 1$ we can proceed by induction and assume that it holds for any partition of $n-1$. Then we have that

$$\begin{aligned} f^\alpha &= \textstyle\sum_{i:\alpha_i>\alpha_{i+1}} f^{\alpha^{i-}} \\ &= \textstyle\sum_{i:\alpha_i>\alpha_{i+1}} h^\alpha_{i,1}(n-1)!\frac{\prod_{k'>k}(h^\alpha_{k,1}-h^\alpha_{k',1})}{\prod_{k=1}^h h^\alpha_{k,1}!}\prod_{k\neq i}\frac{h^\alpha_{i,1}-h^\alpha_{k,1}-1}{h^\alpha_{i,1}-h^\alpha_{k,1}} \\ &= \textstyle\sum_{i=1}^h h^\alpha_{i,1}(n-1)!\frac{\prod_{k'>k}(h^\alpha_{k,1}-h^\alpha_{k',1})}{\prod_{k=1}^h h^\alpha_{k,1}!}\prod_{k\neq i}\frac{h^\alpha_{i,1}-h^\alpha_{k,1}-1}{h^\alpha_{i,1}-h^\alpha_{k,1}} \\ &= \textstyle n!\frac{\prod_{k'>k}(h^\alpha_{k,1}-h^\alpha_{k',1})}{\prod_{k=1}^h h^\alpha_{k,1}!}\frac{1}{n}\sum_{i=1}^h h^\alpha_{i,1}\prod_{k\neq i}\frac{h^\alpha_{i,1}-h^\alpha_{k,1}-1}{h^\alpha_{i,1}-h^\alpha_{k,1}} \\ &= \textstyle\frac{n!}{\prod_{(i,j)\in\alpha}h^\alpha_{i,j}}\frac{1}{n}\sum_{i=1}^h h^\alpha_{i,1}\prod_{k\neq i}\frac{h^\alpha_{i,1}-h^\alpha_{k,1}-1}{h^\alpha_{i,1}-h^\alpha_{k,1}}. \end{aligned}$$

In order to prove the hook formula it is then enough to prove that

$$\sum_{i=1}^h h^\alpha_{i,1}\prod_{k\neq i}\frac{h^\alpha_{i,1} - h^\alpha_{k,1} - 1}{h^\alpha_{i,1} - h^\alpha_{k,1}} = n.$$

Let $g(x) = \prod_{k=1}^h(x - h^\alpha_{k,1})$. Then for any $1 \leq i \leq h$ we have that

$$g(h^\alpha_{i,1} - 1) = \prod_{k=1}^h(h^\alpha_{i,1} - 1 - h^\alpha_{k,1}) = -\prod_{k\neq i}(h^\alpha_{i,1} - 1 - h^\alpha_{k,1})$$

and

$$g'(h^\alpha_{i,1}) = \sum_{j=1}^h\prod_{k\neq j}(h^\alpha_{i,1} - h^\alpha_{k,1}) = \prod_{k\neq i}(h^\alpha_{i,1} - h^\alpha_{k,1}),$$

so that

$$\sum_{i=1}^h h^\alpha_{i,1}\prod_{k\neq i}\frac{h^\alpha_{i,1} - h^\alpha_{k,1} - 1}{h^\alpha_{i,1} - h^\alpha_{k,1}} = \sum_{i=1}^h\frac{-h^\alpha_{i,1}g(h^\alpha_{i,1} - 1)}{g'(h^\alpha_{i,1})}.$$

As $\sum_{i=1}^h h^\alpha_{i,1} = \sum_{i=1}^h \alpha_i + h - i = n + \sum_{j=0}^{h-1} j = n + \binom{h}{2}$, in order to prove the hook-formula it is enough to prove the following lemma

Lemma 57. *If $x_1, \dots, x_r$ are non-zero and pairwise distinct and*

$$g(x) = \prod_{i=1}^{r} (x - x_i)$$

we have that

$$\sum_{i=1}^{r} -x_i \frac{g(x_i - 1)}{g'(x_i)} = \sum_{i=1}^{r} x_i - \binom{r}{2}.$$

Proof. Let $a = \sum_{i=1}^r x_i$ and $b = \sum_{1 \le i < j \le r} x_i x_j$. It is easy to see that

$$g(x) = x^r - a x^{r-1} + b x^{r-2} + f(x)$$

where $f(x)$ is a polynomial of degree $\le r-3$ (here we don't need the x_i to be non-pairwise different). As $g(x-1) = \prod_{i=1}^r (x - (x_i + 1))$ and $\sum_{i=1}^r (x_i + 1) = r + \sum_{i=1}^r x_i = a + r$ and

$$\sum_{1 \le i < j \le r} (x_i + 1)(x_j + 1) = b + \sum_{j \ne i} x_i + \sum_{i \ne j} x_j + \sum_{i=1}^{r} (r - i) = b + (r-1)a + \binom{r}{2},$$

we have by the formula we just found that

$$g(x-1) = x^r - (a+r)x^{r-1} + \left(b + (r-1)a + \binom{r}{2} \right) x^{r-2} + f_1(x)$$

for some f_1 of degree at most $r-3$. We can see that

$$x^2 g(x-1) = \left(x^2 - rx + \binom{r}{2} - a \right) g(x) + h(x)$$

where $h(x)$ is a polynomial of degree $\le r-1$. When $x = 0$ this gives $\left(\binom{r}{2} - a\right) g(0) + h(0) = 0$ and as $g(0) = \prod_i (-x_i) \ne 0$ as the x_i are non-zero, we have that $h(0)/g(0) = \binom{r}{2} - a = \binom{r}{2} - \sum_i x_i$. Also as for $1 \le i \le r$ we have that $g(x_i) = 0$ we need to have that

$$h(x_i) = x_i^2 g(x_i - 1) = \sum_{k=1}^{r} x_k^2 g(x_k - 1) \prod_{j \ne k} \frac{x_i - x_j}{x_k - x_j}$$

as the x_j are pairwise different and when $k \ne i$ then $(x_i - x_i)/(x_k - x_i) = 0$ appears in the product. As the x_i are pairwise distinct and as both h and

$$\sum_{k=1}^{r} x_k^2 g(x_k - 1) \prod_{j \ne k} \frac{x - x_j}{x_k - x_j} = \sum_{k=1}^{r} \frac{x_k^2 g(x_k - 1)}{g'(x_k)} \prod_{j \ne k} (x - x_j)$$

are polynomial of degree at most $r-1$ and they take the same value on all the x_i, which are r different numbers, as $g'(x_k) = \sum_{i=1}^{k}\prod_{j\neq i}(x_k - x_j) = \prod_{j\neq k}(x_k - x_j)$, we need to have that they are the same polynomial. Now we have that

$$\begin{aligned} h(0) &= \textstyle\sum_{k=1}^{r} \frac{x_k^2 g(x_k-1)}{g'(x_k)} \prod_{j\neq k}(-x_j) = -\sum_{k=1}^{r} \frac{x_k g(x_k-1)}{g'(x_k)} \prod_{j=1}^{r}(-x_j) \\ &= \textstyle -\sum_{k=1}^{r} \frac{x_k g(x_k-1)}{g'(x_k)} g(0) \end{aligned}$$

and so we have that

$$-\sum_{k=1}^{r} \frac{x_k g(x_k-1)}{g'(x_k)} = \frac{h(0)}{g(0)} = \binom{r}{2} - \sum_{i=1}^{r} x_i$$

and so $\sum_{k=1}^{r} \frac{x_k g(x_k-1)}{g'(x_k)} = \sum_{i=1}^{r} x_i - \binom{r}{2}$ as we wanted to prove. □

Having proven this lemma we then also have that the hook-formula is true. □

12 p-vanishing classes

In this section we want to show that for any prime p, there are some conjugacy classes of S_n on which all irreducible characters of degree divisible by p vanish. The first part of this section is based on part 4 of [3].

Definition 32 (p-vanishing conjugacy class). *A conjugacy class of S_n is called p-vanishing if all irreducible characters of degree divisible by p vanish on it.*

A partition is p-vanishing if it is the cycle partition of a p-vanishing conjugacy class of S_n and an element $\pi \in S_n$ is p-vanishing if its conjugacy class is p-vanishing.

Definition 33 (p-adic decomposition). *Let n be a positive integer and p be a prime. The p-adic decomposition of n is given by*

$$n = a_0 + pa_1 + \ldots + p^k a_k$$

with $0 \leq a_i \leq p-1$, $a_k \neq 0$.

As the $0 \leq a_i \leq p-1$ and $a_k \neq 0$ it is easy to see that we can write n like $a_0 + pa_1 + \ldots + p^k a_k$ in a unique way, so that the p-adic decomposition of n is unique. Throughout this section we will use the notation that the a_i are the coefficients of the p-adic decomposition of n.

Definition 34 (p-adic type conjugacy class)**.** *A partition $(c_1, \ldots, c_h)$ of n is of p-adic type if it satisfies*

$$\sum_{\substack{j:p^i|c_j,\\ p^{i+1}\not| c_j}} c_j/p^i = a_i$$

for every $i \geq 1$ ($a_i = 0$ for $i > k$).

A conjugacy class or an element of S_n are of p-adic type if their cycle partition is of p-adic type.

For example if $n = 11$ we have that $n = 8+2+1$ and so the only partition of n of 2-adic type is $(8, 2, 1)$, while as $n = 9 + 2 \cdot 1$ the partitions of n of 3-adic type are $(9, 2)$ and $(9, 1, 1)$.

If $(c_1, \ldots, c_h)$ is of p-adic type and we write $c_j = p^{h_j} d_j$ with $p \not| d_j$ (which we can always do for $j \leq h$ if we assume $c_h > 0$) we have by the definition of a partition of p-adic type that $\sum_{j:h_j=i} d_j = a_i \leq p - 1$ and so all $d_j \leq p - 1$. So we easily have that $(c_1, \ldots, c_h)$ is of p-adic type if and only if we can write it as

$$\left(p^k d_{k_1}, \ldots, p^k d_{k_{h_k}}, p^{k-1} d_{k-1_1}, \ldots, p^k d_{k-1_{h_{k-1}}}, \ldots, d_{0_1}, \ldots, d_{0_{h_0}}\right)$$

where for each $1 \leq 1 \leq k$ we have that

$$\left(d_{i_1}, \ldots, d_{i_{h_i}}\right) \vdash a_i$$

and $d_{i_{h_i}} > 0$ (let $i_{h_i} = 0$ if $a_i = 0$).

Lemma 58. *$\pi \in S_n$ is of p-adic type if and only if for any i we have that $w_{p^i}(\pi) = a_i + a_{i+1}p + \ldots + a_k p^{k-i}$.*

Proof. If $w_{p^i}(\pi) = a_i + a_{i+1}p + \ldots + a_k p^{k-i}$ for any i we have that the sum of the lengths of the cycles of π divisible by p^j but not by p^{j+1} is equal to

$$p^j w_{p^j}(\pi) - p^{j+1} w_{p^{j+1}}(\pi) = p^j a_j$$

and so π is of p-adic type in this case.

If instead π is of p-adic type and $\alpha(\pi) = (c_1, \ldots, c_h)$ we have that

$$w_{p^i}(\pi) = \sum_{l \geq i} \sum_{\substack{j:p^l|c_j,\\ p^{l+1}\not| c_j}} c_j/p^i = \sum_{l \geq i} p^{l-i} a_l = a_i + \ldots + a_k p^{k-i}$$

and so the lemma is proved. $\square$

The next theorem, which will be proved after some lemmas, is a generalization of theorem 4.1 of [3].

Theorem 59. *Any p-adic type conjugacy class of S_n is p-vanishing.*

Definition 35. *If α is a partition define*

$$\overline{\alpha_i} = w_{p^i}(\alpha) - p w_{p^{i+1}}(\alpha).$$

Lemma 60. *If $\overline{\alpha_i}$ are as defined before we have that $\overline{\alpha_i} \in \mathbb{N}$ for each i and*

$$w_{p^j}(\alpha) = \sum_{i \geq j} p^{i-j} \overline{\alpha_i}.$$

Proof. As by corollary 46 we have that removing a p^{i+1} hook is equivalent to removing a certain sequence of p hooks all of length p^i, it follows by the definition of $w_{p^i}(\pi)$ and $w_{p^{i+1}}(\pi)$ that each $\overline{\alpha_i} \in \mathbb{N}$.

As $\overline{\alpha_i} = 0$ for $i > k$ as when $i > k$ we have that both $w_{p^i}(\pi)$ and $w_{p^{i+1}}(\pi)$ are 0, we have that

$$w_{p^j}(\pi) = w_{p^j}(\pi) - p^{k-j+1} w_{p^{k+1}}(\pi) = \overline{\alpha_j} + \ldots + p^{k-j}\overline{\alpha_k} = \sum_{i \geq j} p^{i-j} \overline{\alpha_i}$$

as we wanted to prove. □

The next lemma is proposition 4.6 of [3].

Lemma 61. *Let $\alpha \vdash n$.*

i) If $\overline{\alpha_k} \neq a_k$ we cannot recursively remove a_k hooks of length p^k from α.

ii) If $\overline{\alpha_k} = a_k$ we can recursively remove a_k hooks of length p^k from α. The resulting partition is $\alpha_{(p^k)}$ and we have that $\overline{\left(\alpha_{(p^k)}\right)_i} = \overline{\alpha_i}$ for $0 \leq i < k$ and $\overline{\left(\alpha_{(p^k)}\right)_k} = 0$.

Proof. As $\overline{\alpha_i} = 0$ for $i > k$ we have by the previous lemma that $\overline{\alpha_k} = w_{p^k}(\alpha)$. So in order to prove $i)$ it is enough to show that for any α, $\overline{\alpha_k} \leq a_k$, as then when $\overline{\alpha_k} \neq a_k$ we actually need to have that $w_{p^k}(\alpha) = \overline{\alpha_k} < a_k$ and so in this case we cannot recursively remove a_k hooks of length p^k from α. As $n = a_0 + pa_1 + \ldots + p^k a_k$ is the p-adic decomposition of n we have that all $a_i < p$ and so we need to have that

$$a_0 + pa_1 + \ldots + p^{k-1}a_{k-1} \leq (p-1)\left(1 + p + \ldots + p^{k-1}\right) = p^k - 1 < p^k,$$

that is we have that $n < (1 + a_k)p^k$. As all $\overline{\alpha_k} \geq 0$ and by the previous lemma with $j = 0$ we need to have that $n = \sum_i \overline{\alpha_i} p^i$ (as $w_1(\beta) = |\beta|$ for any partition β) we in particular need to have that $\overline{\alpha_k}p^k \leq n < (1+a_k)p^k$ and so we have that $\overline{\alpha_k} \leq a_k$ for any $\alpha \vdash n$ and so part i) is proved.

Assume now that $\overline{\alpha_k} = a_k$. Then we have that $a_k = w_{p^k}(\alpha)$, again as $w_{p^{k+1}}(\alpha) = 0$, and so we can recursively removed a_k hooks of length p^k from α and the resulting partition needs to be $\alpha_{(p^k)}$. By corollary 48 and as for any $0 \leq j \leq k$ we have that the removal of a p^k-hook corresponds to the removal of p^{k-j} hooks of length p^j, we have that for any $0 \leq j \leq k$

$$w_{p^j}\left(\alpha_{(p^k)}\right) = w_{p^j}(\alpha) - p^{k-j}w_{p^k}(\alpha).$$

So we have that

$$w_{p^k}\left(\alpha_{(p^k)}\right) = 0$$

(which we could already have obtained as $\alpha_{(p^k)}$ cannot contain any hook of length p^k) and for $0 \leq i < k$ we have that

$$\begin{aligned}
\overline{\left(\alpha_{(p^k)}\right)_i} &= w_{p^i}\left(\alpha_{(p^k)}\right) - pw_{p^{i+1}}\left(\alpha_{(p^k)}\right) \\
&= w_{p^i}(\alpha) - p^{k-i}w_{p^k}(\alpha) - pw_{p^i+1}(\alpha) + pp^{k-j}w_{p^k}(\alpha) \\
&= w_{p^i}(\alpha) - pw_{p^i+1}(\alpha) \\
&= \overline{\alpha_i}.
\end{aligned}$$

□

The results that we have proven from the definition of the $\overline{\alpha_i}$ until now hold for any p, not necessarily for p a prime, even if we will be using them only in the case when p is prime.

Lemma 62. *If m is maximal such that $p^m|n!$ we have that*

$$m = (n - a_0 - a_1 - \ldots - a_k)/(p-1).$$

Proof. As $n = a_0 + pa_1 + \ldots + p^k a_k$ is the p-adic decomposition of n it is easy to see that for each $1 \leq i \leq k$ the number of numbers between 1 and n which are divisible by p^i is equal to $[n/p^i] = (n - a_0 - pa_1 - \ldots - p^{i-1}a_{i-1})/p^i$, where for any real number x, $[x]$ is the largest integer not bigger than x. So we have that

$$\begin{aligned}
m &= \textstyle\sum_{i=1}^k (n - a_0 - pa_1 - \ldots - p^{i-1}a_{i-1})/p^i \\
&= \textstyle\sum_{i=1}^k p^{k-i}(n - a_0 - pa_1 - \ldots - p^{i-1}a_{i-1})/p^k \\
&= \textstyle\sum_{i=1}^k p^{k-i}n/p^k - \sum_{j=0}^k \sum_{i=j+1}^k p^{k-i}p^j a_j/p^k \\
&= n(\textstyle\sum_{i=0}^{k-1} p^i)/p^k - \sum_{j=0}^k a_j(\sum_{i=0}^{k-j-1} p^i)/p^{k-j} \\
&= (n - a_0 - a_1 - \ldots - a_k)/(p-1) - n/(p^k(p-1)) + \textstyle\sum_{j=0}^k a_j/(p^{k-j}(p-1)) \\
&= (n - a_0 - a_1 - \ldots - a_k)/(p-1) - (n - a_0 - pa_1 - \ldots - a_k p^k)/(p^k(p-1)) \\
&= (n - a_0 - a_1 - \ldots - a_k)/(p-1)
\end{aligned}$$

as for any h we have that

$$\left(1+p+\ldots+p^{h-1}\right)(p-1)=p^h-1$$

and so

$$\left(1+p+\ldots+p^{h-1}\right)/p^h=\left(p^h-1\right)/\left(p^h(p-1)\right)=1/(p-1)-1/\left(p^h(p-1)\right).$$

□

Lemma 63. *If m is maximal such that $p^m|\prod_{(i,j)\in\alpha}h^\alpha_{i,j}$ we have that*

$$m=(n-\overline{\alpha_0}-\overline{\alpha_1}-\ldots-\overline{\alpha_k})/(p-1).$$

Proof. By theorem 47 we have that the number of hooks of α with length divisible by p^j is equal to $w_{p^j}(\alpha)$. By lemma 60 we then have that the number of hooks with length divisible by p^j is equal to $\sum_{i\geq j}\overline{\alpha_i}p^{i-j}$. Also as $\overline{\alpha_i}=0$ for $i>k$ we actually have that this number is equal to $\sum_{i=j}^k\overline{\alpha_i}p^{i-j}$. In particular, as no hook of α can be longer than n, and so no hook can be divisible by p^j for $j>k$, we have that

$$\begin{aligned}
m &= \textstyle\sum_{j=1}^k\sum_{i=j}^k\overline{\alpha_i}p^{i-j}\\
&= \textstyle\sum_{i=1}^k\overline{\alpha_i}\sum_{j=1}^i p^{i-j}\\
&= \textstyle\sum_{i=1}^k\overline{\alpha_i}\sum_{l=0}^{i-1} p^{l}\\
&= \textstyle\sum_{i=1}^k\overline{\alpha_i}(p^i-1)/(p-1)\\
&= \left(\textstyle\sum_{i=1}^k\overline{\alpha_i}p^i-\overline{\alpha_i}\right)/(p-1)\\
&= (n-\overline{\alpha_0}-\overline{\alpha_1}-\ldots-\overline{\alpha_k})/(p-1)-\left(n-\overline{\alpha_0}-p\overline{\alpha_1}-\ldots-p^k\overline{\alpha_k}\right)/(p-1)\\
&= (n-\overline{\alpha_0}-\overline{\alpha_1}-\ldots-\overline{\alpha_k})/(p-1)
\end{aligned}$$

again as for any h we have that $\left(1+p+\ldots+p^{h-1}\right)(p-1)=p^h-1$ and as $n=\overline{\alpha_0}+p\overline{\alpha_1}+\ldots+p^k\overline{\alpha_k}$ by lemma 60 with $j=0$. □

Lemma 64. *If $\alpha\vdash n$ and the coefficients $\overline{\alpha_i}$ are as in definition 35 we have that the maximal m for which p^m divides the degree of χ^α, the irreducible representation of S_n indexed by α, is*

$$\left(\sum_{i\geq 0}\overline{\alpha_i}-\sum_{i\geq 0}a_i\right)/(p-1).$$

Proof. This follows easily from lemmas 62 and 63 and from the hook formula.

□

Proof of theorem 59. Assume that p divides the degree of χ^α and that π is of p-adic type. By lemma 64 we need to have that

$$\left(\sum_{i\geq 0}\overline{\alpha_i}-\sum_{i\geq 0}a_i\right)/(p-1)\neq 0.$$

In particular there exists some j for which $\overline{\alpha_j}\neq a_j$. Let i be maximal such that $\overline{\alpha_i}\neq a_i$. Such an i exists as $\overline{\alpha_j}=a_j=0$ for $j>k$. By recursively applying lemma 61 we have that $\overline{\alpha_i}<a_i$, in particular, as π is of p-adic type, by lemmas 58 and 60 we have that

$$w_{p^i}(\alpha)=\overline{\alpha_i}+\overline{\alpha_{i+1}}p+\ldots+\overline{\alpha_k}p^{k-i}<a_i+a_{i+1}p+\ldots+a_kp^{k-i}=w_{p^i}(\pi),$$

as $\overline{\alpha_j}=a_j$ for $j>i$ and so by theorem 51 we have that $\chi^\alpha(\pi)=0$ and so, as this holds for any χ^α of degree divisible by p, we have that any p-adic type conjugacy class is also p-vanishing. □

Until now in this section we have been proving that conjugacy classes of p-adic type are also p-vanishing. In the next part we will try to classify p-vanishing conjugacy classes. This work has been originated on a question of Navarro about which conjugacy classes are 2-vanishing. In order to do this we will study partitions and see which partitions are p-vanishing.

Lemma 65. *If $\alpha\vdash n$ is such that $\overline{\alpha_i}\neq a_i$ for some i, where the $\overline{\alpha_i}$ are defined as in definition 35, then $\sum_i\overline{\alpha_i}\neq\sum_i a_i$. In particular in this case p divides the degree of χ^α.*

Proof. The last part follows immediately from lemma 64 and the first part of the lemma, so we only need to prove the first part. Also in order to prove the first part of the theorem it is enough to prove that if $n=\sum_{i\geq 0}b_ip^i$, where all $b_i\in\mathbb{N}$ and that if $\sum_i b_i\leq\sum_i a_i$ then $b_i=a_i$ for all i, as then we would have that if $\sum_i\overline{\alpha_i}=\sum_i a_i$ then $\overline{\alpha_i}=a_i$ for all i. Also it is easy to see that by definition of k whenever $\sum_i b_ip^i=n$ and $b_i\in\mathbb{N}$ for each i we need to have that $b_i=0$ for all $i>k$.

The fact that $b_i=a_i$ for all i if $n=\sum_{i\geq 0}b_ip^i$, all $b_i\in\mathbb{N}$ and $\sum_i b_i\leq\sum_i a_i$ clearly holds when $k=0$. So assume that it holds for $k-1$ and that $\sum_i b_ip^i=n=\sum_i a_ip^i$, $\sum_i b_i\leq\sum_i a_i$ and $b_i\in\mathbb{N}$ for each i. As $n-a_0$ and $n-b_0$ are both divisible by p we have that b_i and a_i are equivalent mod p. Also as $0\leq a_0<p$ and $b_0\geq 0$ we then need to have that $b_0=a_0+pc$, for some $c\in\mathbb{N}$. Let

$$b_i'=\begin{cases}b_i & i\neq 0,1\\ b_1+c & i=1\\ a_0 & i=0.\end{cases}$$

Then we have that $b'_i \in \mathbb{N}$ for all i,

$$\sum_i b'_i p^i = a_0 + (b_1 + c)p + \sum_{i\geq 2} b_i p^i = b_0 - cp + b_1 p + cp + \sum_{i\geq 2} b_i p^i = \sum_i b_i p^i = n$$

and

$$\sum_i b'_i = a_0 + b_1 + c + \sum_{i\geq 2} b_i = b_0 + b_1 + c(1-p) + \sum_{i\geq 2} b_i \leq \sum_i b_i \leq \sum_i a_i.$$

Also as $b'_0 = a_0$ we also have that $\sum_{i\geq 1} b'_i \leq \sum_{i\geq 1} a_i$ and as the p-adic decomposition of $(n-a_0)/p$ is $a_1 + a_2 p + \ldots + a_k p^{k-1}$ and

$$\sum_{i\geq 1} b'_i p^{i-1} = \left(\sum_{i\geq 1} b'_i p^i\right)/p = (n - b'_0)/p = (n-a_0)/p$$

we can conclude by induction that $b'_i = a_i$ for each $i \geq 1$. So, as we already now that $b'_0 = a_0$, we have that $b'_i = a_i$ for all $i \geq 0$. As if c was ≥ 1 we would have that $\sum_i b'_i < \sum_i b_i \leq \sum_i a_i = \sum_i b'_i$, we need to have that $c = 0$ and so $b_i = b'_i = a_i$ for all i and so the lemma is proved. □

Definition 36. *$\alpha \vdash n$ is of class $m \geq 0$ if it isn't possible to recursively remove from α a sequence of hooks with hook-lengths given by the partition $\big((p^k)^{a_k}, (p^{k-1})^{a_{k-1}}, \ldots, (p^m)^{a_m}\big)$.*

Lemma 66. *If α is of class m for some m, we have that the degree of χ^α is divisible by p.*

Proof. This follows by recursively applying lemma 61 and by lemma 65. □

Theorem 67. *If $(c_1, \ldots, c_h) \vdash n$, with $c_h \geq 1$, is p-vanishing, $p \neq 2, 3$ and i is maximal such that $p^i | n$, then $c_h \geq p^i$. Also the same result is true if $p = 2$, $i \neq 1, 2$ or if $p = 3$ and $i \neq 1$.*

If $p = 2$ and $i = 2$ then $c_h \geq 4$ or $(c_1, \ldots, c_h)$ is either $(2, 1, 1)$ or ends by $(d, 2, 1, 1)$, for some $d \geq 4$ and both of these possibilities actually occur.

If $p = 2, 3$, $i = 1$ and $c_h < p$, then $c_h = 1$.

Proof. If $i = 0$ there is nothing to prove, as then we would have that $p^i = 1$. Also the theorem is trivial in the case where $p = 2$ and $i = 1$. So assume that $i \neq 0$ (and that $i \neq 1$ if $p = 2$).

The proof of the theorem will now proceed considering the following cases:

Case 1: $2 \leq c_h < p^i$,

Case 2: $p^i \geq 4$ and $(c_1, \ldots, c_h)$ ends by $(1,1,1,1)$, $(c,1)$, $(c,1,1)$, $(2,1,1,1)$, $(2,2,1)$ or $(2,2,1,1)$ with $c \geq 3$,

Case 3: $p^i \geq 4$, $(c_1, \ldots, c_h)$ ends by $(c,1,1,1)$, $(c,2,1,1)$ with $c \geq 3$, $(c_1, \ldots, c_h) \neq (3,3,1,1,1)$ and doesn't end by $(d,3,3,1,1,1)$ or $(d,2,1,1)$ with $d \geq 4$,

Case 4: $p^i \geq 4$ and $(c_1, \ldots, c_h)$ ends by $(c,2,1)$ with $c \geq 3$,

Case 5: $p^i \geq 5$, $(c_1, \ldots, c_h) = (3,3,1,1,1)$ or ends by $(d,3,3,1,1,1)$ or $(d,2,1,1)$ with $d \geq 4$, $(c_1, \ldots, c_h) \neq (4,2,1,1)$ and doesn't end by $(e,4,2,1,1)$ with $e \geq 5$,

Case 6: $p^i \geq 5$ and $(c_1, \ldots, c_h) = (4,2,1,1)$ or ends by $(e,4,2,1,1)$ with $e \geq 5$,

Case 7: $p^i = 4$ and $(c_1, \ldots, c_h)$ ends by $(d,3,3,1,1,1)$,

Case 8: If $p^i = 4$ there exists $(c_1, \ldots, c_h)$ ending with $(d,2,1,1)$ with $d \geq 4$ which is 2-vanishing.

As when $p = 3$ and $i = 1$ if we have an exception to the theorem we need to be in case 1 and the cases $(c_1, \ldots, c_h) = (1,1,1), (2,1,1), (2,1)$ are within the special cases of the theorem (when $c_h < p^i$), it can be easily checked that these cases cover all possibilities where $c_h < p^i$.

Case 1.

We will first show that for any n, p and any $i \geq 1$, if $2 \leq c_h < p^i$ then $(c_1, \ldots, c_h)$ is not p-vanishing.

Let $\alpha = (n - c_h, c_h)$. As $c_h < p^i \leq n$ we have that $h \geq 2$ and so

$$0 < c_h \leq c_1 \leq c_1 + \ldots + c_{h-1} = n - c_h$$

and so α is actually a partition. We will show that $\chi^\alpha_{(c_1,\ldots,c_h)} \neq 0$ and χ^α has degree divisible by p. By lemma 66 to show that the degree of χ^α is divisible by p it is enough to show that α is of class i. If $n = p^i$ then as the second row of α contains at least 2 nodes we have that α doesn't contain any hook of length $p^i = n$, as in this case $h^\alpha_{1,1} < p^i$, and so we are done.

