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Multi-Letterplace Ring, Multi-Gradings and Applications

Multi-Letterplace-Ringe, Multi-Graduierungen und Anwendungen

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Eidesstattliche Erklärung

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Aachen, den 11. Juli 2013

(Bastian Haase)

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Contents

§ 1 Introduction

It is the goal of this thesis to extend the knowledge of the (Multi-)Letterplace ring. In [LSL09], the authors used the Letterplace ring to embed the free associative algebra $K\langle X\rangle$ into a Letterplace ring. By use of this embedding, it is possible to compute Gröbner bases in a non-commutative algebra with commutative methods. The authors continued their work in [LSL13], when they achieved more general results and analyzed the results obtained in [LSL09] from a more abstract point of view. This thesis now generalizes many results from this paper, so that they can not only be applied to the Letterplace ring, but also to the Multi-Letterplace ring. Although this thesis consists of six sections, it can be roughly divided into four parts. This first section is an introduction of the basics which are needed for the upcoming sections. After that, sections two and three capture the ideas of [LSL13] and generalize them. These sections yield many useful results for a special class of ideals. Hence, section four and five show two new approaches to make these results applicable to a wider class of ideals. In the last section, a brief application of the theory is given in the context of difference equations.
In this first section, we will introduce the basic knowledge we need in the following sections. We start with the basics of monomials orders, Gröbner bases and the Buchberger criterion. This section is mainly based on [Eis95]. After that, we introduce graded algebras and modules with some interesting propositions. This part mostly relies on [Lan02]. In the last subsection the Multi-Letterplace ring will be defined and a short introduction is given.
In the second section, we take a closer look at the skew monoid ring $S = \Sigma * P$, whereby P is a commutative polynomial ring and Σ is a submonoid of $End(P)$. We will illustrate the concepts by choosing P to be a Multi-Letterplace ring, since this ring is our primary interest. We will present a new criterion for the Buchberger algorithm for graded ideals in S by using our knowledge of Σ. This result requires some thoughts and ideas about suitable orderings on S first. The ideas and the approach in this section rely on [LSL13]. In fact, it is a generalization of the results obtained in section 4 and 5 of this paper. Hence, many proofs were also only slightly changed to fit the new setting.
The third section transfers the results of the second section to a special class of ideals in P. For this purpose, we define a new type of basis called Σ-basis. Once again, we obtain a useful adaption of the Buchberger criterion. We also present an interesting embedding from P to S by introducing the concept of multi-weight functions. We will show that an ideal I in P, which is w-graded, corresponds to a multi-graded ideal in S. By use of this embedding, we are in fact able to transfer homogeneous Gröbner bases in S onto Gröbner Σ-bases in P and vice versa. This section is based on section five of [LSL13] and, similar to the last section, can be seen as a generalization of the results obtained

there. The biggest change is the transition from weight functions to multi-weight functions which ultimately yields multi-graded ideals in S.
After section three, we only focus on the Multi-Letterplace ring. Sections three and four present entirely new results and focus on determining and extending the applicability of the theory presented in sections two and three. Since the results of section three demand the ideal in P to be w-graded, it is interesting to find out which ideals are w-graded. The fourth section mainly deals with this question. We will characterize when a multi-weight function exists such that an element f is w-homogeneous. If we find such a function, the ideal with Σ-basis $\{f\}$ is w-graded. Based on this characterization, we present an algorithm which constructs a suitable multi-weight function for given f. If there is no such function, the algorithm will return a negative result. At the end of this section, a proposition yields a method to determine wether there is a multi-weight function suitable for two elements. This proposition is also proven constructively and yields a similar result for n elements.
The fourth section revealed that there are ideals which are not w-graded for any multi-weight function w. Hence, the fifth section proposes a homogenization of such ideals to solve this problem. The first main result shows that the ideal membership problem can be transferred into the w-graded world whenever the Σ-basis is finite. Since one important advantage of Σ-bases is the fact that it is very often finite even if a normal basis is infinite, this assumption is not too restrictive. The other important result allows us to compute Gröbner bases of the homogenized ideal and project them onto Gröbner bases of the original ideal. For this purpose, the original ideal may even have an infinite Σ-basis.
The last section shows a short application of the Multi-Letterplace ring. We describe how difference equations can be embedded and take a closer look at systems of linear difference equations.
Note that Roberto La Scala's paper [LS13] shows some intersections with this thesis. He focuses on partial difference equations, thus the application of the theory is more on the spot. However, he also uses the Multi-Letterplace ring (although he does not use this name and its appearance is sometimes only implicit) and he also suggests a homogenization. Nonetheless, his approach is not entirely the same. Instead of a multi-weight function he defines a weight-function and an order-function. This approach results in two main differences. First, the weight-function and the order-function are unique, so the process of constructing weight-functions for certain ideals does not arise. In addition, his weight- and order-function result in a grading and not in a multi-grading. This basically relies on the decision to map the weight on $\mathbb{N}$, while we will map it on $\mathbb{N}^m$ in this thesis. Hence, many results in this field are very different. Furthermore, there are also huge differences in the process of homogenization. His basic approach is closer to