So assume that $p^i < n$. As $c_h < p^i \leq p^j$ for each $j \geq i$ if it was possible to recursively remove a sequence of hooks of lengths $((p^k)^{a_k}, \ldots, (p^i)^{a_i})$ from α, we would have that all hooks which are recursively removed from α are of the form $R_{1,l}$ for some l. Assume first that $c_h \leq p^i/2$. Then as we have $h^\alpha_{1,c_h+1} = n - 2c_h \geq n - p^i$, it is easy to see that we can remove a_k hooks of length p^k, then a_{k-1} hooks of length p^{k-1} and so on until we remove a_{i+1} hooks of length p^{i+1} and then remove $a_i - 1$ $(a_i > 0)$ hooks of length p^i from α in a unique way and that in this way we obtain the partition $(p^i - c_h, c_h)$.

As $c_h \geq 2$ we cannot remove the last hook of length p^i and so again α is of class i.

Assume now that $c_h > p^i/2$. As n isn't a power of p as $n \neq p^i$ and by definition of i, we have that any sequence of hooks of lengths $((p^k)^{a_k}, \ldots, (p^i)^{a_i})$ contains at least 2 hooks. Let p^j be the length of the second last hook of such a sequence. Then we have that $h^\alpha_{1,c_h+1} = n - 2c_h \geq n - p^i + p^j$ and so we can now remove all hooks in the sequence apart for the last two in a unique way and that the partition we obtain this way is $(p^i + p^j - c_h, c_h)$. Also as $c_h < p^i \leq p^j$, if there is any hook of length p^j in $(p^i + p^j - c_h, c_h)$ this must be on the first row. If there is no hook of length p^j we are done. Otherwise notice that in order to be possible to remove a hook of length p^i from the partition we would obtain we need to have that the second row of the partition we obtain this way must contain at most 1 node. So, as $c_h \geq 2$ we need to have that the hook that we remove must be either $R^\alpha_{1,1}$ or $R^\alpha_{1,2}$, and in these cases we have that the resulting partitions are given by $(c_h - 1)$ and $(c_h - 1, 1)$ respectively. But $c_h - 1, c_h < p^i$, which means that $h^\alpha_{1,1}, h^\alpha_{1,2} > p^j$ and then it is not possible to remove first a p^j hook and then a p^i hook from $(p^i + p^j - c_h, c_h)$ and so we cannot remove a sequence of hooks of lengths $\left(\left(p^k\right)^{a_k}, \ldots, \left(p^i\right)^{a_i}\right)$ in this case either. So also in this case α is of class i and then we only need to show that $\chi^\alpha_{(c_1,\ldots,c_h)} \neq 0$.

Let m be such that $c_m > c_h$ and $c_{m+1} = c_h$ ($m = 0$ if $c_1 = c_h$). While $j \leq m, h-2$, as then $c_h < c_j$ and $h^{(c_j+\ldots+c_{h-1},c_h)}_{1,c_h+1} = c_j + \ldots + c_{h-1} - c_h \geq c_j$, it is easy to see that we can remove a hook of length c_j form $(c_j + \ldots + c_{h-1}, c_h)$ in a unique way and the resulting partition is $(c_{j+1} + \ldots + c_{h-1}, c_h)$. So we can remove the first $s = \min\{m, h-2\}$ hooks of length c_j from α in a unique way and we obtain the partition $(c_{s+1} + \ldots + c_{h-1}, c_h)$.

Assume now that $m = h - 1$. Then we have that $s = h - 2$ and $(c_{s+1} + \ldots + c_{h-1}, c_h) = (c_{h-1}, c_h)$. As $c_{h-1} > c_h$ there cannot be any hook of length c_{h-1} on the second row and as it is easy to see that $h^{(c_{h-1},c_h)}_{1,2} = c_{h-1}$, as $c_h \geq 2$, we can remove a hook of length c_{h-1} from (c_{h-1}, c_h) in a unique way and we obtain the partition $(c_h - 1, 1)$, for which obviously $h_{1,1} = c_h$. So if $2 \leq c_h < c_{h-1}$ we can recursively remove hooks of length c_j from α in a unique way, and so by the Murnaghan-Nakayama formula we have that in this case $\chi^\alpha_{(c_1,\ldots,c_h)} \neq 0$.

Let now $m < h - 1$. Then $s = m$. After removing the first s hooks of length c_j from α we are left with $(c_{s+1} + \ldots + c_{h-1}, c_h) = ((h - s - 1)c_h, c_h)$ and we need to remove $h - s \geq 2$ hooks of length c_h from this partition. We need to see in how many ways we can do this and see what the sum of the leg-lengths modulo 2 is in any of this cases. If we remove one of the first $h - m - 2$ hooks of lengths c_h from the second row then all other c_h-hooks must be removed from the first row and in this case it is easy to see that all hooks have leg-length 0. This can be done in $h-s-2$ different ways. Otherwise we need to remove the first $h-s-2$ hooks of length c_h from the first row and as $h_{1,c_h+1}^{((h-s-1)c_h,c_h)}=(h-s-2)c_h$ and $h_{1,c_h+1}^{((h-s-1)c_h,c_h)}=0$, we have that the hooks we have removed up to this point have leg-length 0 and this way we obtain (c_h,c_h) from which we need to removed 2 c_h-hooks. As $\chi_{(c_h,c_h)}^{(c_h,c_h)}=\chi_{(c_h)}^{(c_h)}-\chi_{c_h)}^{(c_h-1,1)}=2$. Putting all of this together we have by lemma 54 that $\chi_{(c_1,\ldots,c_h)}^{\alpha}=h-s\neq 0$.

So we have that if $2\leq c_h<p^i$, then there is some irreducible character χ^α of S_n of degree divisible by p such that $\chi_{(c_1,\ldots,c_h)}^{\alpha}\neq 0$ and so $(c_1,\ldots,c_h)$ is not p-vanishing. This proves the theorem for the part where $p=3$ and $i=1$.

Case 2.
Assume now that $(c_1,\ldots,c_h)$ ends by $(1,1,1,1)$, $(c,1)$, $(c,1,1)$, $(2,1,1,1)$, $(2,2,1)$ or $(2,2,1,1)$ with $c\geq 3$, that $i\geq 1$ and that $p\neq 2,3$ or $p=2,3$ and $i\geq 2$. We want to show that also in this case $(c_1,\ldots,c_h)$ is not p-vanishing. First we will show that if $\beta=(n-2,2)$ then $p|\deg(\chi^\beta)$. Notice that by assumption in this part we always need to have $n\geq p^i\geq 4$, so β is actually a partition. Also we have that $p^i\geq 4$. So all parts of $\left((p^k)^{a_k},\ldots,(p^i)^{a_i}\right)$ are bigger then 2 and then as $h_{2,1}^{\alpha}=2$ if we can remove a sequence of hooks of lengths $\left((p^k)^{a_k},\ldots,(p^i)^{a_i}\right)$ all hooks must be on the first row. Then by as $h_{1,3}^{\beta}=n-4\geq\sum_{i=i+1}^{k}a_kp^k+(a_i-1)p^i$ and $l_{1,3}^{\beta}=0$ we can remove all but the last hooks of the sequence from β in a unique way and we obtain $(p^i-2,2)$. As $h_{1,1}^{(p^i-2,2)}=p^i-1$ we cannot remove the last hook of the sequence and so again we have that β is of class i and so $p|\deg\left(\chi^\beta\right)$ by lemma 66.

We will show that if $(c_1,\ldots,c_h)$ ends with $(1,1,1,1)$, $(c,1)$, $(c,1,1)$, $(2,1,1,1)$, $(2,2,1)$ or $(2,2,1,1)$, where $c\geq 3$, then $\chi_{(c_1,\ldots,c_h)}^{\beta}\neq 0$. This is always true if $(c_1,\ldots,c_h)=(1^n)$, as $(c_1,\ldots,c_h)$ is the cycle partition of 1 and $\chi^\beta(1)=\deg(\chi^\beta)\neq 0$, so we can assume that $c_1\geq 2$.

Assume that $(c_1,\ldots,c_h)=(c_1,\ldots,c_j,1^m)$, with $c_j\geq 3$, $m\geq 1$ and $m\neq 3$. As $h_{2,1}^{\beta}=2<c_l$ for $l\leq j$ the first j hooks must always be removed from the first row. First assume that $m\geq 4$. Then as $h_{1,3}^{\beta}=n-4\geq c_1+\ldots+c_j$ we can remove the first j hooks in a unique way obtaining $(m-2,2)$. As all 1-hooks have leg-length 0 and we can always remove some 1-hook from any partition of any positive integer, we can then conclude by lemma 54 that in this case $\chi_{(c_1,\ldots,c_h)}^{\beta}\neq 0$. Assume now that $m=2$. Then as $h_{1,3}^{\alpha}=n-4\geq c_1+\ldots c_{j-1}$ we can remove the first $j-1$ hooks in a unique way and we obtain $(c_j+m-2,2)=(c_j,2)$. Now we have that $h_{1,2}^{(c_j,2)}=c_i-1+1=c_j$ and so as we need to remove the i-th hook from the first row we again can remove

the i-th hook in a unique way and so as again 1-hooks have leg-length 0 we need to have that $\chi^\beta_{(c_1,\ldots,c_h)} \neq 0$. So let now $m = 1$. Then we again have that $h^\alpha_{1,3} = n - 4 \geq c_1 + \ldots c_{j-1}$ and so as the first $j-1$ hooks must be on the first row we can remove the first $j-1$-hooks in a unique way and we now obtain $(c_j + m - 2, 2) = (c_j - 1, 2)$. As $h^{(c_j-1,2)}_{1,1} = c_j$ we can remove also the i-th hook in a unique way, obtaining (1), from which we need to remove a 1-hook, and so again $\chi^\beta_{(c_1,\ldots,c_h)} \neq 0$.

So assume now that $(c_1,\ldots,c_h) = (c_1,\ldots,c_j,2^l,1^m)$, with $l = 1$ and $m \geq 3$ or $l \geq 2$ and $m \geq 1$. First assume that $l = 1$ and $m \geq 3$. As $h^\beta_{1,3} = n - 4 \geq c_1 + \ldots + c_j$ we can remove the first j hooks in a unique way and we obtain $(m, 2)$, as these hooks must be removed from the first row, as any hook in the second row as at most length $2 < c^l$ for any $l \leq j$. If $m = 3$ it is easy to see that there is only one hook of length 2, $H^{(m,2)}_{2,1}$, and so as afterward we only need to remove 1-hooks, which have leg-lengths 0, we need to have that in this case $\chi^\beta_{(c_1,\ldots,c_h)} \neq 0$. If $m > 3$ then we can either remove $R^{(m,2)}_{2,1}$ or $R^{(m,2)}_{1,m-1}$. As anyway the leg-length of this hook is 0 and afterward we only need to remove 1-hooks, we have by lemma 54 that $\chi^\beta_{(c_1,\ldots,c_h)} \neq 0$.

So assume that $l \geq 2$ and $m \geq 1$. Again, by the same reasons as in the previous case, we can remove the first j hooks in a unique way obtaining $(m + 2l - 2, 2)$. If $m \geq 4$ as $h^{(m+2l-2)}_{1,3} \geq 2l$ and as $l^{(m+2l-2)}_{1,3} = 0$ we have that if at any step we remove some of the l hooks of length 2 from the first row, then this hook must have leg-length 0. Also if we remove any 2-hook from the second row it must have also in this case leg-length 0. As any 1-hook as leg-length 0 we then have that the sum of the leg-lengths of the hooks we recursively remove from β is always the same (is always 0) and so again by lemma 54 we need to have that $\chi^\beta_{(c_1,\ldots,c_h)} \neq 0$, as it is possible to remove from $(m + 2l - 2, 2)$ some sequence of l 2-hooks and m 1-hooks.

Assume now that $l \geq 2$ and $m = 3$. After having removed the first j hooks (which can be done in a unique way), we obtain $(2l + 1, 2)$. As $h^{(2l+1,2)}_{1,3} = 2l - 1 > 2(l - 1)$ and $l^{(2l+1,2)}_{1,3} = 0$, we again have as before that the first $l - 1$ hooks of length 2 that we remove from $(2l + 1, 2)$ must have leg-length 0. Now there are two possibilities. The first one is that we obtain (5), if one of the $l - 1$ hooks of length 2 was removed from the second row. This can be done in $l - 1$ different ways. In this case the only 2-hook, $H^{(5)}_{1,4}$, has leg-length 0. Otherwise all of the $l - 1$ 2-hooks were removed from the first row, as we can remove at most one 2-hook from the second row, and in this case we obtain $(3, 2)$. Again the only 2-hook, $H^{(3,2)}_{2,1}$, has leg-length 2. So the sum of the leg-length of the sequence of hooks we removed is constant $(= 0)$ and so again $\chi^\beta_{(c_1,\ldots,c_h)} \neq 0$.

Now assume that $m = 2$. Then $(m+2l-2, 2) = (2l, 2)$. As $h_{1,3} = 2(l-1)$, the first $l-1$ 2-hooks must again have leg-length 0. If one of the 2-hooks was removed from the second row we obtain (4). This can be done in $l-1$ different ways and we can remove the last 2-hook and the 2 1-hooks in a unique way and all of these hooks have leg-length 0. In the second case all $l-1$ hooks are removed from the first row. This can be done in a unique way and we obtain $(2, 2)$. We can now remove the last 2-hook in two different ways, in one we have that the leg-length of the 2-hook is 0 (when we remove $H_{2,1}^{(2,2)}$) and in the other is 1 (when we remove $H_{1,2}^{(2,2)}$). In any case we have that the two 1-hooks can be removed in a unique way from the partition we obtain. So we have that there are $l-1+1 = l$ paths in removing the hooks from β in for which the sum of the leg-lengths is 0 (the first j hooks we removed always have leg-length 0) and one path in which it is 1. By using lemma 54 we then have that $\chi^{\beta}_{(c_1,\ldots,c_h)} = l-1 > 0$ as $l \geq 2$ and so we have again that $\chi^{\beta}_{(c_1,\ldots,c_h)} \neq 0$.

The last case is when $m = 1$. Here $(m+2l-2, 2) = (2l-1, 2)$ and $h_{1,3}^{(2l-1,2)} = 2(l-1)-1 > 2(l-2)$ and $l_{1,3}^{(2l-1,2)} = 0$, so the first $l-2$ 2-hooks must all have leg-length 0. After having removed these hooks we either obtain (5) or $(3, 2)$. If we obtain (5) we must remove the last hooks in a unique way and they would all have leg-length 0, if we obtain $(3, 2)$ we must first remove the hook corresponding to $(2, 1)$ and then that corresponding to $(1, 2)$ and the remove $(1, 1)$. All these three hooks have leg-lengths 0, and so again we have that in any sequence of hooks of lengths $(c_1, \ldots, c_h)$ which are recursively removed from β, all leg-length are 0 and it is possible to remove at least one such sequence of hooks from β, we have again that $\chi^{\beta}_{(c_1,\ldots,c_h)} \neq 0$.

Case 3.

Assume now that $(c_1, \ldots, c_h)$ ends by $(c, 1, 1, 1)$, $(c, 2, 1, 1)$ with $c \geq 3$, $(c_1, \ldots, c_h) \neq (3, 3, 1, 1, 1)$ and doesn't end by $(d, 3, 3, 1, 1, 1)$ or $(d, 2, 1, 1)$ with $d \geq 4$. As $n \geq p^i \geq 4$ (as we have already proved the theorem when $p^i = 1, 2, 3$). None of these situations is possible if $n = 4, 5$, so we actually have that $n > 5$. Also $n = 6$ is not possible as $p^i \geq 4$ need to divide n. So we have that $n \geq 7$. Let $\gamma = (n-3, 3)$. We will show that $p|\deg(\chi^{\gamma})$. As $p^i \geq 4$ and $h_{2,1}^{\gamma} = 3$, it is easy to see that all hooks of the sequence with hook-lengths $((p^k)^{a_k}, \ldots, (p^i)^{a_i})$, which are recursively removed from γ, must be on the first row, so that this can be done in at most one way. Assume first that $p^i \geq 7$. Then as $h_{1,4}^{\gamma} = n-6 \geq \sum_{j=i+1}^{k} a_j p^j + (a_i - 1)p^i$, we can removed all of them apart maybe for the last one and after having done this we obtain $(p^i - 3, 3)$. Now as $h_{1,1}^{(p^i-3,3)} = p^i - 2$ we cannot remove the last hook of the sequence and so in this case by lemma 66 we need to have that p divides the degree of χ^{γ}. So let now $p^i = 4, 5$ and let p^j be the second smallest part of $\left((p^k)^{a_k}, \ldots, (p^i)^{a_i}\right)$. Notice that as $p^i < 7 \leq n$ such a part must exists. Then again it can be seen that all hooks can be remove apart maybe for the last two and this way we obtain $(p^i + p^j - 3, 3)$. If $p^i = 5$ we have that $h_{1,3}^{(p^j+2,3)} = p^j + 1$ and $h_{1,4}^{(p^j+2,3)} = p^j - 1$, so in this case we cannot remove the hook of length p^j. If $p^i = 4$ then $h_{1,3}^{(p^j+1,3)} = p^j$ and so we need to remove now the hook corresponding to $(1,3)$. After having removed this hook we get $(2,2)$ which doesn't have any 4-hook, and so again we cannot remove a sequence of hooks of lengths $\left((p^k)^{a_k}, \ldots, (p^i)^{a_i}\right)$ from γ. So also when $p^i = 4, 5$ we have by lemma 66 that $p | \deg(\chi^\gamma)$.

We will now show that if $(c_1, \ldots, c_h)$ ends by $(c, 1, 1, 1)$ or $(c, 2, 1, 1)$, with $c \geq 3$ and $(c_1, \ldots, c_h) \neq (3, 3, 1, 1, 1)$ or it doesn't end with $(d, 3, 3, 1, 1, 1)$ or $(d, 2, 1, 1)$ for some $d \geq 4$, then $\chi^\gamma_{(c_1,\ldots,c_h)} \neq 0$. First assume that we have $(c_1, \ldots, c_h) = (c_1, \ldots, c_m, d, 1, 1, 1)$ for some $d \geq 4$. Then the first m hooks of lengths c_j must be removed from the first row (as the second row only contains 3 nodes) and as $h^\gamma_{1,4} = n - 6 > c_1 + \ldots + c_m$ we can remove them in a unique way obtaining $(d, 3)$. Now as $d \geq 4 > h_{2,1}^{(d,3)}$ and $h_{1,2}^{(d,3)} = d$ we can remove the hook of length d in a unique way and so as we now only need to remove 1-hooks, which have always leg-length 0, we have by lemma 54 that $\chi^\gamma_{(c_1,\ldots,c_h)} \neq 0$.

So assume first that $(c_1, \ldots, c_h) = \left(c_1, \ldots, c_m, 3^l, 1^3\right)$, with $m \geq 0$ and where $c_m \geq 4$ and $l \geq 1$. As the first m-hooks must again be removed from the first row and $h^\gamma_{1,4} = n - 6 \geq c_1 + \ldots + c_m$ it is easy to see that we can remove these m-hooks in a unique way and we obtain $(3l, 3)$. Now we can either remove one of the first $l - 1$ hooks of length 3 from the second row, and then all other must be removed from the first row and all hooks have leg-length 0, which can be done in $l - 1$ ways, or we remove all the first $l - 1$ hooks of length 3 from the first row and in this case all hooks removed up to this point have leg-length 0. This last case can be done in a unique way and would get $(3, 3)$ from which we must remove first a 3-hook and then 3 hooks of length 1. If we remove the last hook from the second row then we need to remove the 3 1-hooks in a unique way and all hooks would have leg-length 0. Otherwise we need to remove the hook corresponding to $(1, 2)$, which has leg-length 1. Now we obtain $(2, 1)$ and we can remove the 3 1-hooks in 2 different ways. So there are l ways to remove the hooks of length $(c_1, \ldots, c_m, 3^l, 1^3)$ from γ for which the sum of the leg-length is 0 and 2 ways for which the sum of the leg-lengths is 1. In particular by using lemma 54 we have that in this case $\chi^\gamma_{(c_1,\ldots,c_h)} \neq 0$ unless $l = 2$.

Let $(c_1, \ldots, c_h) = (c_1, \ldots, c_m, 3^l, 2, 1, 1)$ for some $l \geq 1$, where $c_m \geq 4$. As $h^\gamma_{2,1} = 3 < c_j$ if $j \leq m$ and $h^\gamma_{1,4} = n - 6 \geq c_1 + \ldots + c_m$ we can recursively remove the first m hooks of a sequence of hooks of lengths $(c_1, \ldots, c_h)$ from γ in a unique way obtaining $(3l+1, 3)$. All these m hooks have leg-length 0. If we remove any of the first $l-1$ 3-hooks from the second row we need to remove all others in a unique way and all hooks would have leg-length 0. This can be done in $l-1$ ways. Otherwise we remove all of the first $l-1$ 3-hooks from the first row obtaining $(4, 3)$. Also these $l-1$ hooks have leg-length 0 and this can be done in a unique way. Now we can remove $R_{2,1}^{(4,3)}$, in which case we can only finish removing the 3 1-hooks in a unique way and also here all hooks have leg-length 0, or we remove $R_{1,3}^{(4,3)}$, which has leg-length 1. In this case we get $(2, 2)$. Now we can remove the last 3 hooks in two different ways, which correspond to the following sequences of nodes $((2, 1), (1, 2), (1, 1))$ or $((1, 2), (2, 1), (1, 1))$. In the first case the sum of the leg-lengths of all the hooks in the sequence is 1, while in the second case it is 2. So using lemma 54 we have that in this case $\chi^{\gamma}_{(c_1,\ldots,c_h)} = l-1+1-1+1 = l \neq 0$.

Case 4.

We will now consider the case of $(c_1, \ldots, c_h)$ ending by $(c, 2, 1)$, with $c \geq 3$. Let now $\lambda = (n-4, 2, 2)$. As $n \geq c+2+1 \geq 6$, we have that $\lambda \vdash n$. We will now show that $p | \deg\left(\chi^{\lambda}\right)$. As $h^{\lambda}_{2,1} = 3$ and $p^i \geq 4$, we need to have that all hooks of a sequence with hook-lengths $\left(\left(p^k\right)^{a_k}, \ldots, (p^i)^{a_i}\right)$ are removed from the first row and so we can remove the sequence of hooks in at most one way. Assume first that $p^i \geq 7$. Then it is easy to see that after having removed all but the last hooks of the sequence (which we can actually do, as can be easily seen by looking at $H^{\lambda}_{1,3}$) we obtain $(p^i-4, 2, 2)$. As $h^{(p^i-4,2,2)}_{1,1} = p^i - 2$ we cannot remove the last hook of the sequence. If instead $p^i = 4, 5$, let p^j be the second smallest part of $\left(\left(p^k\right)^{a_k}, \ldots, (p^i)^{a_i}\right)$ (which must exists). After having removed all but the last 2 hooks we obtain $(p^i+p^j-4, 2, 2)$. Now we have that $h^{(p^i+p^j-4,2,2)}_{1,2} = p^j + p^i - 3 > p^j$ and $h^{(p^i+p^j-4,2,2)}_{1,3} = p^j + p^i - 6 < p^j$ and so in this case we cannot remove the second last hook of the sequence and so λ is of class i and so by lemma 66 we have that $p | \deg\left(\chi^{\lambda}\right)$.

We will show that $\chi^{\lambda}_{(c_1,\ldots,c_h)} \neq 0$ whenever $(c_1, \ldots, c_h)$ ends by $(c, 2, 1)$, for some $c \geq 3$. First assume that $c \geq 4$. Then again as the hooks on the second and third row are all at most of length 3, we need to remove all hooks until that of length c from the first row, and it can be seen that this can be actually be done in a unique way obtaining (1^3). Now from this partition we can again remove in a unique way the last hooks, one of length 2 and one of length 1, and so we have by the Murnaghan-Nakayama formula that in this case $\chi^{\lambda}_{(c_1,\ldots,c_h)} \neq 0$.

So assume now that $(c_1, \ldots, c_h) = (c_1, \ldots, c_j, 3^l, 2, 1)$ for some $l \geq 1$ and $c_m \geq 4$. Again, as $h^{\gamma}_{1,3} = n-6 \geq c_1+\ldots+c_j$ and $h^{\gamma}_{2,1} = 3$, we can remove the first j hooks in a unique way obtaining $(3l-1, 2, 2)$. Also as $h^{(3l-1,2,2)}_{2,2} = 2$

there cannot be any 3-hook on the third row. If we remove one of the l hooks of length 3 from the second row all other 3-hooks must be removed from the first row and it isn't hard to see that after having removed all of the 3-hooks we obtain $(2,1)$, from which we cannot remove any 2-hook. So we must remove all 3-hooks from the first row and in this case, after having removed all 3-hooks, which again can easily be seen to be done in a unique way as they are all on the first row (remove the hooks corresponding to $(1,3l-3),(1,3l-6),\ldots,(1,3),(1,2)$), we obtain (1^3), from which again we can remove the last 2 hooks in a unique way and so we have again by the Murnaghan-Nakayama formula that $\chi^\lambda_{(c_1,\ldots,c_h)}\neq 0$.

Case 5.
We will now consider the cases where $(c_1,\ldots,c_h)=(3,3,1,1,1)$ or ends by $(d,3,3,1,1,1)$ or $(d,2,1,1)$ with $d\geq 4$, $(c_1,\ldots,c_h)\neq(4,2,1,1)$ and doesn't end by $(e,4,2,1,1)$ with $e\geq 5$ for $p^i\geq 5$. If $n=7$ none of these possibilities for $(c_1,\ldots,c_h)$ as above are possible. Let $\delta=(n-4,4)$. δ is a partition of n as $n\geq 8$. We will show that p divides the degree of χ^δ and that in these cases $\chi^\delta_{(c_1,\ldots,c_h)}\neq 0$. First assume that $p^i\neq 5,7$. Then $p^i\geq 8$ and so (unless $n=8$) as $h^\delta_{1,5}=n-8$, $l^\delta_{1,5}=0$, $h^\delta_{2,1}=4$ and all parts of $\left(\left(p^k\right)^{a_k},\ldots,(p^i)^{a_i}\right)$ are bigger then 4, we can remove all but the last part in a unique way obtaining $(p^i-4,4)$ (if $n=8$ we already start like this). Now $h^{(p^i-4,4)}_{1,1}=p^i-3<p^i$ and so we cannot remove the last hook of the sequence we are trying to remove from δ. If $p^i=5,7$ let p^j be the second smallest part of $\left(\left(p^k\right)^{a_k},\ldots,(p^i)^{a_i}\right)$. p^j must exists as $p^i<n$. Again, as $h^\delta_{1,5}\geq n-p^i-p^j$ and $h^\delta_{2,1}=4<p^i\leq p^l$ for any $l\geq i$, we can remove all but the last two hooks of the sequence of hooks of hook-length $\left(\left(p^k\right)^{a_k},\ldots,(p^i)^{a_i}\right)$ in a unique way and we obtain $(p^i+p^j-4,4)$. In order to be able to remove the last hook we need to have that after having removed the p^j-hook there is at most one node on the second row. So we first need to remove either $R^{(p^i+p^j-4,4)}_{1,1}$ or $R^{(p^i+p^j-4,4)}_{1,2}$, as we need to remove a hook on the first row. In the first case we would be left with (3), while in the second case with $(3,1)$. As $3,4<p^i$ this means that $h^{(p^i+p^j-4,4)}_{1,1}$ and $h^{(p^i+p^j-4,4)}_{1,1}$ are bigger then p^j and so δ is of class i and so we again have by lemma 66 that p divides the degree of χ^δ.

Assume now that $(c_1,\ldots,c_h)=(c_1,\ldots,c_m,3,3,1,1,1)$, for some $m\geq 0$ and for which $c_m\geq 4$. First assume that $c_m>4$. Then as $h^\delta_{2,1}=4$ we need to remove the first m hooks from the first row. Also as $h^\delta_{1,5}=n-8>c_1+\ldots+c_m$ and $l^\delta_{1,5}=0$, we can remove these m-hooks in a unique way and we obtain $(5,4)$ and it is easy to see that these m hooks all have leg-length 0. If we now remove the hook corresponding to $(2,2)$ we obtain the partition $(5,1)$ and we next need to remove the hook corresponding to $(1,3)$, which would give the partition $(2,1)$. Now we can remove the 3 1-hooks in two different ways, but in both ways we have that all hooks in the sequence with hooks-length $(c_1, \ldots, c_i, 3, 3, 1, 1, 1)$ have leg-length 0. Otherwise we first need to remove the hook corresponding to $(1, 4)$ obtaining the partition $(3, 3)$. This hook has leg-length 1. If we next remove the hook corresponding to $(2, 1)$ we obtain (3) and so there is a unique way to remove the 3 1-hooks and these last 4 hooks have all leg-length 0. Otherwise the second 3-hook we remove must be the one corresponding to $(1, 2)$, which has leg-length 1 and which would leave the partition $(2, 1)$. There are two ways to remove the final 1-hooks. So putting all of this together there are 2 ways to remove the sequence of hooks for which the sum of the leg-lengths is 0, 1 for which the sum of the leg-lengths is 1 and 2 for which the sum of the leg-lengths is 2. By lemma 54 we then have that $\chi^{\delta}_{(c_1,\ldots,c_h)} = 3 \neq 0$.

If now $(c_1, \ldots, c_h) = (c_1, \ldots, c_m, 4^l, 3, 3, 1, 1, 1)$, for some $m \geq 0$ and for which $c_m > 4$ and $l \geq 1$ we can again remove the first m hooks in a unique way (they all must be on the first row) from δ and these hooks all have leg-length 0. After having done this we can either remove all 4-hooks from the first row, which would leave $(5, 4)$ as before. This can be done in a unique way. Proceeding as before we have there are 4 ways to remove the sequence of hooks for which the sum of the leg-lengths is even and one way for which it is odd. Otherwise we need to remove exactly one of the l hooks of length 4 from the second row and then all other hooks must be removed from the first row and all hooks of the sequence have leg-length 0. Putting all of this together and using lemma 54 we have that $\chi^{\gamma}_{(c_1,\ldots,c_h)} = l + 3 \neq 0$ also in this case.