the classical homogenization, so both yield different but interesting results.

1.1 Monomial Orders and Gröbner Bases

In this subsection, we will introduce the theory of Gröbner bases we will need in the following sections. It mainly bases on the chapter of Gröbner bases in [Eis95]. Throughout this chapter, $X = \{x_i \mid i \in I\}$ will denote the set of variables for a finite or countably infinite set I. Without loss of generality, we can assume that $I \subset \mathbb{N}$ holds and, therefore, I is ordered. Furthermore, $P = k[X]$ denotes the commutative polynomial ring for an arbitrary field k.
Throughout this thesis, $\mathbb{N}$ will denote the set of the non-negative integers $\{0, 1, 2, ...\}$.

1.1.1 Monomial Orders

In the theory of Gröbner bases, monomial orders play an important role. While monomials in a polynomial ring of one variable are ordered naturally, the definition of a suitable order on monomials in more variables is very important. The set of all monomials in P will be denoted as $mon(P)$. If m is a monomial, we find $(\mu_i)_{i\in I}$ with $\mu_i \in \mathbb{N}$ such that $m = \prod_{i\in I} x_i^{\mu_i}$ holds. It is important to note that only finitely many μ_i are not equal to 0. The total degree of m is defined as $tdeg(m) = \sum_{i\in I} \mu_i$.

We will now precisely define when we call an order on a set M *total*.

(1.1) Definition
Let M be a set and $<$ an order on M. We say that $<$ is a total order if

1. $m \leq m$ (reflexivity)
2. $m \leq n \ \wedge \ n \leq m \ \Rightarrow \ m = n$ (antisymmetry)
3. $m \leq n \ \wedge \ n \leq p \ \Rightarrow \ m \leq p$ (transitivity)
4. $m \leq n$ or $n \leq m$ (totality)

for all $m, n, p \in M$.

(1.2) Example
Fix $M = mon(P)$ and let m, n be two arbitrary elements of $mon(P)$. There are $(\mu_i)_{i\in I}, (\lambda_i)_{i\in I} \subset \mathbb{N}^{(I)}$ such that both $m = \prod_{i\in I} x_i^{\mu_i}$ and $n = \prod_{i\in I} x_i^{\lambda_i}$ hold. We put

$$\begin{aligned} m <_{lex} n \ \Leftrightarrow \ & \exists j \in I : \ \mu_j < \lambda_j \\ & \text{and } \mu_i = \lambda_i \text{ for all } i < j \end{aligned}$$

and call this order the *lexicographical order*. It is in fact a total order on $mon(P)$.

(1.3) Example
Consider $M = mon(P)$ again. By combining the lexicographical order with the total degree of the monomial we obtain the *graded lexicographical order*. It is defined precisely via

$$m <_{glex} n \;\Leftrightarrow\; tdeg(m) < tdeg(n) \text{ or } \left(tdeg(m) = tdeg(n) \text{ and } m <_{lex} n\right).$$

It is easy to see that this order is also a total order.