Assume now that $(c_1, \ldots, c_h) = (c_1, \ldots, c_m, 4^l, 2, 1, 1)$, for some $m \geq 0$ and for which $c_m > 4$ and $l \geq 0$. First let $l = 0$. Then $m \geq 1$. Again as $h^{\delta}_{1,5} = n - 8 \geq c_1 + \ldots + c_{m-1}$, $l^{\delta}_{1,5} = 0$ and $h^{\delta}_{2,1} = 4$ we need to remove the first $m - 1$ hooks in a unique way and we obtain $(c_i, 4)$. Now as we need to remove the m-th hook from the first row and as $h^{(c_m,4)}_{1,2} = c_m$ we need to remove this hook and this way we obtain $(3, 1)$, from which we can remove the last hooks in a unique way (there is a unique 2-hook and after having removed it we get (1^2)), and so by the Murnaghan-Nakayama formula we must have that $\chi^{\delta}_{(c_1,\ldots,c_h)} \neq 0$ in this case.

Let now $l \geq 1$. As in this case $h^{\delta}_{1,5} = n - 8 \geq c_1 + \ldots + c_m$, $l^{\delta}_{1,5} = 0$ and $h^{\delta}_{2,1} = 4$ we must now remove the first m hooks from the first row in a unique way and we get $(4l, 4)$. These m hooks all have leg-length 0. Now we can either remove one of the l 4-hooks from the second row, and then all the other must be removed in a unique way (they are all in the first row) and all hooks have leg-length 0. This can be done in l different ways. Or all l hooks of length 4 are removed from the first row. In this case we get $(3, 1)$ and the first $l-1$ 4-hooks have leg-length 0, while the last one has leg-length 1. We can now remove the last hooks in a unique way and these last hooks all have leg-length 0. So by lemma 54 $\chi^{\delta}_{(c_1,\ldots,c_h)} = l-1$ and the $\chi^{\delta}_{(c_1,\ldots,c_h)} \neq 0$ if $l \geq 2$.

Case 6:

The only cases left when $p^i \geq 5$ are when $(c_1,\ldots,c_h) = (4,2,1,1)$ or it ends with $(e,4,2,1,1)$, for some $e \geq 5$.

If $(c_1,\ldots,c_h) = (4,2,1,1)$ then $n = 8$ and $p = 2$. It is easy to check that $2|\deg(\chi^{(3,3,2)})$ and that $\chi^{(3,3,2)}_{(4,2,1,1)} \neq 0$, so that $(4,2,1,1)$ isn't 2-vanishing.

If $(c_1,\ldots,c_h)$ ends with $(e,4,2,1,1)$ with $e \geq 5$ then as the last two rows of $(n-5,3,2)$ don't contain any hook of length $\geq e$ we need to remove all hooks until that of length e from the first row in a unique way and at this point we get $(3,3,2)$ from which we need to remove hooks of lengths $(4,2,1,1)$ and so we can conclude that also in this case $\chi^{(n-5,3,2)}_{(c_1,\ldots,c_h)} \neq 0$ by the the previous part and the Murnaghan-Nakayama formula.

We will now show that if $i \geq 1$, $i \geq 3$ when $p = 2$ and $i \geq 2$ when $p = 3$ are as in the text of the theorem then p divides the degree of $\chi^{(n-5,3,2)}$. As $p^i \geq 5$ and hooks in the last two rows are all at most 4, we need to remove all hooks from the first row. If $p^i \geq 8$, then as $h^{(n-5,3,2)}_{1,4} = n-8 \geq a_k p^k + \ldots a_{i+1}p^{i+1} + (a_i - 1)p^i$ and $l^{(n-5,3,2)}_{1,4} = 0$ we can remove all but the last hooks of the sequence of hooks with lengths $\left((p^k)^{a_k}, \ldots, (p^i)^{a_i}\right)$ in a unique way and we obtain $(p^i-5,3,2)$. As we have that $h^{(p^i-5,3,2)}_{1,1} = p^i - 3$ we cannot remove the last hook. If instead $p^i \leq 7$, let p^j be the second smallest part of $\left((p^k)^{a_k}, \ldots, (p^i)^{a_i}\right)$, which must exist as now $p^i < n$. As $p^i \geq 5$ and $p^j \geq p^i$ we have that $p^i + p^j \geq 8$ and so we can in any case remove all but the last two hooks of the sequence and we obtain $(p^i + p^j - 5, 3, 2)$. If $p^i = 7$ then we have that $(p^i + p^j - 5, 3, 2) = (p^j + 2, 3, 2)$. As $h^{(p^j+2,3,2)}_{1,4} = p^j - 1$, $h^{(p^j+2,3,2)}_{1,3} = p^j + 1$ and any hook of length p^j must be on the first row we cannot remove a p^j-hook. Otherwise we need to have that $p^i = 5$. In this case $(p^i + p^j - 5, 3, 2) = (p^j, 3, 2)$ and $h^{(p^j,3,2)}_{1,3} = p^j - 1$, $h^{(p^j,3,2)}_{1,2} = p^j + 1$ and $h^{(p^j,3,2)}_{2,1} = 4$, so we cannot remove any hook of length p^j. So in any case $(n-5,3,2)$ is of class i and so by lemma 66 we can conclude that p divides the degree of $\chi^{(n-5,3,2)}$.

So the theorem is proved for $p \neq 2$ or for $p = 2$ and $i \neq 2$.

Case 7.

We will now show that if $p = 2$ and $i = 2$ then $(c_1,\ldots,c_h)$ cannot end with $(d,3,3,1,1,1)$, for $d \geq 4$ when $(c_1,\ldots,c_h)$ is p-vanishing. We will first show that 2 divides the degree of $\chi^{(n-6,3,1,1,1)}$. As $n \geq 12$ as $4|n$, $8 \not| n$ and $n > 3+3+1+1+1 = 9$ we have that $n - 6 \geq 6$ and so $(n-6,3,1,1,1)$ is a partition of n. Again let 2^j the second smallest part of $\left((2^k)^{a_k}, \ldots, (2^i)^{a_i}\right)$.

Here we need to have that $2^j \geq 8$, as $2^j > 2^i = 4$ (as $a_l = 0, 1$ for any l when $p = 2$) and so as $h_{1,4} = n - 9 \geq n - p^i - p^j$ and as all hooks not on the first row are at most of length 6, we must remove all hooks apart for the last two from the first row and we obtain $(2^i + 2^j - 6, 3, 1, 1, 1) = (2^j - 2, 3, 1, 1, 1)$. Now $h_{1,2}^{(2^j-2,3,1,1,1)} = 2^j - 2$ and $h_{1,1}^{(2^j-2,3,1,1,1)} = 2^j + 2$, so we cannot remove any hook of length 2^j from $(2^j - 2, 3, 1, 1, 1)$ and so we need to have by lemma 66 that 2 divides the degree of $\chi^{(n-6,3,1,1,1)}$.

We will show that $\chi^{(n-6,3,1,1,1)}_{(c_1,\ldots,c_h)} \neq 0$ whenever $(c_1, \ldots, c_h)$ is ending by $(d, 3, 3, 1, 1, 1)$ with $d \geq 4$. As the hooks not on the first row are either 6 or at most 3, we can remove all hooks of the lengths that appear in $(c_1, \ldots, c_h)$ which are bigger then 3 and not equal to 6 in a unique way, if we remove them in decreasing order of length. Now, if there were l 6's in $(c_1, \ldots, c_h)$ we are left with $(6l + 3, 3, 1, 1, 1)$. If we remove some of the 6-hooks from the second row we have that this hook has leg-length 3 and all other hooks must be removed in a unique way and they all have leg-length 0. This can be obtained in l different ways. Otherwise we remove all 6-hooks from the first row and it is easy to see that they must all have leg-length 0 and that we obtain $(3, 3, 1, 1, 1)$. As $\chi^{(3,3,1,1,1)}_{(3,3,1,1,1)} = -3$ we have that

$$\chi^{(n-6,3,1,1,1)}_{(c_1,\ldots,c_h)} = -l - 1 - 1 - 1 - 1 + 1 = -l - 3 \neq 0$$

for any l and so $\chi^{(n-6,3,1,1,1)}_{(c_1,\ldots,c_h)} \neq 0$ for any $(c_1, \ldots, c_h)$ ending by $(d, 3, 3, 1, 1, 1)$ with $d \geq 4$.

Case 8.

So by theorem 59 in order to finish proving the theorem we only need to show that when $p = 2$ and $i = 2$ then there is some 2-vanishing $(c_1, \ldots, c_h)$ ending by $(d, 2, 1, 1)$ for some $d \geq 4$. Let $n = 2^{m_1} + \ldots + 2^{m_l}$, with $m_j < m_{j+1}$ for each j, be the 2-adic decomposition of n ($m_1 = i = 2$ and $m_l = k$). Let $(c_1, \ldots, c_h) = (2^{m_l}, \ldots, 2^{m_2}, 2, 1, 1)$. We want to show that in this case $\chi^\alpha_{(c_1,\ldots,c_h)} = 0$ for any χ^α of even degree. By lemmas 61 and 65 if α is a partition of n such that $2|\deg(\chi^\alpha)$, we cannot remove a sequence of hooks of lengths $(2^{m_l}, \ldots, 2^{m_1})$ from α. So assume that β is obtained by α by removing a sequence of hooks of lengths $(2^{m_l}, \ldots, 2^{m_2})$. Then $\beta \vdash 4$ and β cannot contain any 4-hook. So $\beta = (2, 2)$. It is easy to see that we can remove a sequence of one 2-hook and two 1-hooks from β in two different ways, one with sum of the leg-lengths equal to 0 and one with the sum of the leg-lengths equal to 1. So any time we can remove a sequence of hooks of lengths $(2^{m_l}, \ldots, 2^{m_2})$ from α we can remove from what we obtain a sequence of one 2-hook and two 1-hooks in two different ways and these ways have different sum of the leg-lengths modulo 2. So using lemma 54 we need to have that $(c_1, \ldots, c_h)$ is 2-vanishing and so the theorem is proved. □

Lemma 68. *If for some j, $a_0 + \ldots + a_j p^j \neq 0, n$ and $\alpha = (c, 1^{n-c})$, with $a_0 + \ldots + a_j p^j \leq n - c < p^{j+1}$, we have that p divides the degree of χ^α.*

Proof. As $n \neq a_0 + \ldots + a_j p^j$ we need to have that $n \geq p^{j+1}$ (as then we need to have that $a_i \neq 0$ for some $i > j$ and so $n \geq a_i p^i \geq p^{j+1}$, actually $n > p^{j+1}$, but $n \geq p^{j+1}$ is enough here) and so α is a partition of n, as $n - c < p^{j+1}$. Also as $h^\alpha_{2,1} = n - c < p^{j+1}$, if we can remove a path of hooks of lengths $((p^k)^{a_k}, \ldots, 1^{a_0})$, we need to remove all the hooks of length $\geq p^{j+1}$ from the first row. Let p^m be the minimal of such lengths. Notice that p^m exists as $n \neq a_0 + \ldots + a_j p^j$. As

$$\begin{aligned} h^\alpha_{1,2} &= c - 1 = n - (n - c) - 1 \geq n - p^{j+1} \geq n - p^m \\ &\geq a_k p^k + \ldots + a_{m+1} p^{m+1} + (a_m - 1) p^m \end{aligned}$$

and $l^\alpha_{1,2} = 0$, we can remove from α in a unique way a sequence of hooks with lengths $((p^k)^{a_k}, \ldots, (p^{m+1})^{a_{m+1}}, (p^m)^{a_m - 1})$ and we obtain

$$(c - (a_k p^k + \ldots + a_{m+1} p^{m+1} + (a_m - 1) p^m), 1^{n-c}) = (a_0 + \ldots + a_j p^j + p^m - n + c, 1^{n-c}).$$

Now we have that in this partition $h_{2,1} = n - c < p^{j+1} \leq p^m$,

$$h_{1,1} = a_0 + \ldots + a_j p^j + p^m > p^m$$

as $a_0 + \ldots + a_j p^j \neq 0$ and

$$h_{1,2} = a_0 + \ldots + a_j p^j + p^m - n + c - 1 = p^m - 1 + (a_0 + \ldots + a_j p^j) - (n - c) \leq p^m - 1$$

and so there is this partition has no hook of length p^m. So we have that we cannot recursively remove from α a sequence of hooks with hook-lengths $((p^k)^{a_k}, \ldots, (p^m)^{a_m})$, that is α is of degree m and so by lemma 66 we have that p divides the degree of α. □

Theorem 69. *Let i be minimal such that $a_i \neq 0$. If $(c_1, \ldots, c_h)$ is p-vanishing with $c_h \geq 1$ we have that $c_h \leq a_i p^i$.*

Proof. Assume that $c_h > a_i p^i$. As $\sum c_j = n = \sum a_j p^j$ we then need to have that $n \neq a_i p^i$, by the minimality of i such that $a_i \neq 0$. So we have that $(n - a_i p^i, 1^{a_i p^i})$ is actually a partition of n. By the previous lemma with $j = i$ we have that p divides the degree of χ^α and it is easy to see by the Murnaghan-Nakayama formula that $\chi^{(n-a_i p^i, 1^{a_i p^i})}_{(c_1, \ldots, c_h)} = (-1)^{a_i p^i}$ (all $c_j > a_i p^i = h^{(n-a_i p^i, 1^{a_i p^i})}_{2,1}$ and so all hooks must be removed from the first row). □

Corollary 70. *Let i be maximal such that $p^i | n$ (i minimal such that $a_i \neq 0$). If $a_i = 1$, $i \neq 1$ if $p = 3$ and $i \neq 1, 2$ if $p = 2$ and if $(c_1, \ldots, c_h) \vdash n$ is p-vanishing and $c_h \geq 1$, we need to have that $c_h = p^i$.*

Proof. By theorem 67 we have that in this case c_h needs to be at least p^i, while by theorem 69 we have that this part needs to be at most $a_i p^i = p^i$, as such an i is also minimal such that $a_i \neq 0$. $\square$

Until now we have proved results on the smallest part of a p-vanishing partition. Now we will turn to study the largest parts of a p-vanishing partition. In the following k' is any non-negative integer, so we are not assuming that $k' \leq k$, where k is maximal such that $a_k \neq 0$.

Lemma 71. *Let $n = a + bp^{k'}$ with $0 \leq a \leq p^{k'} - 1$ and $b \geq 0$. If we cannot recursively remove a sequence of b hooks of length $p^{k'}$ from $\alpha \vdash n$ then χ^α is of class k' and so the degree of χ^α is divisible by p.*

Proof. This follows easily from the definition of a partition of class k', lemma 66 and as removing a sequence of hooks of lengths $((p^k)^{a_k}, \ldots, (p^{k'})^{a_{k'}})$ is equivalent to remove a sequence of $b = \sum_{j \geq k'} a_j p^{j-k'}$ hooks all of length $p^{k'}$. $\square$

Definition 37. *For a given partition $(c_1, \ldots, c_h)$ and a given $k' \in \mathbb{N}$, define*

$$d_{k'} = \sum_{c_j \geq p^{k'}} c_j,$$

that is $d_{k'}$ is the sum of the parts of $(c_1, \ldots, c_h)$ which are greater or equal to $p^{k'}$.

From theorem 72 until conjecture 76 we will let $d = d_{k'} = \sum_{c_j \geq p^{k'}} c_j$.

Theorem 72. *Let $n = a + bp^{k'}$ where $0 \leq a \leq p^{k'} - 1$ and $b \geq 0$. Let $(c_1, \ldots, c_h)$ be p-vanishing with $c_h > 0$. If $d = bp^{k'}$ then c_j is a multiple of $p^{k'}$ whenever $c_j \geq p^{k'}$.*

Proof. If $b = 0$ the theorem clearly holds, as then, as for all $a \leq p^{k'} - 1$, we have that $n < p^{k'}$ and so in this case no part of $(c_1, \ldots, c_h)$ can be $\geq p^{k'}$. So we can assume that $b > 0$.

Assume that $d = bp^{k'}$ but not all for all j for which $c_j \geq p^{k'}$ we have that c_j is a multiple of $p^{k'}$. We want to show that in this case $(c_1, \ldots, c_h)$ isn't p-vanishing. Let l be maximal such that $c_l \geq p^{k'}$ and $p^{k'} \not| c_l$. Notice that we cannot have that $p^{k'} = 1$ in this case. Also by definition of d and l we need to have that $bp^{k'} = d \geq 2c_l$ (as there needs to be some $l' < l$ such that $p^{k'} \not| c_{l'}$ and so $d \geq c_{l'} + c_l \geq 2c_l$). The proof of this theorem will be divided in the following cases:

Case 1: $n = bp^{k'}$,

Case 2: $n \neq bp^{k'}$ and $c_l = cp^{k'} + e$ with $a \leq e < p^{k'}$,

Case 3: $n \neq bp^{k'}$ and $c_l = cp^{k'} + e$ with $1 \leq e < a$.

These cases cover all the possibilities, as c_l isn't divisible by $p^{k'}$.

Case 1.

Assume that $n = bp^{k'}$. In this case $c_j \geq p^{k'}$ for all $j \leq h$. Consider $\alpha \vdash n$ given by $(n - c_l, 2, 1^{c_l - 2})$. As $c_l > p^{k'} \geq 2$ and $n - c_l \geq c_l > 2$ we have that α is a partition of n. Let $m = \min\{j : c_j = c_l\} \cup \{h-1\}$. By definition of l it is easy to see that $l \geq 2$ (as c_l appears in the summation $d = \sum_{c_j \geq p^{k'}} c_j$, as $p^{k'}|d$ and l is maximal such that $p^{k'} \not| c_l$, there must exists some $j < l$ such that $p^{k'} \not| c_j$), so that we have $h \geq 2$ and then $m \geq 1$. As all $c_j \geq 2$, $j \leq h$, we have that $h^{\alpha}_{1,3} = n - c_l - 2 \geq c_1 + \ldots + c_{m-1}$ $(m - 1 < l, h - 1)$. As we also have $h^{\alpha}_{2,1} = c_l < c_j$ for all $j \leq m - 1$ and as $l^{\alpha}_{1,3} = 0$ we can then remove the first $m - 1$ hooks of any sequence with of hooks with lengths $(c_1, \ldots, c_h)$ in a unique way from α and we obtain $(c_m + \ldots + c_h - c_l, 2, 1^{c_l - 2})$. If $c_m > c_l$ we need to have that $m = h - 1$ and $l = h$ and then the partition we obtain is $(c_{h-1}, 2, 1^{c_l - 2})$. Here $h_{2,1} < c_{h-1}$ and $h_{1,2} = c_{h-1}$, so we can remove also the next hook of the sequence in a unique way, which would leave (1^{c_l}), for which $h_{1,1} = c_l$. So in this case we have by the Murnaghan-Nakayama formula that $\chi^{\alpha}_{(c_1,\ldots,c_h)} = \pm 1 \neq 0$. Next assume that $c_m = c_l$ and $l = h$. So we now need to remove $l - m + 1 \geq 2$ hooks all of lengths c_l from $(c_m + \ldots + c_h - c_l, 2, 1^{c_l - 2}) = ((l - m)c_l, 2, 1^{c_l - 2})$. If we remove one of the first $l - m - 1$ of these hooks from the second row then this hook must be $H_{2,1}$ and we then need to remove all other hooks in a unique way (they all must be removed from the first row). Also the hook we would remove from the second row would have leg-length $c_l - 2$ and all others would have leg-length 0. This can be done in $l - m - 1$ different ways. Otherwise all the first $l - m - 1$ hooks of length c_l must be removed from the first row and they all have leg-length 0 (as in $((l - m)c_l, 2, 1^{c_l - 2})$, $h_{1,3} \geq (l - m - 1)c_l$ and $l_{1,3} = 0$). Now we would get $(c_l, 2, 1^{c_l - 2})$. Now we can either first remove $H_{2,1}$ and then $H_{1,1}$ or first $H_{1,2}$ and then $H_{1,1}$. In the first case the last two leg-lengths are $c_l - 2$ and 0, while in the second case they are 1 and $c_l - 1$. So by lemma 54 we have that in this case $\chi^{\alpha}_{(c_1,\ldots,c_h)} = (-1)^{c_l - 2}(l - m) + (-1)^{c_l} = (-1)^{c_l}(l - m + 1) \neq 0$ as $m \leq l$. The last case is when $c_m = c_l$ and $l < h$. If we remove one of the c_l-hooks from the second row then this hook would need to be $H_{2,1}$, it would have leg-length $c_l - 2$ and all other hooks would need to be removed from the first row and they would all have leg-length 0. This can be done in $l - m + 1 \geq 1$ different ways. Otherwise all c_l-hooks must be removed from the first row. Assume this is possible. Then as the partition we obtain must be a partition of a non-zero multiple of $p^{k'}$ by definition of l, the last hook we removed cannot have been

$H_{1,1}$ or $H_{1,2}$ (in the first case we would be left with either (0) or (1), while in the second case with (1^{c_l}) and c_l isn't a multiple of $p^{k'}$). So the partition we obtained is of the form $(e,2,1^{c_l-2})$, for some $e\geq 2$ such that $p^{k'}|e+c_l$. We will show that we cannot remove a sequence of $(e+c_l)/p^{k'}$ hooks of length $p^{k'}$ from $(e,2,1^{c_l-2})$, which also will prove that α is of class k' by lemma 71. As any maximal sequence of $p^{k'}$-hooks that we remove from $(e,2,1^{c_l-2})$ consists of the same number of hooks, it is enough to show that there is a maximal sequence of $p^{k'}$-hooks which are recursively removed from $(e,2,1^{c_l-2})$ which contains less than $(e+c_l)/p^{k'}$ hooks. Let m_1 and m_2 maximal such that $m_1p^{k'}\leq e-2$ and $m_2p^{k'}\leq c_l-2$ respectively. As $(m_1+m_2)p^{k'}\leq e+c_l-4$ we have that $m_1+m_2<(e+c_l)/p^{k'}$. Now we can recursively remove first m_1 $p^{k'}$-hooks from the first row of $(e,2,1^{c_l-2})$ and then m_2 $p^{k'}$-hooks from the first column of the resulting partition and it is easy to see that this way we obtain $(e-m_1p^{k'},2,1^{c_l-m_2p^{k'}-2})$. In this partition we have that $h_{2,2}=1\neq p^{k'}$, $h_{1,1}=e+c_l-(m_1+m_2)p^{k'}-1$, which isn't divisible by $p^{k'}$ as $p^{k'}|e+c_l$, $h_{1,2}=e-m_1p^{k'}\leq p^{k'}+1$, $h_{2,1}=c_l-m_2p^{k'}\leq p^{k'}+1$ by definition of m_1 and m_2, so that $h_{1,i}\leq h_{1,3}=h_{1,2}-2<p^{k'}$ and $h_{i,1}\leq h_{3,1}=h_{2,1}-2<p^{k'}$ for any $i\geq 3$ (when $(1,i)$ or $(i,1)$ is a node of $(e-m_1p^{k'},2,1^{c_l-m_2p^{k'}-2})$) and as $p^{k'}\not|e,c_l$, we then have that no hook of $(e-m_1p^{k'},2,1^{c_l-m_2p^{k'}-2})$ has length equal to $p^{k'}$ and so any maximal sequence of $p^{k'}$-hooks which are recursively removed from $(e,2,1^{c_l-2})$ contains $m_1+m_2<(a+c_l)/p^{k'}$ hooks and so we have that also when $m=h-1$ and $c_h\neq c_l$ then $\chi^\alpha_{(c_1,\ldots,c_h)}\neq 0$ and that we always have that when $n=bp^{k'}$ then the degree of χ^α is divisible by p. So we have proved the case where $n=bp^{k'}$.

Case 2.

Assume now that $n\neq bp^{k'}$ and that, when we write $c_l=cp^{k'}+e$, with $0\leq e<p^{k'}$, then $e\geq n-bp^{k'}=a$. Notice that as c_l isn't a multiple of $p^{k'}$, $e\neq 0$, so this is always the case when $n-bp^{k'}=1$. Let $\beta=(n-c_l,1^{c_l})$. As $n>bp^{k'}>c_l$, as c_l is not a multiple of $p^{k'}$ and $c_l\leq d=bp^{k'}$, we have that β is a partition of n. Let m be minimal such that $c_m=c_l$. As $n\neq bp^{k'}$ and $d=bp^{k'}$, we need to have that $m\leq l<h$. So as

$$h^\beta_{1,2}=n-c_l-1\leq n-c_l-c_h\leq c_1+\ldots+c_{m-1},$$

$l^\beta_{1,2}=0$ and $h_{2,1}=c_l<c_j$ for $j<m$, we can remove the first $m-1$ hooks of a sequence with hook-lengths $(c_1,\ldots,c_h)$ from β in a unique way and we obtain $(c_{m+1}+\ldots+c_h,1^{c_l})$ (as $c_m=c_l$). If we remove some of the c_l-hooks from the second row, this hook must be $H_{2,1}$ and it must have leg-length c_l-1. All other hooks must be removed from the first row and they must have leg-length 0. This can be done in $l-m+1\geq 1$ different ways. Otherwise we must have removed all c_l-hooks from the first row. If this is

possible, after having removed these hooks we must obtain a partition of the form $(f, 1^{c_l})$. As this partition is obtained from β by removing a sequence of hooks, all from the first row, whose sum of the lengths is a multiple of $p^{k'}$, we need to have that $f + c_l \equiv n \equiv n - ap^{k'} \mod p$. So as removing a $qp^{k'}$-hook is equivalent to removing a certain sequence of q hooks of length $p^{k'}$, in order to show that $\chi^{\beta}_{(c_1,\ldots,c_h)} \neq 0$ and that p divides the order of χ^{β}, it is enough to show that whenever $\lambda = (f, 1^g)$, such that $f + g = sp^{k'} + t$ and $f = rp^{k'} + u$, for some $s, r \geq 0$ and $1 \leq t \leq u < p^{k'}$, then we cannot remove from λ a sequence of s hooks of length $p^{k'}$. As $1 \leq t < p^{k'}$ we have that $h^{\lambda}_{1,1} = f + g$ is not divisible by $p^{k'}$. So whenever $h^{\lambda}_{i,j} = p^{k'}$, then we have that $(i, j) = (1, c)$ or $(i, j) = (c, 1)$ for some $c \geq 2$. We can remove a hook from the first row if and only if $f > p^{k'}$ and in this case $h_{1,f-p^{k'}+1} = p^{k'} - 1 + 1 = p^{k'}$, so after having removed this hook and we obtain $(f - p^{k'}, 1^g)$ and now $f - p^{k'} + b = (s - 1)p^{k'} + t$ and $g = rp^{k'} + u$. We can remove a hook from the first column if and only if $g \geq p^{k'}$ and in this case $h_{g-p^{k'}+2} = g - 1 - (g - p^{k'} + 2) + 1 = p^{k'}$ and after having removed this hook and we obtain $(f, 1^{g-p^{k'}})$ and we have that $f + g - p^{k'} = (s - 1)p^{k'} + t$ and $f - p^{k'} = (r - 1)p^{k'} + u$. So we can remove a $p^{k'}$-hook from λ if and only if $f > p^{k'}$ or $g \geq p^{k'}$ and in any way we remove any such hook from it we obtain a partition of the form $(f', 1^{g'})$, where $f' + g' = (s - 1)p^{k'} + t$ and $g' = r'p^{k'} + u$. So we have that after having removed any maximal sequence of $p^{k'}$-hooks from λ we obtain a partition $(f'', 1^{g''})$, where $1 \leq f'' \leq p^{k'}$ and $0 \leq g'' < p^{k'}$ and f'' and g'' satisfy $f'' + g'' = s'p^{k'} + t$ and $g'' = r''p^{k'} + u$, for some $s', r'' \geq 0$. As $g'' < p^{k'}$ we need to have that $r'' = 0$ and so $g'' = u$. We want to show that $s' \geq 1$, as then we cannot remove any $p^{k'}$-hook from $(f'', 1^{g''})$ and as we then need to have that this partition is obtain from λ by removing less then s $p^{k'}$-hooks we have that we cannot remove any sequence of s $p^{k'}$-hooks from λ. But the fact that $s' \geq 1$ follows from the fact that $t \leq u$ and $e \geq 1$, and so we have that if $n \neq bp^{k'}$, $c_l = cp^{k'} + f''$, with $n - ap^{k'} \leq f'' < p^{k'}$ and $\beta = (n - c_l, 1^{c_l})$, then p divides the order of χ^{β} and $\chi^{\beta}_{(c_1,\ldots,c_h)} \neq 0$, in particular $(c_1, \ldots, c_h)$ is not p-vanishing in this case.

Case 3.