(1.4) Remark
These two orders correspond to two total orders on $\mathbb{N}^m$. For $l, n \in \mathbb{N}^m$, you can define two total orderings via

$$\begin{aligned} l <_{lex} n \;\;\Leftrightarrow\;\; & \exists j \in \{1, .., m\}\; l_j < n_j \\ & \text{and } l_i = n_i \text{ for all } i < j \end{aligned}$$

and

$$l <_{glex} n \;\Leftrightarrow\; \sum_i l_i < \sum_i n_i \text{ or } \left(\sum_i l_i = \sum_i n_i \text{ and } l <_{lex} n\right).$$

We will also call these orders the lexicographical order and the graded lexicographical order. It will always be clear from context which one is actually meant.

For the purpose of Gröbner basis computations, we are often interested in the termination of the corresponding algorithms. Therefore, we need to make sure that every nonempty subset of monomials has a least element.

(1.5) Definition
Let $<$ be a total ordering on a set M. We say that $<$ is a well-order if every nonempty subset of M has a least element with respect to $<$.

(1.6) Remark
Note that both the lexicographical and the graded lexicographical order suffice this condition.

We will also require a compatibility with the multiplication in P.

(1.7) Definition
Let $<$ be a total ordering on a set $mon(P)$. We say that $<$ is compatible with multiplication if

1. $f < g \Leftrightarrow fm < gm$
2. $f < fp$

for all $f, g, m, p \in mon(P)$ with $p \neq 1$.

(1.8) Remark
Considering their definition, it is easy to check that $<_{lex}$ and $<_{glex}$ are compatible with multiplication.

Since we are interested in orderings which satisfy all these conditions, the next definition summarizes them.

(1.9) Definition
Let $<$ be an order on $mon(P)$. If $<$ is a total well-order which is compatible with multiplication, we call $<$ a monomial order.

We will now define the concepts of gcd and lcm on monomials.

(1.10) Definition
Let m, n be two arbitrary elements of $mon(P)$. There are $(\mu_i)_{i\in I}, (\lambda_i)_{i\in I} \subset \mathbb{N}^{(I)}$ such that both $m = \prod_{i\in I} x_i^{\mu_i}$ and $n = \prod_{i\in I} x_i^{\lambda_i}$ hold. We define the *greatest common divisor* and the *least common multiple* as

$$gcd(m,n) = \prod_{i\in I} x^{\min\{\mu_i,\lambda_i\}}$$
$$lcm(m,n) = \prod_{i\in I} x^{\max\{\mu_i,\lambda_i\}}$$

1.1.2 Gröbner Bases of Ideals

We will now briefly define the basics of Gröbner bases of ideals and state the most important propositions. Fix any $f = \sum_{i=1}^{n} a_i m_i \in P$ with $a_i \in k \setminus \{0\}$ and $m_i \in mon(P)$. From now on, we assume that P is endowed with a monomial order $<$. Thus, choose $g \in I$ such that $m_g = \max_i\{m_i\}$ is holds. In addition, let G be a subset of P. We will use the following abbreviations:

Symbol	Notation
$lm(f)$	m_g
$lc(f)$	a_g
$lt(f)$	$lc(f) \cdot lm(f)$
$lm(G)$	$\{lm(f) \mid f \in G \setminus \{0\}\}$
$LM(G)$	ideal generated by $lm(G)$

In the polynomial ring $k[x]$ and in general in any Euclidean ring, there is a Euclidean division. However, since $k[X]$ is not even a principal ideal domain, we can not expect the existence of such a division in $k[X]$. The following theorem is a very popular solution to this problem.

(1.11) Proposition
For any $G \subset P$ and $f \in P$ there is an expression

$$f = \sum_{i=1}^{n} h_i g_i + f',\ h_i, f' \in P, g_i \in G$$

with $lm(f) \geq lm(h_i g_i)$. In addition, no monomial of f' is contained in $LM(G)$. Any such f' is called a remainder of f with respect to G.

Proof
cf. [Eis95]
The proof in [Eis95] demands G to be finite, but we will explain in the next subsection, when we present the division algorithm for modules, that this assumption may be omitted. □

(1.12) Remark
Note that the remainder of f is neither unique nor unique up to units.

We are now able to define Gröbner bases.

(1.13) Definition
Let I be an arbitrary ideal of P. Then $G \subset P$ is a Gröbner basis of I if $LM(G) = LM(I)$.

(1.14) Remark
Any Gröbner basis of an ideal I is also an ordinary basis of I.

If G is a basis of I, there is a very useful characterization of Gröbner bases we will use in this thesis. The following definition is important for this characterization.

(1.15) Definition
Let f, g be two elements of P with $m = lm(f), n = lm(g)$ and $l = lcm(m, n)$. Then, the s-polynomial is defined as

$$s(f,g) = lc(g)\frac{l}{m}f - lc(f)\frac{l}{n}g.$$

We will now state the famous Buchberger Criterion for ideals.

(1.16) Proposition (Buchberger criterion)
Let G be a basis of I. Then, G is a Gröbner basis of I if and only if

$$\forall\, f, g \in G: \text{ a remainder of } s(f,g) \text{ with respect to } G \text{ is zero.}$$

Proof
cf. [Eis95]
The original proof was only given for a finite subset G. Anyway, this assumption is not needed, which will be explained in more detail in the next paragraph when we cite the Buchberger criterion for modules. □

1.1.3 Gröbner Bases of Modules

We will extend the theory presented in the last subsection for submodules of free modules. Hence, we will assume that F is a free P-module with basis $\{e_j\}_{j\in J}$ for a finite or countably infinite set J.

First, we need some adjusted definitions of known concepts. Note that these definitions often depend on the choice of the basis.

(1.17) Definition
The set of monomials of F is defined as

$$mon(F) = \{me_j \mid m \in mon(P),\ j \in J\}.$$

If $f \in mon(F)$ and $s \in k \setminus \{0\}$, we call sf a *term*.

The definition of monomials allows us to define the term *monomial order* for modules.

(1.18) Definition
A total well-order $<$ on $mon(F)$ is called monomial if for any $m_1, m_2 \in mon(F)$ and $n \in mon(P) \setminus \{1\}$

$$m_1 < m_2 \;\Rightarrow\; m_1 < nm_1 < nm_2$$

holds.

(1.19) Example
Recall that we can assume that $J \subset \mathbb{N}$ holds and, thus, we can use the natural order of $\mathbb{N}$ on J. Let $\prec$ be a monomial order on $mon(P)$. For two monomials me_i and ne_j we define a monomial order via

$$me_i < ne_j \;\Leftrightarrow\; i < j \text{ or } (i = j \text{ and } m \prec n)\,.$$

From now on, we assume that F is endowed with a monomial order. The next definition introduces divisibility in $mon(F)$.

(1.20) Definition
Let $\widetilde{m} = sme_i$ and $\widetilde{n} = rne_j$ be two terms of F. We say that $\widetilde{m}$ divides $\widetilde{n}$ if $i = j$ and there are $a \in k$ and $p \in mon(P)$ such that $smap = rn$ holds. We write $\frac{\widetilde{n}}{\widetilde{m}} = ape_i$. Furthermore, we define

$$gcd(me_i, ne_j) = \delta_{ij} gcd(m, n)e_i,$$
$$lcm(me_i, ne_j) = \delta_{ij} lcm(m, n)e_i.$$

Note that the definitions of the leading term, the leading monomial and the leading coefficient in P can be naturally transferred to F by use of the definitions of this subsection. Hence, also Gröbner bases and s-polynomials are defined analogously. Thus, we can restate the division with remainder in the case of modules by only making a slight adaption.

(1.21) Proposition
For any $G \subset F$ and $f \in F$ there is an expression

$$f = \sum_{i=1}^{n} h_i g_i + f', \; h_i \in P, g_i \in G, f' \in F$$

with $lm(f) \geq lm(h_i g_i)$. In addition, no monomial of f' is contained in $LM(G)$. Any such f' is called a remainder of f with respect to G.