In this case we have again that $n \neq bp^{k'}$ but differently from the previous case we now have that $c_l = cp^{k'} + e$ for some $1 \leq e < n - bp_{k'}$. Notice that $a = n - bp^{k'} \geq 2$ in this case. Here let $\gamma = (n - c_l, n - bp^{k'}, 1^{c_l - n + bp^{k'}})$. As $c_l < bp^{k'}$, $n \neq bp^{k'}$ and $n - bp^{k'} < p^{k'} < c_l$ we have that γ is a partition of n. We want to show that p divides the degree of χ^{γ} and that in this case $\chi^{\gamma}_{(c_1,\ldots,c_h)} \neq 0$. We will actually show that this holds whenever $1 \leq e < p^{k'}$ and $e \neq n - bp^{k'}$. As $h^{\gamma}_{2,1} = c_l$ is not divisible by p and $h^{\gamma}_{2,2} = n - bp^{k'} - 1 < p^{k'}$, any

$p^{k'}$-hook of γ must be either on the first row or is of the form $(q, 1)$, for some $q \geq 3$. If we remove a $p^{k'}$-hook of the form $H^{\gamma}_{q,1}$, we are left with a partition of the form $(f, n - bp^{k'}, 1^g)$, for some f, g such that $f + g$ is divisible by $p^{k'}$ (the partition we obtain is a partition of $n - p^{k'} = (b-1)p^{k'} + n - bp^{k'}$) but g isn't divisible by $p^{k'}$, as $n - bp^{k'} + g + p^{k'} = c_l \not\equiv n - bp^{k'} \mod p^{k'}$. Also we still get a partition of the form $(f, n - bp^{k'}, 1^g)$ with $p^{k'}|(f+g)$, $p^{k'} \not| g$ whenever we remove a $p^{k'}$-hook from the first row of γ and $h^{\gamma}_{1,n-bp^{k'}+1} \geq p^{k'}$ (in this case $g = c_l - n + bp^{k'}$). So repeating this argument, while $h_{1,n-bp^{k'}+1} \geq p^{k'}$ we have that whichever $p^{k'}$ hook we remove from the partition we obtained at the previous stage we always get a partition of the form $(f, n - bp^{k'}, 1^g)$ with $p^{k'}|(f+g)$, $p^{k'} \not| g$. Also as long as $h_{1,n-bp^{k'}+1} \geq p^{k'}$ we always can remove a $p^{k'}$-hook from this partition, as $l_{1,n-bp^{k'}+1} = 0$. So we can assume that $h_{1,n-bp^{k'}+1} < p^{k'}$, that is $n - bp^{k'} \leq f < n - (b-1)p^{k'}$. Now if $g \geq p^{k'}$ we can still remove a hook of the form $H_{q,1}$, for some $q \geq 3$, getting again a partition of the same kind as before, or we can remove, if possible, a $p^{k'}$-hook from the first row. If $f = n - bp^{k'} + p^{k'} - 1$ we have that $h_{1,n-bp^{k'}+1} = p^{k'} - 1$ and $h_{1,n-bp^{k'}} = p^{k'} + 1$, so we cannot remove $p^{k'}$-hooks from the first row and so, as the only possible $p^{k'}$-hooks which we can recursively remove from this point are those of the form $H_{q,1}$, for $q \geq 3$, after having removed as many $p^{k'}$-hooks as possible we would get a partition of the form $(n - bp^{k'} + p^{k'} - 1, n - bp^{k'}, 1^g)$, and as $n - bp^{k'} \neq 0$ we have that this is a partition of a number which is at least $p^{k'}$, so by definition of b (b is the biggest integer such that $bp^{k'} \leq n$) we cannot remove a sequence of b $p^{k'}$-hooks from γ and so in this case we have that p divides the degree of χ^{γ} by lemma 71.

So assume now that when we obtain the partition $(f, n - bp^{k'}, 1^g)$ with $n - bp^{k'} \leq f < n - (b-1)p^{k'}$ we have that

$$f \neq n - bp^{k'} + p^{k'} - 1 = n - (b-1)p^{k'} - 1.$$

Then $f < n - (b-1)p^{k'} - 1$. If $f \geq p^{k'}$ it can be easily seen that in this partition $h_{1,f-p^{k'}+2} = p^{k'}$, as in this case $2 \leq f - p^{k'} + 2 \leq n - bp^{k'}$. Also as we now either have to remove this hook or a hook of the form $H_{q,1}$, $q \geq 3$, we have that $h_{1,f-p^{k'}+2}$ is constant as long as we don't remove the corresponding hook. So, again as if we remove $H_{q,1}$, $q \geq 3$ we still get a partition of the kind $(f, n - bp^{k'}, 1^g)$, we have that at some point we would need to remove $H_{1,f-p^{k'}+2}$, and after having done this we obtain $(n - bp^{k'} - 1, f - p^{k'} + 1, 1^g)$, with $p^{k'}|(f+g)$ and $p^{k'} \not| f, g$. Now we have that $h_{2,1} = f + g - p^{k'} - 1$ and $h_{1,1} = n - bp^{k'} + g$ are not divisible by $p^{k'}$. As we also have that $h_{1,2} = n - bp^{k'} - 1 < p^{k'}$ we have that any $p^{k'}$-hook of $(n - bp^{k'} - 1, f - p^{k'} + 1, 1^g)$ must be of the form $H_{q,1}$, for some $q \geq 3$. After having removed if possible this hook (which exists if and only if $g \geq p^{k'}$), we are still left with a partition of this kind and so we can repeat the argument until we obtain a partition of the kind $(n-bp^{k'}-1, f-p^{k'}+1, 1^g)$ with $g < p^{k'}$, which doesn't have any $p^{k'}$-hook. This partition is a partition of $n-bp^{k'}-1+f-p^{k'}+1+g > n-bp^{k'}$ as $f \geq p^{k'}$, $p^{k'} \not| f$ and g cannot be negative. So as if it would be possible to remove a sequence of b $p^{k'}$-hooks from γ we would obtain a partition of $n-bp^{k'}$, we cannot remove such a sequence in this case either and so we have that also in this case χ^γ has degree divisible by p.

So assume now that when we obtain $(f, n-bp^{k'}, 1^g)$ with $n-bp^{k'} \leq f < n-(b-1)p^{k'}$, we have that $f < p^{k'}$. In this case $h_{1,2} = f < p^{k'}$ and $h_{1,1} = n-(n-bp^{k'}-1) = bp^{k'}+1$, which isn't divisible by $p^{k'}$, so in this case there is no $p^{k'}$-hook on the first row. Proceeding as in the case where $f = n-(b-1)p^{k'}-1$ we get to the case where $g < p^{k'}$, in which case are no more $p^{k'}$-hooks. The partition we now have is $(f, n-bp^{k'}, 1^g)$, where both $f, g < p^{k'}$ and $f+g$ is divisible by $p^{k'}$. Also as $f \geq n-bp^{k'}$ we have that $f+g > 0$, so this partition is a partition of a number bigger than $n-bp^{k'}$ and so again we have that p divides the degree of χ^γ. So we have that in any case the degree of χ^γ is divisible by p and so we only have left to show that when $n-bp^{k'} \geq 2$, $c_l = cp^{k'}+e$ for some $1 \leq e < p^{k'}$ and $e \neq n-bp^{k'}$, we have that $\chi^\gamma_{(c_1,\ldots,c_h)} \neq 0$.

Now as $h^\gamma_{2,1} = c_l$, $h^\gamma_{1,n-bp^{k'}+1} \geq \sum_{c_j > c_l} c_j$ and $l^\gamma_{1,n-bp^{k'}+1} = 1$, if m is maximal such that $c_m > c_l$ (let $m = 0$ when $c_1 = c_l$), we can remove the first m hooks of a sequence of hooks of lengths $(c_1, \ldots, c_h)$ from γ in a unique way, obtaining a partition of the kind $(f, n-bp^{k'}, 1^{c_l-n+bp^{k'}})$. After having done this we can remove one of the c_l-hooks from the second row, in which case this hook must be $H_{2,1}$ and has leg-length $c_l-n+bp^{k'}$, and then all other hooks must be removed in a unique way (they are all on the first row) and they all must have leg-length 0. This can be done in at least one way and by the considerations we just made on the leg-lengths we have that the sum of the leg-lengths is constant in all these cases. Otherwise, if possible we must have removed all the hooks of length c_l from the first row. By the maximality of l such that $c_l \geq p^{k'}$ and $p^{k'} \not| c_l$ and as $d = n-bp^{k'}$, the partition we obtained this way must be a partition of a number of the form $n-bp^{k'}+sp^{k'}$, for some $s \geq 0$. If one of the hooks we removed was $H_{1,1}$, then the partition we obtained must be (f), for some $f < n-bp^{k'}$, which gives a contradiction. Also as if at some step we removed some hook of the form $H_{1,q}$, with $q \leq n-bp^{k'} = a$ as all hooks we removed are on the first row after having removed this hook we must have obtained $(n-bp^{k'}-1, q-1, 1^{c_l-n+bp^{k'}})$ and as now $h_{1,2} = n-bp^{k'}-1, n-bp^{k'}-2 < p^{k'} < c_l$, this hook must have been the last one we removed. So the partition we obtain removing, if possible, all hooks of length $\geq c_l$ from the first row of γ must either be a partition of the form $(f, n-bp^{k'}, 1^{c_l-n+bp^{k'}})$ or $(n-bp^{k'}-1, q, 1^{c_l-n+bp^{k'}})$, from which we first need to remove a sequence of hooks of lengths multiple of $p^{k'}$ and then some hooks of total length equal to $n-bp^{k'}$. We will show that if $(f, n-bp^{k'}, 1^{c_l-n+bp^{k'}})$ or $(n-bp^{k'}-1, q, 1^{c_l-n+bp^{k'}})$ is a partition of $n-bp^{k'}+sp^{k'}$, then it is not possible to remove from this partition a sequence of s $p^{k'}$-hooks. The first case is done just like when we proved that p divides the degree of χ^γ ($f=(n-bp^{k'}+sp^{k'})-c_l$), so it is enough to show this for the second case. In this case we have that

$$q = n-bp^{k'}+sp^{k'}-n+bp^{k'}+1-c_l+n-bp^{k'} = n-bp^{k'}+sp^{k'}-c_l+1,$$

$$h_{1,2} = n-bp^{k'}-1, n-bp^{k'}-2 < p^{k'},$$
$$h_{1,1} = n-bp^{k'}+sp^{k'}-q+1 = c_l > p^{k'}$$

and so any $p^{k'}$-hook must be of the form $H_{t,1}$, $t \geq 3$. As removing this hook, if it exists, we obtain $(n-bp^{k'}-1, q, 1^g)$ for $g = c_l-n+bp^{k'}-p^{k'}$, and we have that $H_{1,2}$ remains the same, $H_{1,1}$ is decreased by $p^{k'}$ and so it still isn't divisible by $p^{k'}$ by definition of l, we have that we can remove $p^{k'}$-hooks only as long as $g' \geq p^{k'}$ and we always obtain a partition of the from $(n-bp^{k'}-1, q, 1^{g'})$ for some $g' = c_l-n+bp^{k'}-ip^{k'}$ for some i. As we have that $c_l \not\equiv n-bp^{k'}$ mod p, we also need to have that g' is always positive. So after having removed some $p^{k'}$-hooks from $(n-bp^{k'}-1, q, 1^{c_l-n+bp^{k'}})$ we obtain $(n-bp^{k'}-1, q, 1^{g'})$, with $1 \leq g' < p^{k'}$. This partition doesn't have any more $p^{k'}$-hooks and it is a partition of $n-bp^{k'}-1+q+g' > n-bp^{k'}$. In particular we cannot remove a sequence of s $p^{k'}$-hooks from $(n-bp^{k'}-1, q, 1^{c_l-n+bp^{k'}})$, and so when we can remove all hooks with lengths $\geq c_l$ of the sequence with lengths $(c_1, \ldots, c_h)$ from γ from the first row we cannot finish removing such a sequence of hooks. So whenever removing such a sequence of hooks from γ we always need to remove one c_l-hook from the second row and all other hooks from the first row and so by the previous considerations we have by lemma 54 that if $n-bp^{k'} \geq 2$, $c_l = cp^{k'}+e$ for some $1 \leq e < p^{k'}$ and $e \neq n-bp^{k'}$ then $\chi^\gamma_{(c_1,\ldots,c_h)} \neq 0$ and so as we have finished considering also case 3, we have that the theorem is proved. □

Theorem 73. *Let $n = a+bp^{k'}$ with $0 \leq a \leq p^{k'}-1$ and $b \geq 0$. Let $(c_1, \ldots, c_h)$ be p-vanishing. We have that $d \leq bp^{k'}$ in the following cases:*

- $p \neq 3$,
- $p = 3$, $k' \geq 2$,
- $p = 3$, $k' = 1$, $n \not\equiv 2$ *mod* 3.

In addition we have in the case where $p = 3$, $k' = 1$:

- *If $n = 2$ then $d = 0$,*
- *If $n = 5$ and $d > 3$ then $(c_1, ...) = (4, 1)$,*
- *If $n = 8$ and $d > 6$ then $(c_1, ...) = (4, 3, 1)$.*

Proof. Here too we can assume that $b > 0$, as if $b = 0$ then $n < p^{k'}$ and so the theorem clearly holds in this case. Also as the particular cases (when $n = 2, 5, 8$, $p = 3$ and $k' = 1$) can be easily checked by finding the character table of S_n (the case $n = 2$, $p = 3$ and $k' = 1$ follows also by the fact that in this case $b = 0$), we will only prove the first part of the theorem. This is done by considering the following cases:

Case 1: $n = bp^{k'}$,

Case 2: $n \neq bp^{k'}$ and $d = n$,

Case 3: $a = n - bp^{k'} \neq 0, p^{k'} - 1$ and $bp^{k'} < d < n$,

Case 4: $n - bp^{k'} = p^{k'} - 1 \neq 0$, $bp^{k'} < d < n$ and $n - d > d - bp^{k'}$,

Case 5: $n - bp^{k'} = p^{k'} - 1 \neq 0$, $bp^{k'} < d < n$ and $n - d < d - bp^{k'}$,

Case 6: $n - bp^{k'} = p^{k'} - 1 \neq 0$, $bp^{k'} < d < n$, $n - d = d - bp^{k'}$ and $n - bp^{k'} \geq 4$,

Case 7: $n - bp^{k'} = p^{k'} - 1 \neq 0$, $bp^{k'} < d < n$, $n - d = d - bp^{k'}$ and $n - bp^{k'} = 2$.

It is easy to see that this way we cover all cases where $d > bp^{k'}$, as in cases 6 and 7 $n - bp^{k'}$ needs to be even and non-zero.

Case 1.

Let $n = bp^{k'}$. Then we have that

$$d = \sum_{c_j \geq p^{k'}} c_j \leq \sum_i c_i = n = bp^{k'}$$

for any $(c_1, \ldots, c_h) \vdash n$, so the theorem clearly holds in this case too.

Case 2.

Assume now that $n \neq bp^{k'}$ and $d = n$. Let $\alpha = \left(bp^{k'}, 1^a\right)$. By lemma 68 we have that p divides the degree of χ^α. We will show that if $d = n$ then $\chi^\alpha_{(c_1,\ldots,c_h)} \neq 0$. When $d = n$ we have that $c_j \geq p^{k'}$ for any $1 \leq j \leq h$. Again as $h^\alpha_{2,1} < p^{k'}$ we have that any hook in a sequence of hooks of lengths $(c_1, \ldots, c_h)$ which are recursively removed from α must be on the first row. So we can remove them in at most one way and so by the Murnaghan-Nakayama formula in order to show that $\chi^{\alpha}_{(c_1,\ldots,c_h)} \neq 0$ it is enough to show that it is actually possible to remove such a sequence. As

$$h^{\alpha}_{1,2} = (c_1, \ldots, c_h)p^{k'} - 1 = n - a - 1 \geq n - p^{k'} \geq c_1 + \ldots + c_{h-1}$$

as $c_h \geq p^{k'}$, we can remove the first $h-1$ hooks of this sequence and obtain $(c_h - a, 1^a)$, for which $h_{1,1} = c_h$ and so we can recursively remove a sequence of hooks of lengths $(c_1, \ldots, c_h)$ and then we have that $\chi^{\alpha}_{(c_1,\ldots,c_h)} \neq 0$ when $d = n$.

Case 3.
Assume now that $a = n - bp^{k'} \neq p^{k'} - 1$, and that $bp^{k'} < d < n$. Let $\beta = \left(bp^{k'} - 1, n - d + 1, 1^{d-bp^{k'}}\right)$. We will show that β is a partition of n, that χ^{β} has degree divisible by p and that $\chi^{\beta}_{(c_1,\ldots,c_h)} \neq 0$ in this case. As $b \neq 0$ we have that $bp^{k'} - 1 \geq p^{k'} - 1$. Also as $d > bp^{k'}$ and $n - bp^{k'} \neq p^{k'} - 1$ (so that we actually have that $n - bp^{k'} < p^{k'} - 1$) and as we need to have that $d < n$, we have that

$$2 \leq n - d + 1 < n - bp^{k'} + 1 < p^{k'} - 1 + 1 = p^{k'}.$$

So as $d - bp^{k'} > 0$ and

$$bp^{k'} - 1 + n - d + 1 + d - bp^{k'} = n$$

we have that β is a partition of n.

We will now show that we cannot remove a sequence of b hooks of length $p^{k'}$ from β, from which we will have that the degree of χ^{β} is divisible by p by lemma 71. First notice that as

$$h^{\beta}_{2,1} = n - d + 1 + d - bp^{k'} = n - bp^{k'} + 1 < p^{k'} - 1 + 1 = p^{k'}$$

we have that any hook of any sequence of $p^{k'}$-hooks which are recursively removed from β must correspond to some node on the first row. So if we can remove any sequence of b $p^{k'}$-hooks from β we can do this in a unique way. As $n - d + 1 < n - bp^{k'} + 2 < p^{k'} - 1 + 2 = p^{k'} + 1 \leq bp^{k'} + 1$ we have that $n - d + 2 \leq bp^{k'} - 1$ and so $(1, n-d+2) \in \beta$. Also as $l^{\beta}_{1,n-d+2} = 0$ and

$$\begin{aligned} h^{\beta}_{1,n-d+2} &= bp^{k'} - 1 - (n - d + 2) + 1 = -(n - bp^{k'}) + d - 2 \\ &> -(p^{k'} - 1) + bp^{k'} - 2 = (b-1)p^{k'} - 1 \end{aligned}$$

so that $h^{\beta}_{1,n-d+2} \geq (b-1)p^{k'}$ and as $l^{\beta}_{1,n-d+2} = 0$ we can recursively remove $b-1$ hooks of length $p^{k'}$ from β and this way we obtain $\left(p^{k'} - 1, n - d + 1, 1^{d-bp^{k'}}\right)$.

Now as $d > bp^{k'}$ we have that this partition has at least three rows, and so in it we have that $h_{1,1} \geq p^{k'} - 1 + 2 = p^{k'} + 1$. As in this partition we also have that $h_{2,1} \leq h^{\beta}_{2,1} < p^{k'}$ (the inequality is actually an equality but we don't need that here) and $h_{1,2} = p^{k'} - 1 - 1 + 1 = p^{k'} - 1$ we have that this last partition doesn't contain any hook of length $p^{k'}$ and so, as it is the only partition we can obtain from β by recursively removing $b-1$ hooks of length $p^{k'}$, we cannot remove a sequence of b $p^{k'}$-hooks from β and so we have that $p|\deg\left(\chi^{\beta}\right)$.

So we only need to show that $\chi^{\beta}_{(c_1,\ldots,c_h)} \neq 0$ in order to finish the case when $n - bp^{k'} \neq p^{k'} - 1$ and $bp^{k'} < d < n$. Let l be such that $c_l \geq p^{k'}$ and $c_{l+1} < p^{k'}$. As $d > bp^{k'}$ we need to have that $c_1 \geq p^{k'}$, so such an l exists and $l \geq 1$. Then

$$\begin{aligned} h^{\beta}_{1,n-d+2} &= bp^{k'} - 1 - (n-d+2) + 1 \\ &= n - (n - bp^{k'}) - (n-d) - 2 \\ &> n - (p^{k'} - 1) - \textstyle\sum_{j>l} c_j - 2 \\ &= c_1 + \ldots + c_l - p^{k'} - 1 \\ &\geq c_1 + \ldots + c_{l-1} - 1 \end{aligned}$$

and so $h^{\beta}_{1,n-d+2} \geq c_1 + \ldots + c_{l-1}$ and then as $l^{\beta}_{1,n-d+2} = 0$ and $h^{\beta}_{2,1} < p^{k'} \leq c_j$ for $j \leq l$, we can remove the first $l-1$ hooks of any sequence of hooks with hook lengths $(c_1, \ldots, c_h)$ in a unique way and we obtain $\left(c_l - 1 - d + bp^{k'}, n - d + 1, 1^{d-bp^{k'}}\right)$ (what we obtain must be a partition of $c_l + \ldots + c_h$ and the all rows apart from the first one must be as in β). In this partition we have that $h_{1,1} = c_l - 1 - d + bp^{k'} + 1 + d - bp^{k'} = c_l$, so we can now remove a hook of length c_l in a unique way and we obtain $(n-d)$, from which we can remove the last hooks of the sequence in a unique way. So by the Murnaghan-Nakayama formula we have that $\chi^{\beta}_{(c_1,\ldots,c_h)} \neq 0$ when $n - bp^{k'} \neq p^{k'} - 1$ and $bp^{k'} < d < n$.

Case 4.

Let now $d, bp^{k'} \neq n$, $n - bp^{k'} = p^{k'} - 1$, $d > bp^{k'}$ and $n - d > d - bp^{k'}$. Let $\gamma = \left(bp^{k'} + 1, 2^{d-bp^{k'}}, 1^{n-2d+bp^{k'}-1}\right)$. As $d - bp^{k'} \geq 1$,

$$n - 2d + bp^{k'} - 1 = (n-d) - (d - bp^{k'}) - 1 \geq 0$$

and $bp^{k'} + 1 \geq p^{k'} + 1 \geq 3$ ($p^{k'}$ must be at least 2 as $n \neq bp^{k'}$) we have that γ is actually a partition. Also as $bp^{k'} + 1 + 2(d - bp^{k'}) + n - 2d + bp^{k'} - 1 = n$ we have that $\gamma \vdash n$. We will now show that $p|\deg\left(\chi^{\gamma}\right)$. As $h^{\gamma}_{2,1} = d - bp^{k'} + n - 2d + bp^{k'} = n - d < p^{k'}$, $h^{\gamma}_{1,3} = bp^{k'} - 1 \geq (b-1)p^{k'}$ and $l^{\gamma}_{1,3} = 0$, we can recursively remove $b-1$ hooks of length $p^{k'}$ from γ in a unique way obtaining

$\left(p^{k'}+1, 2^{d-bp^{k'}}, 1^{n-2d+bp^{k'}-1}\right)$. Now we have that $h_{1,2}=p^{k'}+d-bp^{k'}>p^{k'}$ (as $d>bp^{k'}$), $h_{1,3}=p^{k'}-1$ and $h_{2,1}<p^{k'}$ and so we cannot remove any more $p^{k'}$-hooks and so we have that the degree of χ^γ is divisible by p by lemma 71.

Also if l is again such that $c_l\geq p^{k'}$ and $c_{l+1}<p^{k'}$ as then $h_{2,1}^\gamma<p^{k'}$, $h_{1,3}^\gamma=bp^{k'}-1=n-p^{k'}\geq c_1+\ldots+c_{l-1}$ (as $c_l\geq p^{k'}$) and $l_{1,3}^\gamma=0$, we can remove from γ the first $l-1$ hooks of a sequence of hooks with lengths $(c_1,\ldots,c_h)$ and we obtain $\left(c_l+bp^{k'}-d+1, 2^{d-bp^{k'}}, 1^{n-2d+bp^{k'}-1}\right)$. As here we have that $h_{2,1}<p^{k'}\leq c_l$ and $h_{1,2}=c_l+bp^{k'}-d+1-1+d-bp^{k'}=c_l$ we can remove the next hook in a unique way and after having done this we obtain $\left(1^{n-d}\right)$, from which we can remove the remaining hooks in a unique way. So by the Murnaghan-Nakayama formula we get that $\chi^\gamma_{(c_1,\ldots,c_h)}\neq 0$ in the case where $d, a'_{k'}p^{k'}\neq n$, $n-bp^{k'}=p^{k'}-1$, $d>bp^{k'}$ and $n-d>d-bp^{k'}$ and so also in this case $(c_1,\ldots,c_h)$ isn't a p-vanishing partition.

Case 5.
Assume now that $d, bp^{k'}\neq n$, $n-bp^{k'}=p^{k'}-1$, $d>bp^{k'}$ and that we now have $n-d<d-bp^{k'}$. Let $\delta=\left(bp^{k'}+1, 2^{n-d}, 1^{2d-n-bp^{k'}-1}\right)$. As $bp^{k'}+1\geq 3$, $n-d\geq 1$, $2d-n-bp^{k'}-1=(d-bp^{k'})-(n-d)-1\geq 0$ and $bp^{k'}+1+2(n-d)+2d-n-bp^{k'}-1=n$, we have that $\delta\vdash n$. We will now show that p divides $\deg(\chi^\delta)$. As $h_{2,1}^\delta=n+d+2d-n-bp^{k'}-1+1=d-bp^{k'}<p^{k'}$, $l_{1,3}^\delta=0$ and $h_{1,3}^\delta=bp^{k'}-1\geq(b-1)p^{k'}$ we can remove $b-1$ hooks of length $p^{k'}$ from δ in a unique way and we obtain $\left(p^{k'}+1, 2^{n-d}, 1^{2d-n-bp^{k'}-1}\right)$ and as for this partition we have that $h_{2,1}<p^{k'}$, $h_{1,2}=p^{k'}+n-d>p^{k'}$, as $n-d\geq 1$, and $h_{1,3}=p^{k'}-1$, we cannot remove any more $p^{k'}$-hooks and then we have that p divides the degree of χ^δ again by lemma 71.

Now as $h_{1,3}^\delta=bp^{k'}-1=n-p^{k'}\geq c_1+\ldots+c_{l-1}$ and as again $l_{1,3}^\delta=0$ and $h_{2,1}^\delta<p^{k'}\leq c_j$ for $j\leq l$, we can recursively remove from δ the first $l-1$ hooks of a sequence of hooks $(c_1,\ldots,c_h)$ in a unique way and this way we obtain $\left(c_l-d+bp^{k'}+1, 2^{n-d}, 1^{2d-n-bp^{k'}-1}\right)$. In this partition we have that $h_{1,1}=cl-d+bp^{k'}+1+n-d+2d-n-bp^{k'}-1=c_l$, so we can now remove in a unique way also the l-th hook of the sequence. It is easy to see that after having removed this hook we obtain $\left(1^{n-d}\right)$, from which we can remove the remaining hooks in a unique way. So by the Murnaghan-Nakayama formula we have that in this case $\chi^\delta_{(c_1,\ldots,c_h)}\neq 0$ and so $(c_1,\ldots,c_h)$ isn't p-vanishing when $d, bp^{k'}\neq n$, $n-bp^{k'}=p^{k'}-1$, $d>bp^{k'}$ and $n-d<d-bp^{k'}$.

Case 6.
The last case we need to consider is when $d, bp^{k'}\neq n$, $n-bp^{k'}=p^{k'}-1$, $d>bp^{k'}$ and $n-d=d-bp^{k'}$. Assume now that $n-bp^{k'}\geq 4$. We can easily

see that this case can only occur when p is odd, as $p^{k'} \geq 2$ as $n \neq bp^{k'}$ and $n - bp^{k'} = p^{k'} - 1$ must be even as $n - d = d - bp^{k'}$. Let $\lambda = (bp^{k'}, 2^{n-d})$. As $bp^{k'} \geq p^{k'} \geq 3$, $n - d \geq 1$ and $bp^{k'} + 2(n - d) = bp^{k'} + n - d + d - bp^{k'} = n$ we have that λ is a partition of n. We will show that χ^λ has degree divisible by p and that in this case $\chi^\lambda_{(c_1,\ldots,c_h)} \neq 0$. As

$$h^\lambda_{2,1} = n - d + 1 \leq 2(n - d) = n - bp^{k'} = p^{k'} - 1,$$

$l^\lambda_{1,3} = 0$ and $h^\lambda_{1,3} = bp^{k'} - 2 \geq (b - 1)p^{k'}$ we can remove in a unique way a sequence of $b - 1$ hooks all of length $p^{k'}$ from λ. After having done this we obtain $(p^{k'}, 2^{n-d})$ and we now have $h_{2,1} \leq h^\lambda_{2,1} < p^{k'}$, $h_{1,2} = p^{k'} - 1 + n - d > p^{k'}$ as $n - d = (n - bp^{k'})/2 \geq 2$ and $h_{1,3} = p^{k'} - 2$ ($p^{k'} \geq 5$ as $p^{k'} - 1 = n - bp^{k'} \geq 4$, so that $(1, 3) \in (p^{k'}, 2^{n-d})$). So we cannot remove any more $p^{k'}$-hooks, in particular we cannot recursively remove from λ any sequence of b hooks of length $p^{k'}$ and so we have by lemma 71 that p must divide the degree of χ^λ.

Here too we have that $h^\lambda_{1,3} = bp^{k'} - 2 \geq n - c_l - \ldots - c_h = c_1 + \ldots + c_{l-1}$ as $c_{l+1} + \ldots + c_h = n - d \geq 2$ and as again $h^\lambda_{2,1} < p^{k'} \leq c_j$ for $j \leq l$ and $l^\lambda_{1,3} = 0$ we can remove the first $l - 1$ hooks of a sequence of hooks with length $(c_1, \ldots, c_h)$ in a unique way and we obtain $(c_l + bp^{k'} - d, 2^{n-d})$, for which $h_{1,1} = c_l + b - d + n - d = c_l$ as $n - d = d - bp^{k'}$. So we can remove also the next hook of this new sequence in a unique way and after having removed this hook we are left with (1^{n-d}), from which we again can remove the last hooks in a unique way and so, as again we can recursively remove a sequence of hooks with length $(c_1, \ldots, c_h)$ from λ in a unique way we have by the Murnaghan-Nakayama formula that $\chi^\lambda_{(c_1,\ldots,c_h)} \neq 0$ from which follows that also when $d, bp^{k'} \neq n$, $n - bp^{k'} = p^{k'} - 1$, $d > bp^{k'}$, $n - d = d - bp^{k'}$ and $n - bp^{k'} \geq 4$ we have that $(c_1, \ldots, c_h)$ isn't p-vanishing.

Case 7.

The only case we have left is when $d, bp^{k'} \neq n$, $n - bp^{k'} = p^{k'} - 1$, $d > bp^{k'}$, $n - d = d - bp^{k'}$ and $n - bp^{k'} = 2$ and $n - d = d - bp^{k'}$=1, as $n - bp^{k'}$ must be even and non-zero. In this case we have that $p^{k'} - 1 = 2$, so $p^{k'} = 3$, in which case we need to have that $k' = 1$ and we need to have that $a = 2$, so that $n \equiv 2 \bmod 3$ in this case and so we have that the theorem is proved. □

Definition 38. *If $k' \in \mathbb{N}$ we define $a'_{k'}$ by*

$$a'_{k'} = \sum_{j \geq k'} a_j p^{j-k'}.$$

It can be easily seen that

$$n = a_0 + \ldots + a_{k'-1}p^{k'-1} + \left(\sum_{j \geq k'} a_j p^{j-k'}\right) p^{k'} = a_0 + \ldots + a_{k'-1}p^{k'-1} + a'_{k'}p^{k'}.$$

In lemma 74, theorem 75 and conjecture 76 we will let $a'_{k'} = \sum_{j\geq k'} a_j p^{j-k'}$.