Proof
cf. [Eis95].
The proof is given for a finitely generated module F and a finite set G. In fact, such an assumption is not needed, since the order is monomial.
We put $f_1 := f$. If $lm(f_1) \in LM(G)$, we find $g_{i_1} \in G$ such that $lm(g_{i_1})$ divides $lm(f)$. Hence, we define $f_2 := f_1 - \frac{lt(f_1)}{lt(g_{i_1})} g_{i_1}$ and remark that $lm(f_1) > lm(f_2)$ holds. We iterate this process as long as $f_i \neq 0$ and $lm(f_i) \in LM(G)$ hold. This process terminates, because the elements f_i induce a strictly descending chain

$$lm(f_1) > lm(f_2) > ...$$

which has to be finite since $<$ is a well-ordering.
Thus, we obtain $f = \sum_{i=1}^{n} h_i g_i + \widehat{f}$ with $g_i \in G$, $h_i \in P$ and $lm(f) \geq lm(h_i g_i)$. We also

know that $lm(\widehat{f})$ is not contained in $LM(G)$. However, there might be a monomial of $\widehat{f}$ contained in $LM(G)$. In this case, we continue the reduction process. Assume $\widehat{f} = \sum_i a_i m_i$ with $a_i \in k \setminus \{0\}$ and $m_i \in mon(P)$. Without loss of generality, we may assume that $m_i > m_{i+1}$ holds. Denote j the minimal index such that $m_j \in LM(G)$ holds. We continue similarly to the first reduction process and obtain $\widehat{f_2}$. Note that this reduction does not affect the terms of $\widehat{f}$ whose monomial is greater than m_j. After this reduction, there might still be monomials of $\widehat{f_2}$ in $LM(G)$. If this is the case, we apply the process to the greatest of these monomials. The assumption that there are infinitely many such monomials occuring in this process yields again an infinitely strictly descending chain of monomials which contradicts the fact that the order is a well-order. Hence, we obtain termination and the last $\widehat{f_n}$ is equal to f'. □

We will now state the famous Buchberger Criterion for modules.

(1.22) Proposition (Buchberger criterion)
Let G be a basis of $M \subseteq F$. Then, G is a Gröbner basis of M if and only if

$$\forall\, f, g \in G: \text{ a remainder of } s(f,g) \text{ with respect to } G \text{ is zero.}$$

Proof
cf. [Eis95].
Note that the proof is given when F is finitely generated and G is also finite. But, the proof still holds true without these assumptions, since the monomial order allows us to deduce the infinite case from the finite case. We have already seen that the division algorithm, which is crucial for this proof, still holds true and the rest of the proof does not make use of the additional assumptions. □

1.2 Graded Algebras and Modules

In this subsection, we will recall some basic definitions and results concerning the grading of algebras and modules. Note that any ring mentioned in this thesis is a ring with identity.

(1.23) Definition
Let A be a commutative ring. We say that A is *graded*, if there is a direct sum composition of A:

$$A = \bigoplus_{\sigma \in \Sigma} A_\sigma$$

whereby Σ is a monoid with operation $\star$ and A_σ are additive abelian groups. This composition has to be compatible with the ring multiplication, i.e. for any $\sigma, \tau \in \Sigma$ the equation

$$A_\sigma A_\tau \subset A_{\sigma\star\tau}$$

holds.
An element $f \in A_\sigma$ is called homogeneous (with respect to the grading) of *degree* σ.

(1.24) Remark
Note that A_{id}, endowed with the multiplication in A, is a subring of A since $A_{id}A_{id} \subset A_{id}$ holds. Hence, any A_σ can be interpreted as an A_{id}-module.

(1.25) Remark
Assume that A is a R-algebra for any commutative non-graded ring R and assume also that the mapping $R \to A,\ r \mapsto r \cdot 1_A$ is injective. By use of the identification $r = r \cdot 1_A$, this means that we may assume $R \subset A$. Then, we say that A is graded if and only if it is graded as a ring. In this case, we endow R with the trivial grading $R = R_{id}$ and obtain $R \subset A_{id}$. Hence, all A_σ are R-modules.