Lemma 74. *If $(c_1, \ldots, c_h)$ is p-vanishing, $n \neq a'_{k'}p^{k'}$ and $d < a'_{k'}p^{k'}$, then*

$$\sum_{c_j < n - a'p^{k'}} c_j > n - a'_{k'}p^{k'}.$$

Proof. Assume that for some $j \neq h$ we have that $n - a'_{k'}p^{k'} \leq c_j < p^{k'}$ and $\sum_{i=j+1}^{h} c_i \leq c_j$ let $\alpha = (n - c_j, 1^{c_j})$. χ^α has degree divisible by p by lemma 68. Also as $h^\alpha_{2,1} = c_j$ when removing any sequence of hooks of lengths $(c_1, \ldots, c_h)$ from α we need to remove all hooks of length $> c_j$ from the first row. Let m maximal such that $c_m > c_j$ ($m = 0$ if $c_1 = c_j$). Now we have that

$$h^\alpha_{1,2} = n - c_j - 1 \geq n - c_j - c_h \geq c_1 + \ldots + c_m$$

and as $l^\alpha_{1,2} = 0$ we can then remove in a unique way the first m hooks of the sequence and we obtain $((m-j-1)c_j + e, 1^{c_j})$, where $1 \leq e = \sum_{i=j+1}^{h} c_i \leq c_j$. If we remove one of the first $m - j - 1$ hooks of length c_j (there are at least $m - j$ values for i such that $c_i = c_j$) from the second row it is easy to see that this hook has leg-length $c_j - 1$ and all other hooks must be removed from the first row and have leg-length 0. Otherwise the first $m - j - 1$ hooks of length c_j are all removed from the first row, and we can easily see that after having done this we would be left with $(e, 1^{c_j})$. As $1 \leq e \leq c_j$ we have that in this partition $h_{1,2} = e - 1 < c_j$, $h_{1,1} = e + c_j > c_j$ and so as $h_{2,1} = c_j$ we have that this is the only hook of this partition with length c_j. Also in this case this hook has leg-length $c_j - 1$ and all the other hooks have leg-length 0, so by lemma 54 we have that in this case $\chi^\alpha_{(c_1,\ldots,c_h)} \neq 0$.

If $\sum_{c_j < n - a'_{k'}p^{k'}} c_j \leq n - a'_{k'}p^{k'}$ and $n - d = \sum_{c_j < p^{k'}} c_j > n - a'_{k'}p^{k'}$ we need to have that $(c_1, \ldots, c_h)$ has at least one part of length between $n - a'_{k'}p^{k'}$ and $p^{k'} - 1$. Let l be maximal such that $n - a'_{k'}p^{k'} \leq c_l \leq p^{k'} - 1$. If $l < h$ we can conclude by the previous part with $j = l$ that $(c_1, \ldots, c_h)$ is not p-vanishing in this case. If $l = h$ we also need to have by theorem 69 that $c_l \leq n - a'_{k'}$ so that $c_l = c_h = n - a'_{k'}$ and as $n - d > n - a'_{k'}p^{k'}$ we need to have that $c_h = n - a'_{k'}p^{k'} \leq c_{h-1} < p^{k'}$ and so in this case we can conclude again by the previous part, now with $j = h - 1$, that $(c_1, \ldots, c_h)$ is not p-vanishing. □

Theorem 75. *Let $p = 2, 3$. If $(c_1, \ldots, c_h)$ is p-vanishing we need to have that $d \geq a'_{k'}p^{k'}$ if*

- *$p = 2$ and $k' \neq 1, 2$,*
- *$p = 2$, $k' = 2$ and $n \not\equiv 4$ mod 8,*

- *$p = 2$, $k' = 1$ and n is odd or divisible by 8,*
- *$p = 3$ and $k' \neq 1$,*
- *$p = 3$, $k' = 1$ and $n \not\equiv 3, 5, 6, 8$ mod 9.*

In addition if $n < 8$ and $p = 2$ or if $n < 9$ and $p = 3$ we have that if for some k' we have that $d < a'_{k'}p^{k'}$, then $(c_1, \ldots, c_h)$ is one of the following:

- *$(1, 1) \vdash 2$, $(2, 1, 1) \vdash 4$ or $(4, 1, 1) \vdash 6$ if $p = 2$,*
- *$(2, 1) \vdash 3$, $(1, 1, 1) \vdash 3$, $(2, 1, 1, 1) \vdash 5$, $(3, 2, 1) \vdash 6$, $(3, 1, 1, 1) \vdash 6$ or $(3, 2, 1, 1, 1) \vdash 8$ if $p = 3$.*

Proof. The last statement can be checked using the Murnaghan-Nakayama formula and the hook formula, that is first by finding by the hook-formula all irreducible characters of degree divisible by p and then by the Murnaghan-Nakayama formula finding those equivalence classes on which they vanish. So we will only prove the first part of the theorem.

The theorem is trivial when $a'_{k'} = 0$, as then $p^{k'} > n$. We may thus assume that $a'_{k'} \neq 0$. Also the theorem clearly holds when $k' = 0$ as in this case $p^{k'} = 1$ and so we have that for any partition $d = n = a'_{k'}p^{k'}$.

Assume that $n \neq a'_{k'}p^{k'}$. In order to prove the theorem we only need to show, by the previous lemma, that if $\sum_{c_j < n - a'_{k'}p^{k'}} c_j > n - a'_{k'}p^{k'}$ then $(c_1, \ldots, c_h)$ is not p-vanishing, for those k' for which we want to prove the theorem for the given p and n. The proof of the theorem will be divided in the following cases:

Case 1: $p = 2$, n odd and $k' = 0$ or $8|n$ and $k' = 0, 1, 2$ or $n \equiv 4$ mod 8 and $k' = 3$,

Case 2: $p = 2$, $n \equiv 2$ mod 4 and $k' = 2$,

Case 3: $p = 2$, $n = a'_{k'}2^{k'}$ and $k' \geq 3$ or $n \neq a'_{k'}2^{k'}$ and the theorem holds for $k' - 1$,

Case 4: $p = 3$, $k' \geq 2$ and the theorem holds for $k' - 1$,

Case 5: $p = 3$, $n \equiv 0, 1, 4, 7$ mod 9 and $k' = 1$,

Case 6: $p = 3$, $n \equiv 3, 6$ mod 9 and $k' = 2$,

Case 7: $p = 3$, $n \equiv 2$ mod 9 and $k' = 1$ or $n \equiv 5, 8$ mod 9 and $k' = 2$.

As we have already noticed that the theorem always holds for $k' = 0$ we have that these cases include all the other cases that we need to consider.

For cases 1, 2 and 3 let $p = 2$.

Case 1.
As we have already seen that the theorem always holds for $k' = 0$, it needs to hold in particular for n odd and $k' = 0$. Also when n is divisible by 8 we have that the theorem holds for $k' = 0, 1, 2$ by theorem 67. When $n \equiv 4 \bmod 8$ and $k' = 3$ we have that the theorem easily holds by theorem 67 and lemma 74 (here $n \neq 2^{k'} a'_{k'}$ as $a_{k'-1} = a_2 = 1$).

Case 2.
We will now show that the theorem holds for $n \equiv 2 \bmod 4$ and $k' = 2$. Again by the lemma we only need to show that whenever $\sum_{c_j=1} c_j > 2$, then $(c_1, \ldots, c_h)$ isn't p-vanishing. In order to do this we will use $(n-2, 1, 1)$, $(n-3, 2, 1)$ and $(n-3, 1, 1, 1)$. As we are assuming that $n \equiv 2 \bmod 4$ and $n \geq \sum_{c_j=1} c_j \geq 3$ we need to have that $n \geq 6$, so that these are actually partitions of n. The degree of $\chi^{(n-2,1,1)}$ and $\chi^{(n-3,1,1,1)}$ are divisible by 2 by lemma 68. Using the hook formula it can be easily seen that the degree of $\chi^{(n-3,2,1)}$ is $n(n-2)(n-4)/3$ (the degree of χ^α, for any $\alpha \vdash n$ is equal to the product of the numbers between 1 and n which are not equal to the hook-lengths of the hooks on the first row of α divided by the product of the hook-lengths of the hooks on the lower rows of α) and as n is even we then have that also this degree is divisible by 2.

So let now $(c_1, \ldots, c_h) = (c_1, \ldots, c_l, 3^r, 2^s, 1^t)$ with $c_l \geq 4$ and assume that $t \geq 3$. First assume that $s \neq (t-1)(t-2)/2$. In this case we can show that $\chi^{(n-2,1,1)}_{(c_1,\ldots,c_h)} \neq 0$. To see this notice that whenever we are removing from $(n-2, 1, 1)$ a sequence of hooks with lengths $(c_1, \ldots, c_h)$, we need to remove all hooks of length at least 3 them from the first row. As by assumption $t \geq 3$, we can remove the first $h - s - t$ hooks of this sequence in a unique way and we obtain $(2s + t - 2, 1, 1)$. Now, as again $t \geq 3$, it isn't hard to see that we can either remove one 2-hook from the second row and all other hooks from the first row. This can be done in s different ways and the sum of the leg-lengths in this case is 1. Otherwise we need to remove all 2-hooks from the first row, which would leave $(t-2, 1, 1)$ and then remove t 1-hooks from this partition. In this case we have that the sum of the leg-length is 0. It can be easily seen that $\{h^{(t-2,1,1)}_{1,i} : 2 \leq i \leq t-2\} = \{1, 2, \ldots, t-3\}$ and then using the hook formula we have that

$$\chi^{(t-2,1,1)}(1) = \frac{t!}{2t \prod_{i=1}^{t-3} i} = \frac{(t-1)(t-2)}{2}.$$

As we are assuming that $s \neq (t-1)(t-2)/2$ we have by lemma 54 that

$\chi^{(n-2,1,1)}_{(c_1,\ldots,c_h)} = (t-1)(t-2)/2 - s \neq 0$ and so in this case $(c_1,\ldots,c_h)$ isn't p-vanishing.

Assume now that $s = (t-1)(t-2)/2$ and $r \neq t(t-2)(t-4)/3$. As $t(t-2)(t-4)/3 < 0$ when $t = 3$ this can never happen in this case. We will show that in this case $\chi^{(n-3,2,1)}_{(c_1,\ldots,c_h)} \neq 0$. In this case $\sum_{c_j=1,2} c_j = qt+(t-1)(t-2) \geq 5$ as we are assuming that $t \geq 3$. So as $h^{(n-3,2,1)}_{1,3} = n-5$, $l^{(n-3,2,1)}_{1,3} = 0$ and $h^{(n-3,2,1)}_{2,1} = 3$, we can recursively remove from $(n-3,2,1)$ in a unique way the hooks of length at least 4 of a sequence with lengths $(c_1,\ldots,c_h)$ and after having done this we obtain $(3r+2s+t-3,2,1)$. Now we can remove one of the 3-hook from the second row and all other hooks from the first row. This can be done in r different ways and here the sum of the leg-lengths is equal to 1. Otherwise we need to remove all 3-hooks from the first row, in which case all hooks removed up to this point have leg-length 0, as again $\sum_{c_j=1,2} c_j \geq 5$, and in this case after having removed also all the 3-hooks of the sequence we obtain $(2s+t-3,2,1)$. If $t = 3$ then $s = 1$ and this partition is equal to $(2,2,1)$ and so in this case $\chi^{(n-3,2,1)}_{(c_1,\ldots,c_h)} = -r-1 \neq 0$ for any $r \geq 0$. If $t = 4$ then $s = 3$ and so $(2s+t-3,2,1) = (7,2,1)$ and $\chi^{(n-3,2,1)}_{(c_1,\ldots,c_h)} = -r \neq 0$, as $r \neq t(t-2)(t-4)/3 = 0$.

So assume now that $t \geq 5$. As the hooks in the second and third row of $(2s+t-3,2,1)$ have all odd length, when we remove any sequence of 2-hooks from $(2s+t-3,2,1)$ we need to have that all these hooks are on the first row. As we now want to remove from this partition a sequence s 2-hooks and $h^{(2s+t-3,2,1)}_{1,3} = 2s+t-5 \geq 2s$ as $t \geq 5$ and $l^{(2s+t-3,2,1)}_{1,3} = 0$, we can remove such a sequence of 2-hooks (which all have leg-length 0). After having removed them we get $(t-3,2,1)$. As $\{h^{(t-3,2,1)}_{1,j}\} = \{t-1,t-3,t-5,t-6,\ldots,1\}$ we have by the hook formula that

$$\chi^{(t-3,2,1)}_{(1^t)} = \chi^{(t-3,2,1)}(1) = \frac{t!}{3(t-1)(t-3)\cdot(t-5)!} = \frac{t(t-2)(t-4)}{3}.$$

So putting all of this together we have that when $t \geq 5$, $s = (t-1)(t-2)/2$ and $r \neq t(t-2)(t-4)/3$ then

$$\chi^{(n-3,2,1)}_{(c_1,\ldots,c_h)} = -r + t(t-2)(t-4)/3 \neq 0.$$

As we already know that $\chi^{(n-3,2,1)}_{(c_1,\ldots,c_h)} \neq 0$ when $t = 3,4$, $s = (t-1)(t-2)/2$ and $r \neq t(t-2)(t-4)/3$ we have that in any of these cases $(c_1,\ldots,c_h)$ isn't p-vanishing.

So the only case we have left to consider is when $t \geq 4$, $s = (t-1)(t-2)/2$ and $r = t(t-2)(t-4)/3$. In this case we will show that $\chi^{(n-3,1,1,1)}_{(c_1,\ldots,c_h)} \neq 0$. As here we have that $t \geq 4$, $h_{1,2}^{(n-3,1^3)} = n-4$, $l_{1,3}^{(n-3,1^3)} = 0$ and $h_{2,1}^{(n-3,1^3)} = 3$, we can remove the hooks of length at least 4 of a sequence of hooks with lengths $(c_1,\ldots,c_h)$ that are recursively removed from $(n-3,1,1,1)$ in a unique way. These hooks all have leg-length 0 and after having removed them we obtain $(3r+2s+t-3,1,1,1)$. Now we need to recursively remove r 3-hooks from this partition. We can remove one of the 3-hooks from the second row and all other from the first row, which can be done in r different ways and in which case the sum of all leg-lengths is always equal to 2. Otherwise we need to remove all 3-hooks from the first row, which can be done as $t \geq 4$ and after having done this we obtain $(2s+t-3,1,1,1)$, from which we need to remove s 2-hooks. We can remove one of these 2-hook from the third row and then it can be easily seen that all other 2-hooks must be removed from the first row, in which case after having removed all 2-hooks we obtain $(t-1,1)$. This can be done in s different ways and the sum of all leg-lengths of the hooks removed until now is 1. Otherwise we need to remove also all 2-hooks from the first row, again this can be done as $t \geq 4$, after having done this we obtain $(t-3,1,1,1)$. In this case (which can be done in a unique way) we have that the sum of the leg-lengths of the hooks removed up to this point is 0. Now using the hook formula we can easily see that

$$\chi_{(1^t)}^{(t-1,1)} = \chi^{(t-1,1)}(1) = \frac{t!}{t \cdot (t-2)!} = t-1$$

and

$$\chi_{(1^t)}^{(t-3,1^3)} = \chi^{(t-3,1^3)}(1) = \frac{t!}{3 \cdot 2t \cdot (t-4)!} = \frac{(t-1)(t-2)(t-3)}{6}.$$

Putting all of this together we have that when $t \geq 4$, $s = (t-1)(t-2)/2$ and $r = t(t-2)(t-4)/3$ then

$$\begin{array}{rcl}\chi_{(c_1,\ldots,c_h)}^{(n-3,1^3)} & = & r - s\chi^{(t-1,1)}(1) + \chi^{(t-3,1^3)}(1) \\ & = & \frac{t(t-2)(t-4)}{3} - \frac{(t-1)^2(t-2)}{2} + \frac{(t-1)(t-2)(t-3)}{6} \\ & = & \frac{2t^3-12t^2+16t-3t^3+12t^2-15t+6+t^3-6t^2+11t-6}{6} \\ & = & \frac{-6t^2+12t}{6} \\ & = & -t(t-2)\end{array}$$

and so as in this case we have that $\chi_{(c_1,\ldots,c_h)}^{(n-3,1^3)} \neq 0$ and so $(c_1,\ldots,c_h)$ is not p-vanishing.

So we need to have that if $n \equiv 2 \bmod 4$ and $(c_1,\ldots,c_h)$ is p-vanishing then $\sum_{c_j<4} c_j \leq 2$ and so the theorem holds in this case for $k' = 2$.

Case 3.

If now $a_0 + 2a_1 + \ldots + 2^{k'-1}a_{k'-1} = n - 2^{k'}a'_{k'} = 0$ and $k' \geq 3$ the theorem holds by theorem 67. So assume now that $n - 2^{k'}a'_{k'} \neq 0$ and that

$$\sum_{c_j < 2^{k'-1}} c_j \leq a_0 + 2a_1 + \ldots + 2^{k'-2}a_{k'-2}$$

(that is the theorem holds for $k' - 1$). Using theorem 73 we then have that $\sum_{c_j < p^{k'-1}} c_j = a_0 + a_1 2 + \ldots + a_{k'-2}2^{k'-2}$ and then by theorem 72 we have that c_j is a multiple of $2^{k'-1}$ whenever $c_j \geq 2^{k'-1}$. So we have that for some l, $(c_1, \ldots, c_h) = (2^{k'-1}b_1, \ldots, 2^{k'-1}b_l, c_{l+1}, \ldots, c_h)$ for some

$$(b_1, \ldots, b_l) \vdash (n - a_0 - a_1 2 - \ldots - a_{k'-2}2^{k'-2})/2^{k'-1} = a_{k'-1} + 2a'_{k'}$$

and $c_{l+1} < 2^{k'-1}$. In order to prove the theorem as in this case $n - 2^{k'}a'_{k'} \neq 0$, it is enough to show, by what we already proven, that if

$$\sum_{c_j < a_0 + a_1 2 + \ldots + a_{k'-1}2^{k'-1}} c_j > a_0 + a_1 2 + \ldots + a_{k'-1}2^{k'-1}$$

then $(c_1, \ldots, c_h)$ isn't p-vanishing. Also as

$$\sum_{c_j < 2^{k'-1}} c_j = a_0 + a_1 2 + \ldots + a_{k'-2}2^{k'-2}$$

we have

$$\sum_{c_j < a_0 + a_1 2 + \ldots + a_{k'-1}2^{k'-1}} c_j > a_0 + a_1 2 + \ldots + a_{k'-1}2^{k'-1}$$

if and only if

$$\sum_{2^{k'-1} \leq c_j < a_0 + a_1 2 + \ldots + a_{k'-1}2^{k'-1}} c_j > a_{k'-1}2^{k'-1}$$

if and only if

$$\sum_{b_j < a_{k'-1} + (a_0 + a_1 2 + \ldots + a_{k'-2}2^{k'-2})/2^{k'-1}} b_j > a_{k'-1}.$$

As $a_{k'-1} \leq 1$ and $a_0 + a_1 2 + \ldots + a_{k'-2}2^{k'-2} < p^{k'-1}$ it is enough to prove that if $\sum_{b_j = 1} b_j > 1$, then $(c_1, \ldots, c_h)$ isn't p-vanishing. Notice that in this case we need to have that $a'_{k'} \geq 1$. Let α be the partition with core $\alpha_{(2^{k'-1})} = (a_0 + a_1 2 + \ldots + a_{k'-2}2^{k'-2})$ and quotient $\alpha^{(2^{k'-1})} = ((2a'_{k'}, 1), 0, \ldots, 0)$. It is easy to see that we cannot recursively remove a sequence of $a'_{k'}$ hooks of length 2 from $\alpha^{(2^{k'-1})}$ ($a'_{k'} \neq 0$) and so we cannot remove a sequence of $a'_{k'}$ hooks of length $2^{k'}$ from α and then by lemma 71 we have that p divides the degree of α. As here the quotient contains only a partition which is different from 0, using theorem 55 we have that

$$\chi^{\alpha}_{(c_1,\ldots,c_h)} = \pm\chi^{(a_0+a_1 2+\ldots+a_{k'-2}2^{k'-2})}_{(c_l+1,\ldots,c_h)}\chi^{(2a'_{k'},1)}(b_1,\ldots,b_l) = \pm\chi^{(2a'_{k'},1)}(b_1,\ldots,b_l).$$

Now as $h^{(2a'_{k'},1)}_{2,1} = 1$, $h^{(2a'_{k'},1)}_{1,2} = 2a'_{k'} - 1$ and $\sum_{b_j>1} b_j < 2a'_{k'}$ as we are assuming that $\sum_{b_j=1} b_j > 1$, we can remove the hooks of length bigger than 1 of a sequence with hook-length $(b_1,\ldots,b_l)$ from $(2a'_{k'},1)$ in a unique way and as 1-hooks always have leg-length 0 we then have by lemma 54 that $\chi^{(2a'_{k'},1)}(b_1,\ldots,b_l) \neq 0$ and so we also have that $\chi^{\alpha}_{(c_1,\ldots,c_h)} \neq 0$.

As when $k' \geq 3$ or $n \neq 2^{k'}a'_{k'}$ and $\sum_{c_j<p^{k'-1}} c_j \leq a_0+2a_1+\ldots+2^{k'-2}a_{k'-2}$ we have that $\sum_{c_j<p^{k'}} c_j \leq a_0 + 2a_1 + \ldots + 2^{k'-1}a_{k'-1}$ and as we now that the theorem holds for $k' = 0,1,2$ when $n \equiv 0 \bmod 8$, for $k' = 0$ when $n \equiv 1 \bmod 2$, for $k' = 2$ when $n \equiv 2 \bmod 4$ and for $k' = 3$ when $n \equiv 4 \bmod 8$ we have that the theorem is proved for the case $p = 2$.

Case 4.

In all the remaining part of the proof we will have that $p = 3$. Assume first that $k' \geq 2$ and

$$\sum_{c_j<3^{k'-1}} c_j \leq a_0 + 3a_1 + \ldots + 3^{k'-2}a_{k'-2}.$$

If $n = 3^{k'}a'_{k'}$ we have that the theorem holds by theorem 67, so we can assume that $n \neq 3^{k'}a'_{k'}$. Also we can assume by the first part of the proof that $a'_{k'} \neq 0$. Using theorems 72 and 73 we then have that if $(c_1,\ldots,c_h)$ is p-vanishing then $(c_1,\ldots,c_h) = (3^{k'-1}b_1,\ldots,3^{k'-1}b_l,c_{l+1},\ldots,c_h)$, with $(b_1,\ldots,b_l) \vdash a_{k'-1}+3a'_{k'}$ and $c_{l+1}+\ldots+c_h = a_0+3a_1+\ldots+3^{k'-2}a_{k'-2} < 3^{k'-1}$. In order to prove the theorem it is enough to show, by lemma 74, that if $(c_1,\ldots,c_h)$ is of this form and $\sum_{b_j\leq a_{k'-1}} b_j > a_{k'-1}$, then $(c_1,\ldots,c_h)$ isn't p-vanishing. If $a_{k'-1} = 0$ this is obvious, as no such $(c_1,\ldots,c_h)$ exists. If $a_{k'-1} = 1$ let α have core $\alpha_{(3^{k'-1})} = (a_0+3a_1+\ldots+3^{k'-2}a_{k'-2})$ and quotient $\alpha^{(3^{k'-1})} = ((3a'_{k'},1),0,\ldots,0)$. $\alpha_{(3^{k'-1})}$ is a $3^{k'-1}$-core as $a_0 + 3a_1 + \ldots + 3^{k'-2}a_{k'-2} < 3^{k'-1}$. As we cannot remove a sequence of $a'_{k'}$ 3-hooks from $\alpha^{(3a'_{k'})}$ we have that 3 divides the degree of α by lemma 71. Also as in this case we can apply theorem 55 we get that

$$\chi^{\alpha}_{(c_1,\ldots,c_h)} = \pm\chi^{\alpha_{(3^{k'})}}_{(c_{l+1},\ldots,c_h)}\chi^{(3a'_{k'},1)}_{(b_1,\ldots,b_l)} = \pm\chi^{(3a'_{k'},1)}_{(b_1,\ldots,b_l)}.$$

As $(b_1,\ldots,b_l)$ has at least two parts equal to 1 (as $\sum_{b_j\leq a_{k'-1}} b_j > a_{k'-1}$ and $a_{k'-1} = 1$) and $(3a'_{k'},1)$ only has one node which isn't on the first row, it is easy to see by lemma 54 that in this case $\chi^{(3a'_{k'},1)}_{(b_1,\ldots,b_l)} \neq 0$. In particular $\chi^\alpha_{(c_1,\ldots,c_h)} \neq 0$ when $a_{k'-1} = 1$ and $\sum_{b_j \leq a_{k'-1}} b_j > a_{k'-1}$, so that in this case $(c_1, \ldots, c_h)$ isn't p-vanishing.

So let now $a_{k'-1} = 2$. As we are assuming that $k' \geq 2$, we have that $3^{k'-1} > 2$. Assume that $\sum_{b_j \leq a_{k'-1}} b_j > a_{k'-1} = 2$. First it is easy to see that if α has core $\alpha_{(3^{k'-1})} = (a_0 + 3a_1 + \ldots + 3^{k'-2}a_{k'-2})$ (this is a $3^{k'-1}$-core as $a_0 + 3a_1 + \ldots + 3^{k'-2}a_{k'-2} < 3^{k'-1}$) and quotient $\alpha^{(3^{k'-1})}$ equal to either $((3a'_{k'}, 1, 1), 0, \ldots, 0)$ or $((3a'_{k'}, 1), (1), 0, \ldots, 0)$ then 3 must divides the degree of χ^α (as we cannot remove $a'_{k'}$ 3-hooks from the quotient and so we cannot even remove $a'_{k'}$ hooks of length $3^{k'}$ from α and by lemma 71). As $h^{(3a'_{k'},1,1)}_{2,1} = 2$, $h^{(3a'_{k'},1)}_{2,1} = 1$, $h^{(3a'_{k'},1,1)}_{1,2} = h^{(3a'_{k'},1)}_{1,2} = 3a'_{k'} - 1 \geq \sum_{b_j>2} b_j$, as we are assuming that $\sum_{b_j \leq 2} > 2$ and $\sum_{b_j} = 3a'_{k'} + 2$, and $l^{(3a'_{k'},1,1)}_{1,2} = l^{(3a'_{k'},1)}_{1,2} = 0$, if s is the number of b_j equal to 1 and t is the number of b_j equal to 2, it is easy to see that we can recursively remove the first $l - s - t$ hooks of a sequence with lengths $(b_1, \ldots, b_l)$ from $\alpha^{(3^{k'-1})}$ in a unique way and this way we obtain $((s + 2t - 2, 1, 1), 0 \ldots, 0)$ or $((s + 2t - 2, 1), (1), 0 \ldots, 0)$.

If $s = 0$ let $\alpha^{(3^{k'-1})} = ((3a'_{k'}, 1, 1), 0, \ldots, 0)$. In this case $t \geq 2$ and after having removed from $\alpha^{(3^{k'-1})}$ the hooks with length ≥ 3 of the sequence with lengths $(b_1, \ldots, b_l)$ we have obtained $((2t - 2, 1, 1), 0 \ldots, 0)$. Now we need to remove $t \geq 2$ hooks of length 2 from $(2(t - 1), 1, 1)$. In order to do this we need to remove one of the first $t - 1$ of these hooks from the second row and the other from the first row. The hook we removed from the second row has leg-length 1 and the others have leg-length 0. So using lemma 54 and theorem 67 we have that in this case

$$\begin{aligned}\chi^\alpha_{(c_1,\ldots,c_h)} &= \pm\chi^{(a_0+a_12+\ldots+a_{k'-2}3^{k'-2})}_{(c_l+1,\ldots,c_h)}\chi^{(3a'_{k'},1,1)}(b_1, \ldots, b_l)\\ &= \pm\chi^{(3a'_{k'},1,1)}(b_1, \ldots, b_l) = \pm(1 - t)\end{aligned}$$

and so in this case $(c_1, \ldots, c_h)$ is not p-vanishing.

If $s = 1$ use $\alpha^{(3^{k'-1})} = ((3a'_{k'}, 1), (1), 0, \ldots, 0)$. In this case $(b_1, \ldots, b_l) = (b_1, \ldots, b_{l-1}, 1)$ and $b_{l-1} = 2$. As $b_j \geq 2$ for $j \leq l - 1$ it is easy to see that whenever we recursively remove from $\alpha^{(3^{k'-1})}$ a sequence of hooks with lengths $(b_1, \ldots, b_l)$, we need to remove the first $l - 1$ from the first component of the quotient. As $(3a'_{k'}, 1)$ is a partition of $b_1 + \ldots + b_{l-1}$ in this case, all nodes apart for one are on the first row and $b_{l-1} \geq 2$, it is easy to see that we can actually do this in a unique way. After having done this we have that the new quotient is $(0, (1), 0 \ldots, 0)$, from which we can remove the last hook of the sequence (in a unique way). As we can remove also in a unique way a sequence of hooks of lengths $(c_{l+1}, \ldots, c_h)$ from $\alpha_{(3^{k'-1})} = (a_0 + 3a_1 + \ldots + 3^{k'-2}a_{k'-2})$, we have, by definition of b_j that in this case we can recursively remove in a unique way from α a sequence of hooks with lengths $(c_1, \ldots, c_h)$ and so by the Murnaghan-Nakayama formula we have that in this case $\chi^{\alpha}_{(c_1,\ldots,c_h)} = \pm 1 \neq 0$.