(1.26) Example
Consider the polynomial ring in n variables $A := k[x_1, ..., x_n]$ as a k-algebra. Denote A_d the additive group of all monomials with total degree d. Then, we obtain

$$A = \bigoplus_{i \in \mathbb{N}} A_i$$

and the equation $A_i A_j \subset A_{i+j}$ obviously holds too for any $i, j, \in \mathbb{N}$. In fact, this example motivated the term *grading* in the first place.

We will now introduce a similar concept for modules.

(1.27) Definition
Let A be a graded ring over the monoid Σ and let M be a A-module. We say that M is a *graded module* if we can write M as

$$M = \bigoplus_{\sigma \in \Sigma} M_\sigma$$

such that

$$A_\sigma M_\tau \subset M_{\sigma\star\tau}$$

holds. Here, M_σ are additive abelian groups.
An element $f \in M_\sigma$ is called homogeneous (with respect to the grading) of *degree* σ. Note that every element in M can be uniquely written as a sum of homogeneous elements.

(1.28) Remark
Since $A_{id}M_\sigma \subset M_\sigma$ holds, M_σ is an A_{id}-module for any $\sigma \in \Sigma$.

(1.29) Remark
An ideal $I \subset A$ is called *graded* if $I = \bigoplus_{\sigma\in\Sigma} I_\sigma$ with $I_\sigma = I \cap A_\sigma$.

The next definition is a natural extension of the definition of homogeneous elements.

(1.30) Definition
Let M be a graded module and let G be a subset of M. If $G = \bigcup_{\sigma\in\Sigma} G \cap M_\sigma$, then G is called a *homogeneous* subset of M.

The theory presented in this subsection leads to an interesting application on the ideal membership problem.

(1.31) Proposition
Let A be a graded algebra over a commutative ring R and assume that the mapping $R \to A,\ r \mapsto r \cdot 1_A$ is injective. Fix any ideal $I \subset A$ and let $H \subset I$ be a homogeneous R-basis of I. Denote $H_\sigma = H \cap A_\sigma$ and fix $f = \sum_\sigma f_\sigma \in A$ with $f_\sigma \in A_\sigma$. We obtain that the set $M_f := \{\sigma \in \Sigma \mid f_\sigma \neq 0\} \subset \Sigma$ is finite. Then, $f \in I$ if and only if f is contained in the R-ideal generated by $H' = \bigcup_{\sigma\in M_f} H_\sigma$.

Proof
Recall that $R \subset A_{id}$ holds. Since H is a homogeneous R-basis of I, it follows immediately that H_σ is a R-basis of I_σ. Hence, $\bigcup_{\sigma\in M_f} H_\sigma$ is an R-basis of $\bigoplus_{\sigma\in M_f} I_\sigma$. By definition of M_f, it is obvious that $f \in I$ if and only if $f \in \bigoplus_{\sigma\in M_f} I_\sigma$, which implies the proposition.□

(1.32) Remark
This proposition states that we do not necessarily need to compute the whole basis to decide the membership problem. Consequently, the problem might be decidable even if the basis itself is infinite. We will investigate this problem in the next sections further.

1.3 Multi-Letterplace Ring

In this subsection, we will introduce an algebraic structure which we will call a *Multi-Letterplace ring*. It is a generalization of the *Letterplace ring* which was first introduced in [Fey51] in the context of representation theory. This ring was then used in both

[LSL09] and [LSL13] to compute Gröbner bases in free associative algebras. Especially in [LSL13], the authors used an abstract point of view to understand the Letterplace ring and to find useful embeddings. In this thesis, we will try to generalize these results for the Multi-Letterplace ring. Thus, in the next two sections, we will develop a general theory designed for the Multi-Letterplace ring in combination with a monoid of endomorphisms. So, this ring will be the prototype used to exemplify the results obtained in the next sections.

(1.33) Definition
Let k be a field, $m \in \mathbb{N}$ and let $X = \{x_0, x_1, ...\}$ be a finite or countably infinite set. The *Multi-Letterplace ring* P is defined as

$$P := k[X \times \mathbb{N}^m].$$

Instead of writing $x\left(i, \begin{pmatrix} j_1 \\ \vdots \\ j_m \end{pmatrix}\right)$, we will use the notation $x_i \begin{pmatrix} j_1 \\ \vdots \\ j_m \end{pmatrix}$.