If $s = 2$ let $\alpha^{(3^{k'-1})} = ((3a'_{k'}, 1, 1), 0, \ldots, 0)$. As $\sum_{b_j=1,2} b_j > 2$ and $\sum_{b_j=1} b_j = 2$ in this case, we need to have that $t \geq 1$. It is now easy to see that we can remove the hooks of length bigger than 2 of a sequence with lengths $(b_1, \ldots, b_l)$ from $(3a'_{k'}, 1, 1)$ in a unique way (and all these hooks have leg-length 0) and after having removed these hooks we obtain $(2s, 1, 1)$. It isn't hard to see that in order to remove a sequence of t 2-hooks from this partition we need to remove exactly one of them from the second row and the other from the first row. This can be done in t way and the sum of the leg-lengths of the 2-hooks is always equal to 1. After having done this we get (2), from which we now need to remove 2 1-hooks. So using theorem 55 we have that

$$\chi^{\alpha}_{(c_1,\ldots,c_h)} = \pm \chi^{\alpha_{(3^{k'-1})}}_{(c_{l+1},\ldots,c_h)} \chi^{(3a'_{k'},1,1)}_{(b_1,\ldots,b_l)} = \mp t \neq 0$$

and so also in this case $(c_1, \ldots, c_h)$ isn't p-vanishing.

The last case we need to consider is when $s \geq 3$. Here let $\alpha^{(3^{k'-1})} = ((3a'_{k'}, 1), (1), 0, \ldots, 0)$. As in this case $h^{(3a'_{k'},1)}_{1,2} = 3a'_{k'} - 3 \geq \sum_{b_j>1} b_j$, $l^{(3a'_{k'},1)}_{1,2} = 0$, $h^{(3a'_{k'},1)}_{2,1} = 1$ and $h^{(1)}_{1,1} = 1$, we can recursively remove the hooks of length ≥ 2 of a sequence of hooks with lengths $(b_1, \ldots, b_l)$ from $\alpha^{(3^{k'-1})}$ in a unique way and after having done this we have that the new quotient is $((t-2, 1, 1), (1), 0 \ldots, 0)$, from which we now only need to remove a sequence of t 1-hooks. So the sum of the leg-lengths of all the hooks in any sequence of hooks with lengths $(b_1, \ldots, b_l)$ which are removed from $\alpha^{(3^{k'-1})}$ is always the same and as we can remove such sequence in at least one way and as $\chi^{\alpha_{(3^{k'-1})}}_{(c_{l+1},\ldots,c_h)} = 1$ we have by theorem 55 that $\chi^{\alpha}_{(c_1,\ldots,c_h)} \neq 0$.

So if $p = 3$, $k' \geq 2$ and $\sum_{c_j<3^{k'-1}} c_j \leq a_0 + 3a_1 + \ldots + 3^{k'-2} a_{k'-2}$ (that is the theorem holds for $k'-1$, we have that the theorem holds for k'.

Case 5.
As the theorem always holds for $k' = 0$, it is now enough to prove it for $k' = 1$ when $n \equiv 0, 1, 2, 4, 7 \bmod 9$ and for $k' = 2$ when $n \equiv 3, 5, 6, 8 \bmod 9$ If $n \equiv 1 \bmod 3$ then the theorem holds for $k' = 1$ by lemma 74, as here $a_0 = 1$. Also by theorem 67 we have that the same is true for $n \equiv 0 \bmod 9$.

Case 6.
If $n \equiv 3, 6 \bmod 9$ and the theorem doesn't hold for $k' = 2$, then it cannot hold for $k' = 1$ either by what we just saw in case 4, so in this case we need to have that $\sum_{c_j<3} c_j > a_0 = 0$ and so by theorem 67 we need to have that in this case $c_h = 1$. Using the hook formula it is easy to see that the

degree of $\chi^{(n-2,2)}$ is $n(n-3)/2$ and so it is divisible by 3 in this case. If $(c_1,\ldots,c_h)=(c_1,\ldots,c_l,2^b,1^a)$, with $c_l\geq 3$, it isn't hard to see that

$$\chi^{(n-2,2)}_{(c_1,\ldots,c_h)}=\begin{cases} a(a-3)/2+b & a\geq 4\\ b & a=3\\ -1+b & a=1,2\end{cases}$$

and so we have that if $(c_1,\ldots,c_h)$ is p-vanishing we must have that $a=3$ and $b=0$ or $a=1,2$ and $b=1$. Now consider $(n-4,2,1,1)$. Again using the hook-formula it is easy to see that the degree of $\chi^{(n-4,2,1,1)}$ is given by $n(n-2)(n-3)(n-5)/8$ and as $n\equiv 3,6 \bmod 9$ we have that then the degree of $\chi^{(n-4,2,1,1)}$ is divisible by 3. Write now $(c_1,\ldots,c_h)=(c_1,\ldots,c_{l'},4^e,3^c,2^b,1^a)$, with $c_{l'}\geq 5$. It can be easily seen that $\chi^{(n-4,2,1,1)}_{(c_1,\ldots,c_{l'},4^e,3^c,2,1^2)}=e+1$, that $\chi^{(n-4,2,1,1)}_{(c_1,\ldots,c_{l'},4^e,3^c,2,1)}=e$ and that $\chi^{(n-4,2,1,1)}_{(c_1,\ldots,c_{l'},4^e,3^c,1^3)}=e$ (we can or have to remove at most one 4-hook from the second row and all other hooks of length at least 3 must be removed from the first row). In particular if $(c_1,\ldots,c_h)$ is p-vanishing we must have that e is always 0 and also $a=3$ and $b=0$ or $a=1$ and $b=1$. This, together with lemma 74 shows that the theorem holds for $k'=2$ when $n\equiv 3 \bmod 9$, as then we need to have that $\sum_{c_j<3}c_j\leq 3$.

If $n\equiv 6 \bmod 9$ consider next $(n-5,3,1,1)$. In this case, if $\sum_{c_j<6}c_j>6$, we need to have that $n\geq 15$, so that this is a partition of n in this case. Again using the hook formula it is easy to see that the degree of $\chi^{(n-5,3,1,1)}$ is $n(n-1)(n-3)(n-4)(n-7)/20$, which is the divisible by 3. If now we let $(c_1,\ldots,c_h)=(c_1,\ldots,c_{l''},5^f,3^c,2^b,1^a)$ with $c_{l''}\geq 6$ it can easily be seen that any hook of length at least 6 of a sequence of hooks of lengths $(c_1,\ldots,c_h)$ which are recursively removed from $(n-5,3,1,1)$ must be removed from the first row. As

$$h^{(n-5,3,1,1)}_{1,1}=n-1>c_1+\ldots+c_{l''+f},$$

$$h^{(n-5,3,1,1)}_{1,2}=n-4<c_1+\ldots+c_{l''+f},$$

$l^{(n-5,3,1,1)}_{1,2}=1$ and $l^{(n-5,3,1,1)}_{1,3}=0$, it can be seen that we cannot remove all hooks of length ≥ 5 of such a sequence from the first row. So whenever we are recursively removing a sequence of hooks of lengths $(c_1,\ldots,c_h)$ from $(n-5,3,1,1)$ we must remove one 5-hook from the second row and all other hooks from the first row. So it can be easily seen that $\chi^{(n-5,3,1,1)}_{(c_1,\ldots,c_{l''},5^f,3^c,2,1)}=\chi^{(n-5,3,1,1)}_{(c_1,\ldots,c_{l''},5^f,3^c,1^3)}=f$ and so if $(c_1,\ldots,c_h)$ is p-vanishing we must have that $f=0$.

So if $(c_1,\ldots,c_h)$ is p-vanishing and $n\equiv 6 \bmod 9$ we need to have that $(c_1,\ldots,c_h)=(c_1,\ldots,c_{l''},3^c,2,1),(c_1,\ldots,c_{l''},3^c,1,1,1)$, with $c_{l''}\geq 6$. We will now show that $c \leq 1$. Assume that $c \geq 2$ and consider $(n-6,3,2,1)$. The corresponding irreducible character has degree

$$n(n-1)(n-2)(n-4)(n-6)(n-8)/45$$

which is divisible by 3 as $3|n$ and $9|(n-6)$. As again in this case we have that $n \geq 15$, this is a partition of n. As $c_{l''} \geq 6$, $h_{2,1}^{(n-6,3,2,1)} = 5$,

$$h_{1,4}^{(n-6,3,2,1)} = n-9 \geq c_1 + \ldots + c_{l''}$$

as $c \geq 2$ and $l_{1,4}^{(n-6,3,2,1)} = 0$, we have that $\chi^{(n-6,3,2,1)}_{(c_1,\ldots,c_{l''},3^c,2,1)} = \chi^{(3c-3,2,1)}_{(3^c,2,1)}$ and $\chi^{(n-6,3,2,1)}_{(c_1,\ldots,c_{l''},3^c,1^3)} = \chi^{(3c-3,2,1)}_{(3^c,1^3)}$. If we remove 2 of the first $c-2$ 3-hooks from the second or third row of $(n-6,3,2,1)$ we then must remove all other hooks from the first row. It can be seen that this can be done in $2 \cdot \binom{c-2}{2} = (c-2)(c-3)$ different ways and in each of these ways we have that the sum of the leg-lengths is odd. Otherwise at most one of the 3-hooks can be removed from the second or third row (and all others must be removed from the first row). If we remove one of the first $c-2$ hooks of length 3 from the second row this hook must be $R_{2,2}$. If we remove one of the first $c-2$ 3-hooks from the third row this hook must be $R_{3,1}$. In each of these last two cases, as we need to remove at most one of the first $c-2$ hooks of length3 from the second or third row, we have that the sum of the leg-length of the first $c-2$ 3-hooks we remove from $(3^c-3,3,2,1)$ is 1 and each of these cases can be done in $c-2$ different ways. In the first of these two cases after having recursively removed the first $c-2$ 3-hooks we obtain $(6,1^3)$, while in the second one we obtain $(6,3)$. Otherwise all of the first $c-2$ 3-hooks that we remove from $(3c-3,3,2,1)$ must be on the first row. In this case all these hooks have length 0 and we obtain $(3,3,2,1)$. Putting all of this together, if $a+2b=3$, we have that

$$\begin{aligned}\chi^{(n-6,3,2,1)}_{(c_1,\ldots,c_{l''},3^c,2^b,1^a)} &= \chi^{(3c-3,2,1)}_{(3^c,2^b,1^a)} \\ &= -(c-2)(c-3) - (c-2)\chi^{(6,1^3)}_{(3^2,2^b,1^a)} - (c-2)\chi^{(6,2,1)}_{(3^2,2^b,1^a)} \\ &\quad +\chi^{(3,3,2,1)}_{(3^2,2^b,1^a)}\end{aligned}$$

from which we easily get, as $c \geq 2$, that $\chi^{(n-6,3,2,1)}_{(c_1,\ldots,c_{l''},3^c,2,1)} = -(c-2)(c+1)-2 < 0$ and similarly $\chi^{(n-6,3,2,1)}_{(c_1,\ldots,c_{l''},3^c,2,1)} = -(c-2)(c+3)-4 < 0$ and so if $c \geq 2$ we have that $(c_1,\ldots,c_{l''},3^c,2,1)$ and $(c_1,\ldots,c_{l''},1^3)$ aren't p-vanishing. So by this and what we have already proven we have that when $n \equiv 6 \bmod 9$ then $\sum_{c_j<9} c_j \leq 6$ and so also in this case the theorem holds for $k'=2$.

Case 7.

At last we need to show that the theorem holds for $k'=1$ when $n\equiv 2 \bmod 9$ and for $k'=2$ when $n\equiv 5,8 \bmod 9$. So assume that $n\equiv 2 \bmod 3$. By what we just proved in case 4, we have that if the theorem doesn't hold for $k'=2$ then it cannot hold for $k'=1$ either, that is we need to have that $\sum_{c_j=1} c_j > 3$, so that $(c_1,\ldots,c_h)$ must have at least 3 parts equal to 1 in each of these cases. Also any n for which the theorem might not hold must be bigger or equal to 11, so that all the partitions we will consider in the next part are actually partitions of n. By lemma 68 we have in this case that $\chi^{(n-2,1,1)}$ has degree divisible by 3. Write $(c_1,\ldots,c_h)=(c_1,\ldots,c_l,4^q,3^r,2^s,1^t)$, with $c_l\geq 5$. If $(c_1,\ldots,c_h)$ is p-vanishing we need to have that $s=(t-1)(t-2)/2$ by what we proved in case 2.

Next consider $(n-4,2,1,1)$. Using the hook formula we have that the degree of $\chi^{(n-4,2,1,1)}$ is $n(n-2)(n-3)(n-5)/8$ and so it is divisible by 3 as $n\equiv 2 \bmod 3$. If $t=3$ then $s=1$ and so as $h_{1,2}^{(n-4,2,1,1)}=n-4>\sum_{c_j\geq 3}c_j$, $h_{1,3}^{(n-4,2,1,1)}=n-6<\sum_{c_j\geq 3}c_j$ and $l_{1,3}^{(n-4,2,1,1)}=0$ it can be seen that whenever we recursively remove from $(n-4,2,1,1)$ a sequence of hooks with lengths $(c_1,\ldots,c_h)$, some hook of length ≥ 3 must be removed from a row different from the first row. As such a hook can only have length 4 (it must have length at most 4 as $h_{2,1}^{(n-4,2,1,1)}=4$ and if this hook had length 3 then as the previously removed hooks had length >1 it would need to be removed from (1^4) and after having removed it we would get (1) which gives a contradiction as then we need to remove 1 2-hook and 3 1-hooks) and so we easily have that $\chi_{(c_1,\ldots,c_h)}^{(n-4,2,1,1)}=q$. So if $t=3$ and $(c_1,\ldots,c_h)$ is p-vanishing we need to have that $q=0$. So let now $t\geq 4$. Then $s\geq 3$ and so it can be easily seen that when we recursively remove from $(n-4,2,1,1)$ a sequence of hooks with lengths $(c_1,\ldots,c_h)$ we can either remove all hooks ≥ 3 from the first row or remove one 4-hook from the second row and all other hooks from the first row. So we have that in this case $\chi_{(c_1,\ldots,c_h)}^{(n-4,2,1,1)}=q+\chi_{(2^s,1^t)}^{(2s+t-4,2,1,1)}$. When $t=4$ we have that $s=3$ and so $\chi_{(c_1,\ldots,c_h)}^{(n-4,2,1,1)}=q-10$ and when $t=5$ we have that $s=6$ and $\chi^{(n-4,2,1,1)}(c_1,\ldots,c_h)=q-45$. So let now $t\geq 6$. If we want to recursively remove s 2-hooks from $(2s+t-4,2,1,1)$ (where $s=(t-1)(t-2)/2\geq 3$), we can remove all 2-hooks from the first row, which can be done in a unique way and in which case all 2-hooks have leg-length 0 and after we obtain $(t-4,2,1,1)$. Otherwise we can remove one 2-hook from the third row and all other from the first row, which can be done in s ways and in which case the sum of the leg-lengths of the 2-hooks is 1 and in which case we obtain $(t-2,2)$. Otherwise we need to remove one 2-hook form the third row and one 2-hook from the second row. This can be done in $s(s-1)/2$ ways, here too the sum of the leg-lengths of the 2-hooks is 1 and after having done this we obtain (t). So as now we only need to remove 1-hooks, the degrees of $\chi^{(t-4,2,1,1)}$ and $\chi^{(t-2,2)}$ are $t(t-2)(t-3)(t-5)/8$ and $t(t-3)/2$ respectively and $s=(t-1)(t-2)/2$ we have that putting all of this together when $t \geq 6$

$$\begin{aligned}\chi^{(n-4,2,1,1)}_{(c_1,\ldots,c_h)} &= q+\tfrac{t(t-2)(t-3)(t-5)}{8}-\tfrac{t(t-1)(t-2)(t-3)}{4}-\tfrac{t(t-1)(t-2)(t-3)}{8}\\ &= q-\tfrac{t(t-2)(t-3)(t+1)}{4}.\end{aligned}$$

So we have that if $(c_1,\ldots,c_h)$ is p-vanishing then q must be given by

$$q=\begin{cases}0 & t=3\\ 10 & t=4\\ 45 & t=5\\ \frac{t(t-2)(t-3)(t+1)}{4} & t=6.\end{cases}$$

Let's now consider $(n-5,3,2)$. We have that the degree of $\chi^{(n-5,3,2)}$ is $n(n-1)(n-2)(n-5)(n-7)/24$ and as $n\equiv 2 \mod 3$, so that $3|(n-2),(n-5)$, we have that 3 divides the degree of $\chi^{(n-5,3,2)}$. If $t=3$ then $\sum_{c_j=1,2} c_j=5$ and so it can be seen by looking at the Young diagram of $(n-5,3,2)$ that we cannot remove from this partition all hooks of length at least 3 of a sequence of hooks with lengths $(c_1,\ldots,c_h)$ from the first row. So it can be easily seen that in this case

$$\chi^{(n-5,3,2)}_{(c_1,\ldots,c_h)}=-r\chi^{(3,1^2)}_{(2,1^3)}-q\chi^{(4,1)}_{(2,1^3)}=-2q.$$

If $t\geq 4$ then $s\geq 6$ as $s=(t-1)(t-2)/2$ and so $\sum_{c_j=1,2} c_j>8$ in this case, so it can be easily seen that here

$$\begin{aligned}\chi^{(n-5,3,2)}_{(c_1,\ldots,c_h)} &= -r\chi^{(2s+t-2,1^2)}_{(2^s,1^t)}-q\chi^{(2s+t-1,1)}_{(2^s,1^t)}+\chi^{(2s+t-5,3,2)}_{(2^s,1^t)}\\ &= -r\chi^{(2s+t-2,1^2)}_{(2^s,1^t)}-q\chi^{(t-1,1)}(1)+\chi^{(2s+t-5,3,2)}_{(2^s,1^t)}\\ &= -q(t-1)+\chi^{(2s+t-5,3,2)}_{(2^s,1^t)}.\end{aligned}$$

Using this we have that $\chi^{(n-5,3,2)}_{(c_1,\ldots,c_h)}$ is equal to $-3q+6$ when $t=4$, to $-4q+60$ when $t=5$, to $-5q+270$ when $t=6$ and to $-6q+840$ when $t=7$. When $t\geq 8$ and we are recursively removing s 2-hooks from $(2s+t-5,3,2)$ we can remove either 0,1 or 2 of them from the second and third row. In any case the sum of the leg-lengths of the 2-hooks must be 0. If we don't remove any 2-hook from the lower rows we get $(t-5,3,2)$ and we need to remove the 2-hooks in a unique way. If we remove 1 2-hooks from the lower rows we get $(t-3,3)$ and this can be done in s different ways. If we remove 2 2-hooks from the second and third row we get $(t-1,1)$. This can be done in $s(s-1)/2$ different ways (we need to choose 2 hooks out of s). As the degrees of $\chi^{(t-5,3,2)}$, $\chi^{(t-3,3)}$ and $\chi^{(t-1,1)}$ are $t(t-1)(t-2)(t-5)(t-7)/24$, $t(t-1)(t-5)/6$ and $t-1$ respectively and as $s=(t-1)(t-2)/2$, putting all of this together and simplifying we have that if $t\geq 8$ then $\chi^{(n-5,3,2)}_{(c_1,\ldots,c_h)}=(1-t)(q-t(t-2)(t-3)^2/4)$. So from all of this we get that if $(c_1,\ldots,c_h)$ is p-vanishing then q is equal to 0 if $t=3$, to 2 if $t=4$, to 15 if $t=5$, to 54 if $t=6$, to 140 if $t=7$ and to $t(t-2)(t-3)^2/4$ if $t\geq 8$. As $t(t-2)(t-3)^2\not=t(t-2)(t-3)(t+1)$ for $t\geq 8$ and checking singularly the cases $t\leq 7$ and comparing these numbers with those we got in the previous paragraph, if we have an exception to the theorem for $n\equiv 2 \bmod 3$, then we need to have that $t=3$, $s=1$ and $q=0$ whenever $(c_1,\ldots,c_h)$ is 3-vanishing.

If now $n\equiv 2 \bmod 9$ we have that the degree of $\chi^{(n-3,2,1)}$ is equal to $n(n-2)(n-4)/3$ and so is divisible by 3. Using what we just proved for the case $n\equiv 2 \bmod 3$ and the results we got while considering the case $p=2$ and $n\equiv 2 \bmod 4$ we have that in this case if $(c_1,\ldots,c_h)$ is 3-vanishing then $t\leq 2$ and so in this case the theorem holds for $k'=1$.

If $n\equiv 5 \bmod 9$ by what we have seen until now we need to show that if $(c_1,\ldots,c_h)$ is p-vanishing then $(c_1,\ldots,c_h)=(c_1,\ldots,c_l,3^r,2,1^3)$, with $c_l\geq 5$. We will now show that $r=0$. In order to do this consider $(n-4,2,2)$. The corresponding irreducible character has degree $n(n-1)(n-4)(n-5)/12$ which is divisible by 3 as $9|(n-5)$. Also as $h_{1,2}^{(n-4,2,2)}=n-3>\sum_{c_j\geq 3}c_j$, $h_{1,3}^{(n-4,2,2)}=n-6<\sum_{c_j\geq 3}c_j$, $l_{1,3}^{(n-4,2,2)}=0$ and $h_{2,1}^{(n-4,2,2)}=3$, we have that whenever removing a sequence of $(c_1,\ldots,c_h)$-hooks from $(n-4,2,2)$ we need to remove one 3-hook from the second row and so we have that

$$\chi^{(n-4,2,2)}_{(c_1,\ldots,c_h)}=-r\chi^{(4,1)}_{(2,1^3)}=-2r$$

from which we have that if $(c_1,\ldots,c_h)$ is 3-vanishing we need to have that $\sum_{c_j<5}c_j\leq 5$ and so from lemma 74 we have that the theorem holds for $k'=2$ when $p=3$ and $n\equiv 5 \bmod 9$.

To finish proving the theorem for $p=3$, we now only have left to prove it for $k'=2$ and $n\equiv 8 \bmod 9$. Write now

$$(c_1,\ldots,c_h)=(c_1,\ldots,c_{l'},7^g,6^f,5^e,4^q,3^r,2^s,1^t)$$

with $c_{l'}\geq 8$. We know that if there is any exception to the theorem then this happens when $q=0$, $s=1$ and $t=3$. We will now show that if $(c_1,\ldots,c_h)=(c_1,\ldots,c_{l'},7^g,6^f,5^e,3^r,2,1^3)$ and $(c_1,\ldots,c_h)$ is 3-vanishing, then $r=1$ and $e,f,g=0$, which would then prove the theorem for $p=3$. Start with considering $(n-6,3,3)$. By the hook-formula the corresponding character has degree $n(n-1)(n-2)(n-3)(n-7)(n-8)/144$ and as $n\equiv 8 \bmod 9$

we then have that 3 divides it (as then we also have that $3|(n-2)$). First assume that $r \geq 2$. Then as $h_{2,1}^{(n-6,3,3)} = 4$, $h_{1,4}^{(n-6,3,3)} = n-9 > \sum_{c_j \geq 3} c_j - 6$ and $l_{1,4}^{(n-6,3,3)} = 0$ when we recursively remove from $(n-6,3,3)$ a sequence of hooks of lengths $(c_1,\ldots,c_h)$, we can either remove the first $h-5$ hooks (so that we removed all hooks of the sequence apart for 2 3-hooks and the 2- and 1-hooks) from the first row or remove 1 or 2 of the 3-hooks from the lower rows. As when we remove 2 of the first $r-2$ 3-hooks from the second and third row we need to remove $R_{3,1}$ and $R_{2,1}$ or $R_{2,2}$ and $R_{2,1}$, we can easily see that

$$\begin{aligned}\chi^{(5,3,3)}_{(c_1,\ldots,c_h)} &= \chi^{(5,3,3)}_{(3^2,2,1^3)} + (r-2)\chi^{(8,3)}_{(3^2,2,1^3)} - (r-2)\chi^{(8,2,1)}_{(3^2,2,1^3)} + (r-2)(r-3)\\ &= (r+1)(r-1) \neq 0.\end{aligned}$$

So if $(c_1,\ldots,c_h)$ is p-vanishing we need to have $r = 0,1$. If $r = 0$, then as $h_{1,2}^{(n-6,3,3)} = n-5 = \sum_{c_j \geq 5} c_j$ and $l_{1,4}^{(n-6,3,3)} = 0$, it can be easily seen by just looking at the Young-diagram of $(n-6,3,3)$ that

$$\chi^{(n-6,3,3)}_{(c_1,\ldots,c_h)} = \chi^{(2,2,1)}_{(2,1^3)} = -1$$

from which we have that if $(c_1,\ldots,c_h)$ is p-vanishing then $r = 1$.

Consider now $(n-5,1^5)$. Here we have that the degree of the corresponding character is $(n-1)(n-2)(n-3)(n-4)(n-5)/120$ and so as $n \equiv 8 \mod 9$ we have that it is divisible by 3. As $h_{1,2}^{(n-5,1^5)} = n-6 > n-8 = \sum_{c_j \geq 5} c_j$ and $l_{1,2}^{(n-5,1^5)} = 0$, we have that

$$\chi^{(n-5,1^5)}_{(c_1,\ldots,c_h)} = \chi^{(5e+3,1^5)}_{(5^3,3,2,1^3)} = e\chi^{(8)}_{(3,2,1^3)} + \chi^{(3,1^5)}_{(3,2,1^3)} = e$$

and so as we want $(c_1,\ldots,c_h)$ to be 3-vanishing we need to have that $e = 0$.

If we now consider $(n-7,2^2,1^3)$ we have that $\chi^{(n-7,2^2,1^3)}$ has degree $n(n-1)(n-3)(n-4)(n-5)(n-7)(n-8)/(9\cdot 40)$ from which we have that in this case 3 divides the degree of $\chi^{(n-7,2^2,1^3)}$. As

$$h_{1,3}^{(n-7,2^2,1^3)} = n-9 < n-8 = \sum_{c_j \geq 6} c_j,$$

$l_{1,3}^{(n-7,2^2,1^3)} = 0$, $h_{1,2}^{(n-7,2^2,1^3)} = n-6 > n-8 = \sum_{c_j \geq 6} c_j$ and $h_{2,1}^{(n-7,2^2,1^3)} = 6$, we need to have that

$$\chi^{(n-7,2^2,1^3)}_{(c_1,\ldots,c_h)} = f\chi^{(7,1)}_{(3,2,1^3)} = 2f$$

as then we need to remove one 6-hook from the second row whenever removing a sequence of hooks of lengths $(c_1,\ldots,c_h)$, and so we need to have that $f = 0$ as we are assuming that $(c_1,\ldots,c_h)$ is 3-vanishing.

At last consider $(n-7,2,1^5)$. The corresponding character has degree $n(n-2)(n-3)(n-4)(n-5)(n-6)(n-8)/(3\cdot 280)$, which then is divisible by 3. For reasons just like those in the previous point we have that

$$\chi^{(n-7,2,1^6)}_{(c_1,\ldots,c_h)} = -g\chi^{(8)}_{(3,2,1^3)} = -g$$

and so we need to have that g is also 0 when $(c_1,\ldots,c_h)$ is 3-vanishing.

So if $n\equiv 8 \bmod 9$ and $(c_1,\ldots,c_h)$ is p-vanishing then $\sum_{c_j<8} c_j \leq 8$ and so by what we have already proven we have that also in this case the theorem holds for $k'=2$. As this was the last case we had to consider for $p=3$, we also have that the theorem holds for $p=3$. □

Conjecture 76. *Let $p\neq 2,3$. If $(c_1,\ldots,c_h)$ is p-vanishing we need to have that $d\geq a'_{k'}p^{k'}$.*

Proof. Assume that $\sum_{c_j<p^{k'-1}} c_j \leq a_0+a_1p+\ldots+a_{k'-2}p^{k'-2}$, where $k'-1\geq 0$, and assume that for every $n'=a+bp$, with $0\leq a\leq p-1$ and $b\geq 0$, and $(d_1,\ldots,d_s)\vdash n'$ is p-vanishing we need to have that $\sum_{d_j<p} \leq a$. We want to show that then we have that $\sum_{c_j<p^{k'}} c_j \leq a_0+a_1p+\ldots+a_{k'-1}p^{k'-1}$, that is that $d\geq a'_{k'}p^{k'}$. By theorem 73 (as $p\neq 3$) we have that whenever

$$\sum_{c_j<p^{k'-1}} c_j \leq a_0+a_1p+\ldots+a_{k'-2}p^{k'-2}$$

we actually need to have that

$$\sum_{c_j<p^{k'-1}} c_j = a_0+a_1p+\ldots+a_{k'-2}p^{k'-2},$$

so that we can apply theorem 72 and we get that c_j is a multiple of $p^{k'-1}$ whenever $c_j\geq p^{k'-1}$. Let l be maximal such that $c_l\geq p^{k'-1}$. Then

$$(c_1,\ldots,c_h)=(p^{k'-1}b_1,\ldots,p^{k'-1}b_l,c_{l+1},\ldots,c_h)$$

for some $(b_1,\ldots,b_l)\vdash a'_{k'}p+a_{k'-1}$. Also as the $a_i<p$ (they are the coefficients of the p-adic decomposition of n) we have that

$$c_{l+1}+\ldots+c_h = a_0+\ldots+a_{k'-2}p^{k'-2} < p^{k'-1}.$$

Assume that $\sum_{c_j<p^{k'}} c_j > a_0+a_1p+\ldots+a_{k'-1}p^{k'-1}$. As we are assuming that $\sum_{c_j<p^{k'-1}} c_j = a_0+\ldots+a_{k'-2}p^{k'-2}$ this happens if and only if $\sum_{p^{k'-1}\leq c_j<p^{k'}} c_j > a_{k'-1}p^{k'-1}$, which happens if and only if $\sum_{b_j<p} b_j > a_{k'-1}$.

Now $(b_1, \ldots, b_l)$ is a partition of $a_{k'-1} + a'_{k'}p$ and $0 \leq a_{k'-1} < p$ and $a'_{k'} \geq 0$, so by assumption we can find χ^β, an irreducible character of $S_{a_{k'-1}+a'_{k'}p}$, of degree divisible by p and such that $\chi^\beta_{(b_1,\ldots,b_l)} \neq 0$. Let α be the partition of n with core $\alpha_{(p^{k'-1})} = (a_0 + \ldots + a_{k'-2}p^{k'-2})$ and quotient $\alpha^{(p^{k'-1})} = (\beta, 0, \ldots, 0)$. As the degree of β is divisible by p so is that of χ^α by lemmas 61, 64 and 65 and the fact that removing q-hooks from $\alpha^{(p^{k'-1})}$ (which in this case must be removed from β) corresponds to removing $qp^{k'-1}$-hooks from α. Also as in this case we can use theorem 55 we have that

$$\chi^\alpha_{(c_1,\ldots,c_h)} = \pm\chi^{\alpha_{(p^{k'-1})}}_{(c_{l+1},\ldots,c_h)}\chi^\beta_{(b_1,\ldots,b_l)}$$

as here β is the only non-zero partition in $\alpha^{(p^{k'-1})}$. As $\chi^{\alpha_{(p^{k'-1})}}_{(c_{l+1},\ldots,c_h)} = 1$ as $\alpha_{(p^{k'-1})} = (a_0 + \ldots + a_{k'-2}p^{k'-2})$ and $\chi^\beta_{(b_1,\ldots,b_l)} \neq 0$ we have that $\chi^\alpha_{(c_1,\ldots,c_h)} \neq 0$ in this case and so we have that under these assumptions, if $(c_1, \ldots, c_h)$ is p-vanishing we need to have that $d \geq a'_{k'}p^{k'}$.

As when $n = a'_{k'}p^{k'}$ and $p \neq 2, 3$ we have that the theorem is always satisfied by theorem 67 and by using what we proved earlier for the case when $n \neq a'_{k'}p^{k'}$, we have that in order to prove the theorem for the case where $p \neq 2, 3$ it is then enough, by what we proved until now, to prove the following lemma, which is clearly always satisfied when $a = 0$.

As the cases where $p = 2, 3$ have already been proven, this conjecture is proved up to the next condition. □

Condition 77. *Let $n = a + bp$, where p is a prime different from 2 and 3, $0 \leq a < p$ and $b \geq 0$. If $(c_1, \ldots, c_h)$ is p-vanishing we need to have that $\sum_{c_j<a} c_j \leq a$.*

Using lemma 74 it can be easily seen that this condition is equivalent to conjecture 76 for $k' = 1$.

Even if we cannot prove this condition in the general case, it can be proved when $a = 0, 1, 2$ for any p. For $a = 0, 1$ the condition is trivial, while for $a = 2$ it can be proved as in theorem 75 for the case $p = 2$ and $n \equiv 2 \bmod 4$ (case 3).

Theorem 78. *Let $\pi \in S_n$, $\alpha(\pi) = (c_1, \ldots, c_h)$, with $c_h > 0$, and define $d_{k'} = \sum_{c_j \geq p^{k'}} c_j$ for any $0 \leq k' \leq k$. If $d_{k'} = a_{k'}p^{k'} + \ldots + a_kp^k$ for any $m \leq k' \leq k$ and l is maximal such that $c_l \geq p^m$ we have that π is p-vanishing if and only if the conjugacy class of $S_{n-a_mp^m-\ldots-a_kp^k} = S_{a_0+\ldots+a_{m-1}p^{m-1}}$ with cycle partition $(c_{l+1}, \ldots, c_h)$ is p-vanishing.*

Proof. By theorem 72 we have that whenever $d_{k'} = a_{k'}p^{k'} + \ldots + a_kp^k$ and π is p-vanishing, all c_j which are at $\geq p^{k'}$ must be multiples of $p^{k'}$, in particular in this case we need to have that $w_{p^{k'}}(\pi)=a_{k'}+\ldots+a_kp^{k-k'}$. As this holds for each $k'\geq m$ it is easy to see that $\alpha(\pi)$ is of the form

$$\left(b_1^{(k)}p^k,\ldots,b_{h_k}^{(k)}p^k,\ldots,b_1^{(m)}p^k,\ldots,b_{h_m}^{(m)}p^k,c_{l+1},\ldots,c_h\right),$$

where for each $m\leq k'\leq k$, $(b_1^{(k')},\ldots,b_{h_{k'}}^{(k')})\vdash a_{k'}$, and $c_{l+1}<p^m$. Assume that $\alpha\vdash n$ such that the degree of χ^α is divisible by p and assume that β is obtained by α by removing a sequence of hooks of lengths

$$\left(b_1^{(k)}p^k,\ldots,b_{h_k}^{(k)}p^k,\ldots,b_1^{(m)}p^k,\ldots,b_{h_m}^{(m)}p^k\right).$$

Then β can also be obtained by α by removing a sequence of hooks of lengths $((p^k)^{a_k},\ldots,(p^m)^{a_m})$ and so by lemmas 61, 64 and 65 applied to both α and β we have that p divides also the degree of χ^β and so if the conjugacy class of S_n with cycle partition $(c_{l+1},\ldots,c_h)$ is p-vanishing, we have that $\chi^\beta_{(c_{l+1},\ldots,c_h)}=0$ and so we also have $\chi^\alpha(\pi)=0$ for any χ^α irreducible character of degree divisible by p, that is π is p-vanishing in this case.

Let now $\beta=(b_1,b_2,\ldots)\vdash n-a_mp^m-\ldots-a_kp^k$ be such that χ^β has degree divisible by p. Let $\alpha=(b_1+a_mp^m+\ldots+a_kp^k,b_2,b_3,\ldots)$. Then $\alpha\vdash n$ and as $h_{2,1}^\alpha<n-a_mp^m-\ldots-a_kp^k<p^m$, $h_{1,b_1+1}^\alpha=a_mp^m+\ldots+a_kp^k$ and $l_{1,b_1+1}^\alpha=0$, we can remove from α a sequence of hooks of lengths $((p^k)^{a_k},\ldots,(p^m)^{a_m})$ in a unique way and doing this we obtain β. So again by lemmas 61, 64 and 65 we have that p divides the degree of α. Assume that $\pi\in S_n$ is p-vanishing and that $d_{k'}=a_{k'}p^{k'}+\ldots+a_kp^k$ for any $m\leq k'\leq k$. As again $h_{2,1}^\alpha<n-a_mp^m-\ldots-a_kp^k<p^m\leq c_j$, $j\leq l$, $h_{1,b_1+1}^\alpha=a_mp^m+\ldots+a_kp^k=d_m=c_1+\ldots+c_l$ and $l_{1,b_1+1}^\alpha=0$, we can remove the first l hooks of a sequence with hook-lengths $(c_1,\ldots,c_h)$ from α in a unique way and we obtain β. So by the Murnaghan-Nakayama formula we have that $\chi^\beta_{(c_{l+1},\ldots,c_h)}=\pm\chi^\alpha(\pi)=0$, in particular $\chi^\beta_{(c_{l+1},\ldots,c_h)}=0$ and as this holds for any $\beta\vdash n-a_mp^m-\ldots-a_kp^k$ such that p divides the degree of χ^β we have that in this case the conjugacy class with cycle partition $(c_{l+1},\ldots,c_h)$ is p-vanishing and so the theorem is proved. □

The next theorem completely classify 2-vanishing and 3-vanishing conjugacy classes of S_n.

Theorem 79 (Classification of 2- and 3-vanishing elements of S_n)**.** *Let $p=2,3$ and*

- $p^i=8$ *if* $p=2$,
- $p^i=9$ *if* $p=3$.

Write $n = m+p^i s$ with $0 \leq m < p^i$ and $s \geq 0$. Let $\pi \in S_n$. π is p-vanishing if and only if $\alpha(\pi)$, the cycle partition of π, is of the form $(b_1, \ldots, b_h, e_1, \ldots, e_t)$, where $(b_1, \ldots, b_h) \vdash p^i s$ is the cycle partition of a p-adic type conjugacy class of $S_{p^i s}$ and $(c_1, \ldots, c_t) \vdash m$ is the cycle partition of a p-vanishing conjugacy class of S_m.

Proof. By theorems 73 and 75 we know that if $l \geq i$ then $\sum_{c_j \geq p^l} c_j = a_l + pa_{l+1} + \ldots + a_k p^{k-l}$, where $n = a_0 + a_k p^k$ is the p-adic decomposition of n and then by theorem 72 we need to have that $w_{p^l}(\pi) = a_l + pa_{l+1} + \ldots + a_k p^{k-l}$ and so applying lemma 58 we have that $\alpha(\pi) = (b_1, \ldots, b_h, e_1, \ldots, e_t)$, where $(b_1, \ldots, b_h) \vdash p^i s$ and the corresponding conjugacy class is of p-adic type. Using theorem 78 we also know that $(c_1, \ldots, c_t)$ is the cycle partition of a p-vanishing conjugacy class of S_m and so one direction of the theorem is proved.

The proof of the other direction follows easily from theorem 78. □

The following theorem classify p-vanishing conjugacy classes of S_n, for $p \neq 2, 3$ up to conjecture 76, which is proved up to condition 77.

Theorem 80 (Conjecture on p-vanishing elements of S_n for $p \neq 2, 3$)**.** *Assume conjecture 76. Let $p \neq 2, 3$. Then $\pi \in S_n$ is p-vanishing if and only if it is of p-adic type.*

Proof. Assume that π is p-vanishing. Then by theorems 73 and conjecture 76 for each i we have that $\sum_{c_j \geq p^i} c_j = a_i + pa_{i+1} + \ldots + a_k p^{k-i}$, where $n = a_0 + \ldots + a_k p^k$ is the p-adic decomposition of n. Now by theorem 72 we easily have that $w_{p^i}(\pi) = a_i + pa_{i+1} + \ldots + a_k p^{k-i}$ for each i, and so π is of p-adic type by lemma 58. As we already now the opposite implication by theorem 59, we have proved this theorem. □

As condition 77 holds for $a = 0, 1, 2$ it can be seen by the proof of conjecture 76 up to condition 77 that conjecture 76 holds when all the a_i are equal to 0, 1 or 2, and so we then have that theorem 80 allows us to completely classify p-vanishing element of S_n when $p \neq 2, 3$ and all $a_i = 0, 1, 2$.

13 Sign classes

In this last section we will consider sign classes of S_n. Parts of this section are from [6].

Definition 39 (Sign conjugacy class)**.** *A conjugacy class of S_n is a sign conjugacy class if it takes values 0, 1 or -1 on all irreducible characters of S_n.*

A partition of n is a sign partition if it is the cycle partition of a sign conjugacy class of S_n.

We will start with the following theorem, which is lemma 6 of [6].

Theorem 81. *If $(a_1,\dots,a_h)$ is a sign partition then a_h appears only once in $(a_1,\dots,a_h)$, unless $a_h=1$ in which case it can appear twice.*

Proof. First assume that $a_h>1$ and it appears $m\geq 2$ times in $(a_1,\dots,a_h)\vdash n$. Then it is easy to see that $(n-a_h,a_h)$ is a partition of n and

$$\chi^{(n-a_h,a_h)}_{(a_1,\dots,a_h)}=\chi^{((m-1)a_h,a_h)}_{(a_h^m)}=m\geq 2$$

and so in this case $(a_1,\dots,a_h)$ isn't a sign partition.

If $a_h=1$ and a_h appears $m\geq 3$ times in $(a_1,\dots,a_h)$ we have that

$$\chi^{(n-1,1)}_{(a_1,\dots,a_h)}=\chi^{(m-1,1)}_{(1^m)}=m-1\geq 2$$

and so also in this case $(a_1,\dots,a_h)$ isn't a sign partition. □

We will only state the next theorem

Theorem 82. *If $\alpha\vdash n$ and for $0\leq j\leq k-1$, α_i^j are all the partitions that we can obtain by adding an hook of length k and leg-length j to α, we have that*

$$\sum_{j=0}^{k-1}\sum_i(-1)^j[\alpha_i^j]=\sum_{j=0}^{k-1}(-1)^j[\alpha][(k-j,1^j)].$$

A proof of this can be found in the proofs of lemma 21.5 and theorem 21.1 of [1]. This proof uses the Littlewood-Richardson rule (theorem 2.8.13 of [2]).

Lemma 83. *If no part of $(c_1,\dots,c_h)\vdash n+k$ is equal to k and α_i^j are as in the previous theorem we have that*

$$\sum_{j=0}^{k-1}\sum_i(-1)^j\chi^{\alpha_i^j}_{(c_1,\dots,c_h)}=0.$$

Proof. This lemma follows by the previous theorem, by theorem 10, which gives a formula for the induced character, by the fact that if $\beta\vdash k$ then $\chi^{\beta}_{(k)}\neq 0$ if $\beta\neq(k-j,1^j)$ for some $0\leq j<k$ in which case $\chi^{(k-j,1^j)}_{(k)}=(-1)^j$ (theorem 33) and by characters relations of the second kind (theorem 9). □

The next theorem is theorem 7 of [6].

Theorem 84. *In a sign partition the only part that can be repeated is 1, which can appear at most two times.*

Proof. Write $\alpha = (a_1, \ldots, a_i, a^s, a_{i+s+1}, \ldots, a_h)$, where $a_i > a$ and $a_{i+s+1} < a$ and assume that $s \geq 2$ if $a > 1$ or $s \geq 3$ if $a = 1$ (i could be 0). We want to show that then α is not a sign partition. If $a = 1$ we know that the result is true from the previous theorem, so we can assume that $a \geq 2$. Also still by the previous theorem we can assume that $a_{i+s+1} + \ldots + a_h \neq 0$ (that is $a_h \neq a$). First assume that we can find $\beta \vdash as + a_{i+s+1} + \ldots + a_h$ such that $h^\beta_{2,1} \leq a$ and $\chi^\beta_{(a^s, a_{i+s+1}, \ldots, a_h)} \neq 0, \pm 1$. We will show that in this case α isn't a sign partition. Let $\gamma = (\beta_1 + a_1 + \ldots + a_i, \beta_2, \beta_3, \ldots)$. Then as $h^\gamma_{j,l} \leq h^\gamma_{2,1} = h^\beta_{2,1} \leq a$ whenever $j \geq 2$, if we want to recursively remove from γ a sequence of hooks of lengths $(a_1, \ldots, a_h)$ we need to remove the first i of these hooks from the first row and as $h^\gamma_{1,\beta_1+1} = a_1 + \ldots + a_i$ and $l^\gamma_{1,\beta_1+1} = 0$ we can remove the first i hooks of such a sequence from γ in a unique way and so by the Murnaghan-Nakayama formula we have that

$$\chi^\gamma_\alpha = \chi^\beta_{(a^s, a_{i+s+1}, \ldots, a_h)} \neq 0, \pm 1$$

and so α isn't a sign partition.

We will now show that we can always find such a β. Let $t = a_{i+s+1} + \ldots + a_h$. First assume that $1 \leq t \leq a$. In this case let $\beta = (a(s-1) + t, 1^a)$. As $a(s-1) \leq h^\beta_{2,1} < as$, it is easy to see that if we want to recursively remove s a-hooks from β we need to remove one of them from the second row and the others from the first row and so we have by the Murnaghan-Nakayama formula that

$$\chi^\beta_{(a^s, a_{i+s+1}, \ldots, a_h)} = (-1)^{a-1} s \chi^{(t)}_{(a_{i+s+1}, \ldots, a_h)} = (-1)^{a-1} s \neq 0, \pm 1$$

as $s \geq 2$, and so in this case α isn't a sign partition.

By theorem 81 we can now assume that $a \geq 3$. Assume now that we have $a < t < 2a$. In this case $(t-a)^{(a)} = ((0), \ldots, (0))$ and so we can add an a-hook to $(t-a)$ in a different ways, which give the following partitions $\beta_0 = (t)$, $\beta_j = (a - j, t - a + 1, 1^{j-1})$, for $1 \leq j \leq 2a - t - 1$ and $\beta_j = (t - a, a - j, 1^j)$ for $2a - t \leq j \leq a - 1$. Notice that for each $0 \leq j \leq a - 1$, the a-hook we need to add to $(t - a)$ in order to get β_j has leg-length j. As $a_l < a$ for each $l \geq i + s + 1$, we have by lemma 83 that

$$\sum_{j=0}^{a-1} (-1)^j \chi^{\beta_j}_{(a_{i+s+1}, \ldots, a_h)} = 1 + \sum_{j=1}^{a-1} (-1)^j \chi^{\beta_j}_{(a_{i+s+1}, \ldots, a_h)} = 0$$

and so, as $a \geq 3$, so that the last summation is over at least 2 terms, we can find $1 \leq j \leq a - 1$ such that $(-1)^j \chi^{\beta_j}_{(a_{i+s+1}, \ldots, a_h)} \geq 0$.

If $1 \leq j \leq 2a-t-1$ let $\beta = (t+a(s-1), a-(j-1), 1^{j-1})$. Then $\beta \vdash t+as$ and $h^\beta_{2,1} = a$. As $h^\beta_{1,t+1} = a(s-1)$ and $l^\beta_{1,t+1} = 0$ as $a-(j-1) \leq a < t$, if we want to recursively remove $s-1$ hooks of length a from β we can either remove all of them from the first row or we have to remove one of them from the second row and all others from the first row. So by the Murnaghan-Nakayama formula we have that

$$\chi^\beta_{(a^s,a_{i+s+1},\ldots,a_h)} = \chi^{(t,a-(j-1),1^{j-1})}_{(a,a_{i+s+1},\ldots,a_h)} + (-1)^{j-1}(s-1)\chi^{(t+a)}_{(a,a_{i+s+1},\ldots,a_h)}$$

and as the only a-hooks of $(t, a-(j-1), 1^{j-1})$ are $H^{(t,a-(j-1),1^{j-1})}_{2,1}$ and $H^{(t,a-(j-1),1^{j-1})}_{1,t-a+2}$ (as $a-j+1 \geq a-2a+t+1+1 = t-a+2 > 2$) we have that

$$\begin{aligned}
\chi^\beta_{(a^s,a_{i+s+1},\ldots,a_h)} &= (-1)^{j-1}(s-1) + (-1)^{j-1}\chi^{(t)}_{(a_{i+s+1},\ldots,a_h)} - \chi^{(a-j,t-a+1,1^{j-1})}_{(a_{i+s+1},\ldots,a_h)} \\
&= (-1)^{j-1}s - \chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)} \\
&= (-1)^{j-1}\left(s + (-1)^j\chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)}\right)
\end{aligned}$$

and as $s \geq 2$ and $(-1)^j\chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)} \geq 0$, we have that $\chi^\beta_{(a^s,a_{i+s+1},\ldots,a_h)} \neq 0, \pm 1$ and so $(a_1, \ldots, a_h)$ isn't a sign partition in this case.

If instead $2a-t \leq j \leq a-1$, let $\beta = (t+a(s-1), a-j, 1^j)$. Again $\beta \vdash t+as$, $h^\beta_{2,1} = a$ and if we want to remove $s-1$ a-hooks from β we either need to remove all of them from the first row or remove one of the from the second row and the others from the first row (as again $h^\beta_{1,t+1} = a(s-1)$ and $l^\beta_{1,t+1} = 0$). Also as here $a-j \leq a-2a+t = t-a$, we have that the only a-hooks of $(t, a-j, 1^j)$ are $H_{1,t-a+1}$ and $H_{2,1}$ and so we have that

$$\begin{aligned}
\chi^\beta_{(a^s,a_{i+s+1},\ldots,a_h)} &= \chi^{(t,a-(j-1),1^{j-1})}_{(a,a_{i+s+1},\ldots,a_h)} + (-1)^{j-1}(s-1)\chi^{(t+a)}_{(a,a_{i+s+1},\ldots,a_h)} \\
&= (-1)^j(s-1) + (-1)^j\chi^{(t)}_{(a_{i+s+1},\ldots,a_h)} + \chi^{(t-a,a-j,1^j)}_{(a_{i+s+1},\ldots,a_h)} \\
&= (-1)^j s + \chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)} \\
&= (-1)^j\left(s + (-1)^j\chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)}\right)
\end{aligned}$$

and as again $s \geq 2$ and $(-1)^j\chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)} \geq 0$, we have that also in this case $\chi^\beta_{(a^s,a_{i+s+1},\ldots,a_h)} \neq 0, \pm 1$, from which follows that $(a_1, \ldots, a_h)$ isn't a sign partition in this case either.

The last case we need to consider is when $t \geq 2a$. In this case one of the partitions which appear in $(t-a)^{(a)}$ is of the form (c), for some $c \geq 1$ and all the other partitions appearing in the a-quotient are (0). So we can add an a-hook to $(t-a)$ in $a+1$ ways, which give the partitions (t) and

$\beta_j = (t-a, a-j, 1^j)$, for $0 \le j \le a-1$. Here too we have that the leg-length of the a-hook we add to $(t-a)$ to obtain β_j is j and so again by lemma 83 we have that

$$1 + \sum_{j=0}^{a-1} (-1)^j \chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)} = 0$$

and as $a \ge 3$ we can find $0 \le j \le a-1$ such that $\chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)} \ge 0$. Again let $\beta = (t + a(s-1), a-j, 1^j)$. As here again we have that $t-a+1 > a \ge a-j$, we have by the same calculations as in the previous case that

$$\chi^{\beta}_{(a^s,a_{i+s+1},\ldots,a_h)} = (-1)^j \left(s + (-1)^j \chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)} \right)$$

and as here too $s \ge 2$ and $(-1)^j \chi^{\beta_j}_{(a_{i+s+1},\ldots,a_h)} \ge 0$ we need to have that $\chi^{\beta}_{(a^s,a_{i+s+1},\ldots,a_h)} \ne 0, \pm 1$ and so also in this case $(a_1, \ldots, a_h)$ isn't a sign partition and so the theorem is proved. □

Theorem 85. *If $(a_1, \ldots, a_h)$ is a sign partition and $a_{i+1} + \ldots + a_h \le a_i + 1$ then also $(a_{i+1}, \ldots, a_h)$ is a sign partition.*

Proof. Let $\beta = (\beta_1, \beta_2, \ldots)$ be any partition of $a_{i+1} + \ldots + a_h$ and define $\alpha = (\beta_1 + a_1 + \ldots + a_i, \beta_2, \beta_3, \ldots)$. If $\beta_2 = 0$ then $\beta = (a_{i+1} + \ldots + a_h)$ and so $\chi^{\beta}_{(a_{i+1},\ldots,a_h)} = 1$. If $\beta_2 \ne 0$ then $h^{\alpha}_{2,1} = h^{\beta}_{2,1} < a_{i+1} + \ldots + a_h \le a_i + 1 \le a_j + 1$ for $j \le i$. First assume that $h^{\beta}_{2,1} < a_i$. Then as $h^{\alpha}_{1,\beta_1+1} = a_1 + \ldots + a_i$ and $l^{\alpha}_{1,\beta_1+1} = 0$ we have that $\chi^{\beta}_{(a_{i+1},\ldots,a_h)} = \chi^{\alpha}_{(a_1,\ldots,a_h)} = 0, \pm 1$.

Otherwise we have that $h^{\beta}_{2,1} = a_i$. As

$$h^{\beta}_{2,1} \le \beta_2 + \beta_3 + \ldots = a_{i+1} + \ldots + a_h - \beta_1 \le a_i + 1 - \beta_1$$

and $\beta_1 \ge \beta_2 > 0$ we need to have that $\beta_1 = 1$ and so $\beta = (1^{a_{I+1}+\ldots+a_h})$ and then $\chi^{\beta}_{(a_{i+1},\ldots,a_h)} = \pm 1$ and so we have that the theorem is true. □

The previous theorem is a generalization of one direction of proposition 2 of [6], which is the next theorem.

Theorem 86. *If $(a_1, \ldots, a_h) \vdash n$ is such that $a_1 > a_2 + \ldots + a_h$ we have that $(a_1, \ldots, a_h)$ is a sign partition if and only if $(a_2, \ldots, a_h)$ is a sign partition.*

Proof. Assume that $(a_2, \ldots, a_h)$ is a sign partition and let α be any partition of n. If α doesn't have any a_1-hook we have by the Murnaghan-Nakayama formula that $\chi^{\alpha}_{(a_1,\ldots,a_h)} = 0$. Otherwise as α is a partition of $n < 2a_1$ we need to have that $w_{a_1}(\alpha)=1$, so that α has a unique a_1-hook. Let β be obtained by α by removing this a_1-hook. Then $\beta\vdash a_2+\ldots+a_h$ and

$$\chi^{\alpha}_{(a_1,\ldots,a_h)}=\pm\chi^{\beta}_{(a_2,\ldots,a_h)}=0,\pm1$$

as $(a_2,\ldots,a_h)$ is a sign partition and so also $(a_1,\ldots,a_h)$ is a sign partition.

As the opposite direction of the theorem follows by the previous theorem with $i=1$, we have that the theorem is proved. □

By this last theorem in order to classify all sign partitions, and so also all sign conjugacy classes, it is enough to classify those sign partitions $(a_1,\ldots,a_h)$ for which $a_1\leq a_2+\ldots+a_h$. In all of the following we will assume that $a_h\neq 0$.

Theorem 87. *If (a_1,a_2) is a sign partition then $a_1=a_2$ if and only if $(a_1,a_2)=(1,1)$*

Proof. This is actually an easy corollary of theorem 81, as it is clear that $(1,1)$ is a sign class. □

Theorem 88. *If $a_1\leq a_2+a_3$ then (a_1,a_2,a_3) is a sign partition if and only if $(a_1,a_2,a_3)=(a_1,a_1-1,1)$ and $a_1\geq 2$.*

Proof. If (a_1,a_2,a_3) is a sign partition then by theorem 81 we clearly have that $a_1\geq 2$ (as we are assuming that $a_3\geq 1$). First assume that $a_2+a_3>a_1$. Then we have that

$$\begin{aligned}\chi^{(a_2+a_3,1^{a_1})}_{(a_1,a_2,a_3)} &= (-1)^{a_1-1}\chi^{(a_2+a_3)}_{(a_2,a_3)}+\chi^{(a_2+a_3-a_1,1^{a_1})}_{(a_2,a_3)}\\ &= (-1)^{a_1-1}+(-1)^{a_2-1}\chi^{(a_2+a_3-a_1,1^{a_1-a_2})}\\ &= (-1)^{a_1-1}+(-1)^{a_2-1}(-1)^{a_1-a_2}\\ &= (-1)^{a_1-1}2\neq 0,\pm1\end{aligned}$$

as $h^{(a_2+a_3-a_1,1^{a_1})}_{1,2}=a_2+a_3-a_1-1<a_2$ ($(1,2)\in(a_2+a_3-a_1,1^{a_1})$ as $a_2+a_3>a_1$), and so (a_1,a_2,a_3) is not a sign partition when $a_2+a_3>a_1$. So if (a_1,a_2,a_3) is a sign partition we need to have that $a_2+a_3=a_1$. If $a_1=2,3$ as we are assuming that $a_3\geq 1$, we have that $(a_1,a_2,a_3)=(a_1,a_1-1,1)$ is the only possibility in which this can happen. So assume now that $a_1\geq 4$ and that $a_3\geq 2$. Then

$$\begin{aligned}\chi^{(a_1,3,1^{a_1-3})}_{(a_1,a_2,a_3)} &= (-1)^{a_1-3}\chi^{(a_1)}_{(a_2,a_3)}-\chi^{(2,1^{a_1-2})}_{(a_2,1_3)}\\ &= (-1)^{a_1-3}-(-1)^{a_2-1}\chi^{(2,1^{a_1-a_2-2})}_{(a_3)}\\ &= (-1)^{a_1-3}-(-1)^{a_2-1}(-1)^{a_1-a_2-2}\\ &= (-1)^{a_1-3}2\neq 0,\pm1\end{aligned}$$

and so we have that (a_1, a_2, a_3) isn't a sign partition when it is equal to $(a_1, a_1 - a_3, a_3)$ with $a_3 \geq 2$. So we have that whenever (a_1, a_2, a_3) is a sign partition and $a_1 \leq a_2 + a_3$ then $(a_1, a_2, a_3) = (a_1, a_1 - 1, 1)$ and $a_1 \geq 2$.

Assume now that $(a_1, a_2, a_3) = (a_1, a_1 - 1, 1)$ and $a_1 \geq 2$. As $(a_1 - 1, 1)$ is a sign partition by theorem 86 if $a_1 \geq 3$ or, when $a_1 = 2$ by the fact that $(1, 1)$ is a sign partition, we easily have that $\chi^{\alpha}_{(a_1,a_1-1,1)} = 0, \pm 1$ whenever α contains at most one a_1-hook. So we only need to show that the same holds if α has 2 hooks of length a_1. As α is a partition of $2a_1$ this happens if and only if α isn't a hook (in which case α only has one a_1-hook, as it also contains a $2a_1$-hook) and α can be obtained by adding an a_1-hook to an a_1-hook. Also as $\chi^{(a_1-c,1^c)}_{(a_1-1,1)} = 0$ if $c \neq 0, a_1 - 1$, in which cases $(a_1 - c, 1^c) = (a_1), (1^{a_1})$, we have that if $\chi^{\alpha}_{(a_1,a_1-1,1)} \neq 0, \pm 1$ then α must have 2 a_1-hooks and removing either one of them we need to obtain (a_1) or (1^{a_1}). It is easy to see that if we add an a_1-hook to (a_1) or (1^{a_1}) and we don't obtain an hook partition we must obtain $(a_1, c, 1^{a_1-c})$ or $(a_1 - c + 2, 2^{c-1}, 1^{a_1-c})$ for some $2 \leq c \leq a_1$. For any of these values of c we have that $h^{(a_1,c,1^{a_1-c})}_{1,2} = h^{(a_1-c+2,2^{c-1},1^{a_1-c})}_{2,1} = a_1$ and if we remove the corresponding hook we obtain $(c-1, 1^{a_1-c+1})$ or $(a_1-c+2, 1^{c-2})$. If $c \neq 2$ these partitions are not equal to either (a_1) or (1^{a_1}). So if $\alpha \neq (a_1, 2, 1^{a_1-1})$ we have that $\chi^{\alpha}_{(a_1,a_1-1,1)} \neq 0$ and as

$$\chi^{(a_1,2,1^{a_1-2})}_{(a_1,a_1-1,1)} = (-1)^{a_1-2}\chi^{(a_1)}_{(a_1-1,1)} - \chi^{(1^{a_1})}_{(a_1-1,1)} = (-1)^{a_1-2} - (-1)^{a_1-2} = 0$$

we have that if $a_1 \geq 2$ then $(a_1, a_1 - 1, 1)$ is a sign partition and so the theorem is proved. □

Theorem 89. *If $h \geq 4$ and $(a_1, \ldots, a_h)$ is a sign partition we have that $a_1 \neq a_2 + \ldots + a_h$.*

Proof. First assume that h is even and that $a_1 = a_2 + \ldots + a_h$. Then by the Murnaghan-Nakayama formula we have that

$$\begin{aligned}
\chi^{(a_1,2,1^{a_1-2})}_{(a_1,\ldots,a_h)} &= -\chi^{(1^{a_1})}_{(a_2,\ldots,a_h)} + (-1)^{a_1-2}\chi^{(a_1)}_{(a_2,\ldots,a_h)} \\
&= -(-1)^{\sum_{j=2}^{h}(a_j-1)} + (-1)^{a_1} \\
&= (-1)^{a_1-h} + (-1)^{a_1} \\
&= (-1)^{a_1}2
\end{aligned}$$

and so in this case $(a_1, \ldots, a_h)$ isn't a sign partition.

So assume now that h is odd and again $a_1 = a_2 + \ldots + a_h$. Consider $(a_1, a_1 - a_{h-1} + 1, 1^{a_{h-1}-1})$. As

$$a_{h-1} = a_1 - a_2 - \ldots - a_{h-2} - a_h \leq a_1 - h + 2 \leq a_1 - 3$$

and $a_{h-1} \geq 1$, we have that $2 \leq a_1 - a_{h-1} + 1 \leq a_1$ from which follows that $(a_1, a_1 - a_{h-1} + 1, 1^{a_{h-1}-1})$ is a partition of $a_1 + \ldots + a_h = 2a_1$. Using the Murnaghan-Nakayama formula we have that

$$\chi_{(a_1,\ldots,a_h)}^{(a_1,a_{h-1}+2,1^{a_1-a_{h-1}-2})} = (-1)^{a_1-a_{h-1}-2}\chi_{(a_2,\ldots,a_h)}^{(a_1)} - \chi_{(a_2,\ldots,a_h)}^{(a_{h-1}+1,1^{a_1-a_{h-1}-1})}.$$

By theorem 84 we have that $a_j \neq a_{h-1}$ if $j < h-1$, from which follows that $\chi_{(a_2,\ldots,a_h)}^{(a_{h-1}+1,1^{a_1-a_{h-1}-1})} = (-1)^{(a_2-1)+\ldots+(a_{h-2}-1)+(a_h-1)}$ as when we are removing from $(a_{h-1}+1, 1^{a_1-a_{h-1}-1})$ a sequence of hooks of lengths $(a_2, \ldots, a_h)$ the second last hook we remove needs to be the one corresponding to the node $(1, 2)$ and all other hooks must be on the first column. So we have that

$$\chi_{(a_1,\ldots,a_h)}^{(a_1,a_{h-1}+2,1^{a_1-a_{h-1}-2})} = (-1)^{a_1-a_{h-1}} - (-1)^{a_2+\ldots+a_{h-2}+a_h+h-2} = (-1)^{a_1-a_{h-1}}2$$

as $a_2 + \ldots + a_{h-2} + a_h = a_1 - a_{h-1}$ and h is odd and so $(a_1, \ldots, a_h)$ isn't a sign partition in this case either and so the theorem is proved. $\square$

Theorem 90. *If $h \geq 4$ and $(a_1, \ldots, a_h)$ is a sign partition then we have $a_2 + \ldots + a_h \neq a_1 + 1$, unless if $(a_1, \ldots, a_h) = (3, 2, 1, 1), (5, 3, 2, 1)$.*

Proof. It can be easily checked that $(3, 2, 1, 1)$ and $(5, 3, 2, 1)$ are sign partitions, so we will only prove the other direction. This will be done dividing the proof in the following cases, which is easy to see that cover all possibilities, by theorem 84.

Case 1: h odd,

Case 2: h even and $a_h \geq 2$,

Case 3: h even, $a_{h-1}, a_h = 1$ and $a_{h-2} \geq 3$,

Case 4: h even, $a_{h-1}, a_h = 1$ and $a_{h-2} = 2$,

Case 5: h even, $a_h = 1$ and $a_{h-1} \geq 3$,

Case 6: h even, $a_h = 1$, $a_{h-1} = 2$ and $a_{h-2} \neq 3$,

Case 7: h even, $a_h = 1$, $a_{h-1} = 2$ and $a_{h-2} = 3$.

Case 1.
Assume first that h is odd and $a_2 + \ldots + a_h = a_1 + 1$. By the Murnaghan-Nakayama formula we then have that

$$\begin{aligned}
\chi_{(a_1,\ldots,a_h)}^{(a_1+1,1^{a_1})} &= \chi_{(a_2,\ldots,a_h)}^{(1^{a_1+1})} + (-1)^{a_1-1}\chi_{(a_2,\ldots,a_h)}^{(a_1+1)} \\
&= (-1)^{\sum_{j=2}^{h}(a_j-1)} + (-1)^{a_1-1} \\
&= (-1)^{a_1+1-(h-1)} + (-1)^{a_1-1} \\
&= (-1)^{a_1-1}2
\end{aligned}$$

and so $(a_1,\ldots,a_h)$ isn't a sign partition in this case.

Case 2.

So we can now assume that h is even and $a_2+\ldots+a_h=a_1+1$. At first let $a_h\geq 2$. Notice that in this case we have $a_1=a_2+\ldots+a_h-1>a_h+h-3\geq a_h+1$ as $h\geq 4$. As $a_{h-1}\neq a_h$ by theorem 84, it can be easily seen by the Murnaghan-Nakayama formula that

$$\begin{aligned}
\chi^{(a_1+1,a_h+1,1^{a_1-a_h-1})}_{(a_1,\ldots,a_h)} &= (-1)^{a_1-a_h-1}\chi^{(a_1+1)}_{(a_2,\ldots,a_h)}-\chi^{(a_h,2,1^{a_1-a_h-1})}_{(a_2,\ldots,a_h)}\\
&= (-1)^{a_1-a_h-1}-(-1)^{a_2-1+a_3-1+\ldots+a_{h-2}-1+a_{h-1}-2}\\
&= (-1)^{a_1-a_h-1}-(-1)^{a_2+a_3+\ldots+a_{h-2}+a_{h-1}-h-3-2}\\
&= (-1)^{a_1-a_h-1}+(-1)^{a_2+a_3+\ldots+a_{h-2}+a_{h-1}}\\
&= (-1)^{a_1-a_h-1}2
\end{aligned}$$

as h is even and $a_2+a_3+\ldots+a_{h-2}+a_{h-1}+a_h=a_1-1$.

Case 3.

Assume now that $a_{h-1},a_h=1$ and $a_{h-2}\geq 3$. Then

$$\begin{aligned}
\chi^{(a_1+1,a_1)}_{(a_1,\ldots,a_h)} &= \chi^{(a_1+1)}_{(a_2,\ldots,a_h)}-\chi^{(a_1-1,2)}_{(a_2,\ldots,a_h)}\\
&= 1-\chi^{(a_{h-2},2)}_{(a_{h-2},1,1)}\\
&= 1+\chi^{(1,1)}_{(1,1)}=2
\end{aligned}$$

as $a_j\geq 3$ for $j\leq h-2$ and $H^{(a_{h-2},2)}_{(1,2)}$ is the only hook of $(a_{h-2},2)$ of length a_{h-2} and has leg-length 1. So in this case we have that $(a_1,\ldots,a_h)$ isn't a sign partition.

Case 4.

Let now $a_{h-1},a_h=1$ and $a_{h-2}=2$. If $h=4$ then $a_1=3$ and so $(a_1,\ldots,a_h)=(3,2,1,1)$. If $h\geq 6$ consider $(a_1+1,a_{h-3}+1,1^{a_1-a_{h-3}-1})$. This is a partition as $h\geq 6$ (so that $h-3\neq 1$). Then we have that

$$\begin{aligned}
\chi^{(a_1+1,a_{h-3}+1,1^{a_1-a_{h-3}-1})}_{(a_1,\ldots,a_h)} &= (-1)^{a_1-a_{h-3}-1}\chi^{(a_1+1)}_{(a_2,\ldots,a_h)}-\chi^{(a_{h-3},2,1^{a_1-a_{h-3}-1})}_{(a_2,\ldots,a_h)}\\
&= (-1)^{a_1-a_{h-3}-1}-\chi^{(a_{h-3},2,1^{a_1-a_{h-3}-1})}_{(a_2,\ldots,a_h)}.
\end{aligned}$$

As $h^{(a_{h-3},2,1^{a_1-a_{h-3}-1})}_{1,2}=a_{h-3}<a_i$ for $i<h-3$ by theorem 84 and as $h^{(a_{h-3},2,1^{a_1-a_{h-3}-1})}_{3,1}=a_1-a_{h-3}-1\geq a_2+\ldots+a_{h-4}$ and $a^{(a_{h-3},2,1^{a_1-a_{h-3}-1})}_{3,1}=0$ we have that

$$\begin{aligned}
\chi^{(a_{h-3},2,1^{a_1-a_{h-3}-1})}_{(a_2,\ldots,a_h)} &= (-1)^{a_2-1+a_3-1+\ldots+a_{h-4}-1}\chi^{(a_{h-3},2,1,1)}_{(a_{h-3},2,1,1)}\\
&= (-1)^{a_2+a_3+\ldots+a_{h-4}-h+5}\chi^{(a_{h-3},2,1,1)}_{(a_{h-3},2,1,1)}\\
&= (-1)^{a_1-a_{h-3}-5-h+5}\chi^{(a_{h-3},2,1,1)}_{(a_{h-3},2,1,1)}\\
&= (-1)^{a_1-a_{h-3}}\chi^{(a_{h-3},2,1,1)}_{(a_{h-3},2,1,1)}
\end{aligned}$$

as h is even and $a_{h-2}+a_{h-1}+a_h=4$. By theorem 84 we have that $a_{h-3}\geq 3$. If $a_{h-3}=3$ then $\chi^{(3,2,1,1)}_{(3,2,1,1)}=1$ and if $a_{h-3}=4$ then $\chi^{(4,2,1,1)}_{(4,2,1,1)}=2$. If $a_{h-3}\geq 5$ we have that $h^{(a_{h-3},2,1,1)}_{2,1}=4<a_{h-3}$ and then we have that $(a_{h-3},2,1,1)$ has a unique hook of length a_{h-3} and so $\chi^{(a_{h-3},2,1,1)}_{(a_{h-3},2,1,1)}=-\chi^{(1^4)}_{(2,1,1)}=1$. So in any case $\chi^{(a_{h-3},2,1,1)}_{(a_{h-3},2,1,1)}\geq 1$, from which we have that

$$\chi^{(a_1+1,a_{h-3}+1,1^{a_1-a_{h-3}-1})}_{(a_1,\ldots,a_h)}=(-1)^{a_1-a_{h-3}-1}(1+\chi^{(a_{h-3},2,1,1)}_{(a_{h-3},2,1,1)})\neq 0,\pm 1$$

and so $(a_1,\ldots,a_{h-3},2,1,1)$ isn't a sign partition when $h\geq 6$.

Case 5.

Now assume that $a_h=1$ and $a_{h-1}\geq 3$. Then

$$\begin{aligned}\chi^{(a_1+1,a_1)}_{(a_1,\ldots,a_h)} &= \chi^{(a_1+1)}_{(a_2,\ldots,a_h)}-\chi^{(a_1-1,2)}_{(a_2,\ldots,a_h)}\\ &= 1-\chi^{(a_{h-1}-1,2)}_{(a_{h-1},1)}\\ &= 1+\chi^{(1)}_{(1)}=2\end{aligned}$$

and so $(a_1,\ldots,a_h)$ is not a sign partition.

Case 6.

Let now $a_h=1$, $a_{h-1}=2$ and $a_{h-2}\neq 3$. By theorem 84 if $(a_1,\ldots,a_h)$ is a sign partition we then need to have that $a_{h-2}>3$. As all hooks of $(a_1+1,a_1-1,1)$ which are on the second or third row have length either 3 or 1 and as $a_1>a_h+a_{h-1}+1=1+2+1=4$, we have that

$$\begin{aligned}\chi^{(a_1+1,a_1-1,1)}_{(a_1,\ldots,a_2)} &= -\chi^{(a_1+1)}_{(a_2,\ldots,a_h)}-\chi^{(a_1-2,2,1)}_{(a_2,\ldots,a_h)}\\ &= -1-\chi^{(a_{h-3},2,1)}_{(a_{h-3},2,1)}\\ &= -1+\chi^{(1,1,1)}_{(2,1)}=-2\end{aligned}$$

and so also in this case $(a_1,\ldots,a_h)$ isn't a sign partition.

Case 7.

At last let $a_h=1$, $a_{h-1}=2$ and $a_{h-2}=3$. If $h=4$ we need to have that $a_1=5$ and so $(a_1,\ldots,a_h)=(5,3,2,1)$ in this case. If $h\geq 6$ is even consider $(a_1+1,a_{h-3}+1,1^{a_1-a_{h-3}-1})$. Again this is a partition as $h\geq 6$, so that $h-3\neq 1$. By the same calculation as in the case where $a_{h-1},a_h=1$ and $a_{h-2}=2$ we have that

$$\chi^{(a_1+1,a_{h-3}+1,1^{a_1-a_{h-3}-1})}_{(a_1,\ldots,a_h)}=(-1)^{a_1-a_{h-3}-1}-\chi^{(a_{h-3},2,1^{a_1-a_{h-3}-1})}_{(a_2,\ldots,a_h)}$$

and similarly again to the case where $a_{h-1},a_h=1$ and $a_{h-2}=2$ we have that

$$\chi^{(a_{h-3},2,1^{a_1-a_{h-3}-1})}_{(a_2,\ldots,a_h)}=(-1)^{a_1-a_{h-3}}\chi^{(a_{h-3},2,1^4)}_{(a_{h-3},3,2,1)}$$

(now as $a_{h-2}+a_{h-1}+a_h=6$). Here we need to have that $a_{h-3}\geq 4$ by theorem 84. If $a_{h-3}=4$ then $\chi^{(4,3,2,1^4)}_{(4,3,2,1)}=1$, if $a_{h-3}=5$ then $\chi^{(5,3,2,1^4)}_{(5,3,2,1)}=1$ and if $a_{h-3}=6$ then $\chi^{(6,3,2,1^4)}_{(6,3,2,1)}=2$. So let now $a_{h-3}\geq 7$. As $h^{(a_{h-3},2,1^4)}_{3,1}=6<a_{h-3}$ we have that $\chi^{(a_{h-3},2,1^4)}_{(a_{h-3},3,2,1)}=-\chi^{(1^6)}_{(3,2,1)}=1$ and so we have that $\chi^{(a_{h-3},2,1^4)}_{(a_{h-3},3,2,1)}$ is always ≥ 1, from which we have that

$$\chi^{(a_1+1,a_{h-3}+1,1^{a_1-a_{h-3}-1})}_{(a_1,\ldots,a_h)}=(-1)^{a_1-a_{h-3}-1}(1+\chi^{(a_{h-3},2,1^4)}_{(a_{h-3},3,2,1)})\neq 0,\pm 1$$

and so also in this case $(a_1,\ldots,a_h)$ isn't a sign partition and we then have that the theorem is proved. □

Theorem 91. *If $(a_1,\ldots,a_h)$ is a sign partition with $a_2+\ldots+a_h>a_1$ and $a_3+\ldots+a_h\leq a_1$, then $a_h\leq a_1-a_2$.*

Proof. Assume that $a_2+\ldots+a_h>a_1$, $a_3+\ldots+a_h\leq a_1$ and $a_h>a_1-a_2$. Then we have that

$$\begin{aligned}\chi^{(a_2+\ldots+a_h,1^{a_1})}_{(a_1,\ldots,a_h)} &= (-1)^{a_1-1}\chi^{(a_2,\ldots,a_h)}_{(a_2,\ldots,a_h)}+\chi^{(a_2+\ldots+a_h-a_1,1^{a_1})}_{(a_2,\ldots,a_h)}\\ &= (-1)^{a_1-1}+(-1)^{a_2-1}\chi^{(a_2+\ldots+a_h-a_1,1^{a_1-a_2})}_{(a_3,\ldots,a_h)}\\ &= (-1)^{a_1-1}+(-1)^{a_2-1}\chi^{(a_h-a_1+a_2,1^{a_1-a_2})}_{(a_h)}\\ &= (-1)^{a_1-1}+(-1)^{a_2-1}(-1)^{a_1-a_2}=(-1)^{a_1-1}2,\end{aligned}$$

where the first equality follows from the fact that $a_2+\ldots+a_h>a_1$, the second equality follows from the fact that $a_2+\ldots+a_h-a_1-1<a_2$ as $a_3+\ldots+a_h\leq a_1$ and the third equality from the fact that $a_1-a_2<a_h\leq a_j$ for $3\leq j\leq h$.

So as $\chi^{(a_2+\ldots+a_h,1^{a_1})}_{(a_1,\ldots,a_h)}=(-1)^{a_1-1}2\neq 0,\pm 1$, we have that if $a_2+\ldots+a_h>a_1$, $a_3+\ldots+a_h\leq a_1$ and $a_h>a_1-a_2$ then $(a_1,\ldots,a_h)$ isn't a sign partition and so the theorem is proved. □

Lemma 92. *If $(a_3,\ldots,a_h)$ is a sign partition with*

$$1\leq a_3+\ldots+a_h<a_1-1$$

we have that if $\chi^{\alpha}_{(a_1,a_1-1,a_3,\ldots,a_h)}\neq 0,\pm 1$ then any β-set for α is of the form $X=\{y_1,\ldots,y_{k-2},y_{k-1}+a_1,y_k+a_1\}$, where $Y=\{y_1,\ldots,y_k\}$ is a β-set for a partition of $a_3+\ldots+a_h-1$ and we have that $y_{k-1}+a_1,y_k+a_1\notin Y$ and $y_{k-1}+1\notin Y$ or $y_{k-1}+a_1-1\in Y$ and similarly $y_k+1\notin Y$ or $y_k+a_1-1\in Y$.

Proof. As by theorem 86 we have that $(a_1-1,a_3,\ldots,a_h)$ is a sign partition and as

$$a_1+(a_1-1)+a_3+\ldots+a_h<3a_1,$$

so that for any $\alpha \vdash a_1 + (a_1 - 1) + a_3 + \ldots + a_h$ contains at most 2 hooks of length a_1, we easily get that if α contains at most 1 hook of length a_1 then $\chi^\alpha_{(a_1,a_1-1,a_3,\ldots,a_h)} = 0, \pm 1$. Now again as $a_1 + (a_1 - 1) + a_3 + \ldots + a_h < 3a_1$, we have that α contains 2 hooks of length a_1 if and only if the a_1-quotient of α is $\alpha^{(a_1)} = (\alpha_0, \ldots, \alpha_{a_1-1})$ with two of the α_i equal to (1) and all others equal to (0). From this as

$$a_1 + (a_1 - 1) + a_3 + \ldots + a_h - 2a_1 = a_3 + \ldots + a_h - 1 \geq 0$$

we easily get that if $\chi^\alpha_{(a_1,a_1-1,a_3,\ldots,a_h)} \neq 0, \pm 1$ we can write any β-set of α as $X = \{y_1, \ldots, y_{k-2}, y_{k-1} + a_1, y_k + a_1\}$, with $Y = \{y_1, \ldots, y_k\}$ is a β-set for a partition of $a_3 + \ldots + a_h - 1$ and $y_{k-1} + a_1, y_k + a_1 \notin Y$.

We will show that if $y_k + 1 \in Y$ and $y_k + a_1 - 1 \notin Y$ then we need to have that $\chi^\alpha_{(a_1,a_1-1,a_3,\ldots,a_h)} = 0, \pm 1$. In order to shorten notation if Z is a β-set let $\chi^Z = \chi^{P^*(Z)}$. Assume that $y_k + 1 \in Y$ and $y_k + a_1 - 1 \notin Y$. Then we have that

$$\chi^\alpha_{(a_1,a_1-1,a_3,\ldots,a_h)} = \pm\chi^{\{y_1,\ldots,y_{k-1},y_k+a_1\}}_{(a_1-1,a_3,\ldots,a_h)} + \pm\chi^{\{y_1,\ldots,y_{k-2},y_{k-1}+a_1,y_k\}}_{(a_1-1,a_3,\ldots,a_h)}.$$

If we can prove that $\chi^{\{y_1,\ldots,y_{k-2},y_{k-1}+a_1,y_k\}} = 0$ then we would have that $\chi^\alpha_{(a_1,a_1-1,a_3,\ldots,a_h)} = 0, \pm 1$ in this case, as $(a_1 - 1, a_3, \ldots, a_h)$ is a sign partition. By the Murnaghan-Nakayama formula, to show that $\chi^{\{y_1,\ldots,y_{k-1},y_k+a_1\}} = 0$ it is enough to show that $\{y_1, \ldots, y_{k-1}, y_k + a_1\}$ doesn't have any hook of length $a_1 - 1$. As $y_k + a_1 - (a_1 - 1) = y_k + 1 \in Y$, so that it must also be in $\{y_1, \ldots, y_{k-1}, y_k + a_1\}$, if $\{y_1, \ldots, y_{k-1}, y_k + a_1\}$ has any hook of length $a_1 - 1$ we need to have that $y_j - a_1 + 1 \notin \{y_1, \ldots, y_{k-1}, y_k + a_1\}$ and $y_j - a_1 + 1 \geq 0$ for some $1 \leq j \leq k-1$ by theorem 45. If we would also have that $y_j - a_1 + 1 \neq y_k$, we would then have that Y has also an hook of length $a_1 - 1$, which gives a contradiction as Y is a β-set for a partition of $a_3 + \ldots + a_h - 1 < a_1 - 1$. So we must have that $y_j - a_1 + 1 = y_k$, which gives a contradiction with the fact that we are assuming that $y_k + a_1 - 1 \notin Y$. So when $y_k + 1 \in Y$ and $y_k + a_1 - 1 \notin Y$ then $\{y_1, \ldots, y_{k-1}, y_k + a_1\}$ doesn't have any hook of length $a_1 - 1$ and so we have that in this case $\chi^\alpha_{(a_1,a_1-1,a_3,\ldots,a_h)} = 0, \pm 1$.

Using the symmetry between y_{k-1} and y_k we then also have that if $\chi^\alpha_{(a_1,a_1-1,a_3,\ldots,a_h)} \neq 0, \pm 1$ then we also need to have that $y_{k-1} + 1 \notin Y$ or $y_{k-1} + a_1 - 1 \in Y$ and so the theorem is proved. $\square$

Theorem 93. *If $a \geq 4$ then $(a, a - 1, 2, 1)$ is a sign partition.*

Proof. The case where $a = 4$ can be checked by showing that $\chi^\alpha_{(4,3,2,1)} = 0, \pm 1$ for any $\alpha \vdash 10$.

Assume now that $a \geq 5$. Here we can apply lemma 92 as $(2,1)$ is a sign partition. Using the lemma it is enough to show that $\chi^{\alpha}_{(a,a-1,2,1)} = 0, \pm 1$ when α has β-set $X = \{y_1, \ldots, y_{k-2}, y_{k-1} + a, y_k + a\}$ for some $Y = \{y_1, \ldots, y_k\}$ β-set of a partition of 2, with $y_{k-1} + a, y_k + a \notin Y$ and $y_{k-1} + 1 \notin Y$ or $y_{k-1} + a - 1 \in Y$ and $y_k + 1 \notin Y$ or $y_k + a - 1 \in Y$. As the only partitions of 2 are (2) and $(1,1)$, we can assume that Y is equal to $\{0, \ldots, 2a-2, 2a+1\}$ or $\{0, \ldots, 2a-3, 2a-1, 2a\}$ (we need to have that $0, 1, \ldots, a-1 \in Y$). In the first case we have that the $y \in Y$ such that $y + 1 \notin Y$ or $y + a_1 - 1 \in Y$ are $a-1, a+2, 2a-2, 2a+1$, while in the second case those y are $a-2, a+1, 2a-3, 2a$. So for each of these two possibilities for Y we have $6 = \binom{4}{2}$ possibilities for X and so we need to check that $\chi^{X}_{(a,a-1,2,1)} = 0, \pm 1$ for 12 possible X. From each of these X we can remove an a-hook in two different ways and after having removed an a-hook we can always remove a hook of length $a-1$ in a unique way. Using the Murnaghan-Nakayama formula and theorem 45 we have that

$$\begin{aligned}
\chi^{\{0,\ldots,a-2,a,a+1,a+3,\ldots,2a-1,2a+1,2a+2\}}_{(a,a-1,2,1)} &= (-1)^{a-2}\chi^{\{0,\ldots,a-2,a\ldots,2a-1,2a+1\}}_{(a-1,2,1)} \\
&\quad +(-1)^{a-2}\chi^{\{0,\ldots,a+1,a+3,\ldots,2a-2,2a+1,2a+2\}}_{(a-1,2,1)} \\
&= (-1)^{a-2}(-1)^{a-2}\chi^{\{0,\ldots,2a-3,2a-1,2a+1\}}_{(2,1)} \\
&\quad +(-1)^{a-2}(-1)^{a-4}\chi^{\{0,\ldots,2a-2,2a+2\}}_{(2,1)} \\
&= \chi^{(2,1)}_{(2,1)} + \chi^{(3)}_{(2,1)} = 1
\end{aligned}$$

and similarly it can be proved that also in all the other choices of X that we need to consider we have that $\chi^{X}_{(a,a-1,2,1)} = 0, \pm 1$ and so we have that also when $a \geq 5$, $(a, 1-1, 2, 1)$ is a sign partition and then the theorem is proved. □

Even if in the proof of the theorem we considered both partitions obtained by adding 2 a-hooks to (2) and to $(1,1)$, it is enough to consider only those obtained by adding 2 a-hooks to (2), as the others are their conjugates and $\chi^{\alpha'}(\pi) = \mathrm{sign}(\pi)\chi^{\alpha}(\pi)$.

Theorem 94. *If $a_1 \geq 5$ then $(a_1, a_1 - 1, 3, 1)$ is a sign partition.*

Proof. The case where $a_1 = 5$ can be checked by calculating $\chi^{\alpha}_{(5,4,3,1)}$ for any $\alpha \vdash 13$.

Assume now that $a_1 \geq 6$. In this case we can apply lemma 92 as $(3,1)$ is a sign partition by theorem 86. The only partitions of 3 are (3), $(2,1)$ and $(1,1,1)$ and β-sets for them are $\{0, 1, \ldots, 2a-2, 2a+2\}$, $\{0, 1, \ldots, 2a-3, 2a-1, 2a+1\}$ and $\{0, 1, \ldots, 2a-4, 2a-2, 2a-1, 2a\}$ respectively. If Y is one of these β-sets then those $y \in Y$ such that $y+1 \notin Y$ or $y+a-1 \in Y$ are

$a-1, a+3, 2a-2, 2a+2$ in the first case, $a-2, a, a+2, 2a-3, 2a-1, 2a+1$ in the second case and $a-3, a+1, 2a-4, 2a$ in the last case. So by lemma 92 we only need to check that $\chi^X_{(a,a-1,3,1)} = 0, \pm 1$ when X is obtained by Y by increasing by a two of the given elements of Y. So we only need to consider $\binom{4}{2} + \binom{6}{2} + \binom{4}{2} = 27$ partitions of $2a+3$. We will now show one of these cases

$$\begin{aligned}
\chi^{\{0,\ldots,a-2,a\ldots,2a-1,3a+2\}}_{(a,a-1,3,1)} &= \chi^{\{0,\ldots,a-2,a\ldots,2a-1,2a+2\}}_{(a-1,3,1)} \\
&\quad +(-1)^{a-1}\chi^{\{0,\ldots,2a-2,3a+2\}}_{(a-1,3,1)} \\
&= (-1)^{a-2}\chi^{\{0,\ldots,2a-3,2a-1,2a+2\}}_{(3,1)} \\
&\quad +(-1)^{a-1}\chi^{\{0,\ldots,2a-2,2a+3\}}_{(3,1)} \\
&= \chi^{(3,1)}_{(3,1)} + (-1)^{a-1}\chi^{(4)}_{(3,1)} = (-1)^{a-1}.
\end{aligned}$$

As the other cases can be checked similarly and in all of them we have that $\chi^X_{(a,a-1,3,1)} = 0, \pm 1$, we have that $(a, a-1, 3, 1)$ is a sign partition. □

Also in this theorem like in the previous one we could consider less partitions, as the partitions obtained by adding two a-hooks to $(1, 1, 1)$ are the conjugates of those obtained by adding two a-hooks to (3).

We will finish by stating a conjecture by Olsson (the conjecture at the end of [6]) which would allow by theorem 86 to completely classify sign partitions of n and so also sign conjugacy classes of S_n.

Conjecture 95. *If $(a_1, \ldots, a_h)$ is a sign partition and $a_1 \leq a_2 + \ldots + a_h$ then $(a_1, \ldots, a_h)$ is one of the following:*

- $(1, 1)$, $(3, 2, 1, 1)$ *or* $(5, 3, 2, 1)$,
- $(a, a-1, 1)$ *for some* $a \geq 2$,
- $(a, a-1, 2, 1)$ *for some* $a \geq 4$,
- $(a, a-1, 3, 1)$ *for some* $a \geq 5$.

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