(1.34) Example
Choose $k = \mathbb{R}$ and $m = 3$. We obtain the ring $P := \mathbb{R}[X \times \mathbb{N}^3]$. An element $f \in P$ has a form like

$$f = 2x_3 \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} x_5 \begin{pmatrix} 10 \\ 3 \\ 12 \end{pmatrix} - x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Note that the Multi-Letterplace ring is a commutative ring with identity, since it is induced by a field k.
Consider the element

$$x_i \begin{pmatrix} j_1 \\ \vdots \\ j_n \end{pmatrix},$$

we call x_i the *letter* and $\begin{pmatrix} j_1 \\ \vdots \\ j_n \end{pmatrix}$ the *multi-place* of the element. The term multi-place might seem arbitrary, but it is motivated by the ordinary Letterplace ring: Consider the embedding of the free associative algebra $k\langle X\rangle$ into the Letterplace ring via

$$\tau : k\langle X\rangle \to k[X \times \mathbb{N}],\ x_{i_1} \cdot ... \cdot x_{i_n} \mapsto x_{i_1}(1) \cdot ... \cdot x_{i_n}(n).$$

This is a very important embedding which helps computing Gröbner bases in $k\langle X\rangle$ with commutative methods (cf. [LSL09]). In this case, the *place* of an element stores the information of the position of the element x_{i_j} which would be normally lost by the commutativity of the Letterplace ring. Hence, the term *places* is obvious and the term *multi-place* just follows this notation.

It is clear that computations in the Multi-Letterplace ring might be very complicated if X is infinite. However, embeddings in the Multi-Letterplace often only use a small subring, which allows effective computation. We will try to benefit from this fact by taking appropriate monoids of endomorphisms in consideration. One important class of monoids are the ones generated by (q-)shifts in the i-th component of a multi-place.

§ 2 Skew Monoid Rings

Let P be a k-algebra and Σ a monoid of ring endomorphisms. In this section, we will examine the action of Σ on P by taking the skew monoid ring $S = P * \Sigma$ into consideration. We will especially consider ideals in P which are invariant under the action of Σ. It is our goal to extend the theory presented in [LSL13]. The results and proofs in this chapter are generalizations of the original proofs presented in the cited paper.

To clarify notations, we will use the following symbols:

Symbol	**Meaning**
k	arbitrary field
P	commutative k-algebra
Σ	submonoid of $End_k(P)$
$S = P * \Sigma$	skew monoid ring

We will use the abbreviation

$$f^\sigma = \sigma(f)\sigma \ \ \forall \sigma \in \Sigma, f \in P.$$

We construct S now more precisely. If we denote the operation on Σ by $\circ$, we will use the notation $\sigma\tau := \sigma \circ \tau$. We put $S = P^{(\Sigma)}$, so an element $f \in S$ is a function from Σ to P such that the set $\{\sigma \in \Sigma \mid f(\sigma) \neq 0\}$ is finite. Then, S is a free P-module with basis $\{e_\sigma \mid \sigma \in \Sigma\}$, whereby e_σ denotes the element

$$e_\sigma : \ \Sigma \to P, \ \tau \mapsto \begin{cases} 1 & \tau = \sigma \\ 0 & otherwise \end{cases} .$$

Hence, every $f \in S$ can be uniquely written as

$$f = \sum_{\sigma \in \Sigma} f(\sigma) e_\sigma.$$

This sum is finite, since only finitely many $f(\sigma)$ are not equal zero. We define a multiplication on S via

$$f \cdot g = \left(\sum_{\sigma \in \Sigma} f(\sigma) e_\sigma \right) \cdot \left(\sum_{\tau \in \Sigma} g(\tau) e_\tau \right) := \sum_{\sigma, \tau \in \Sigma} f(\sigma) g(\tau)^\sigma e_{\sigma\tau} = \sum_{\nu \in \Sigma} \left(\sum_{\substack{\sigma, \tau \in \Sigma \\ \sigma\tau = \nu}} f(\sigma) g(\tau)^\sigma \right) e_\nu.$$

We will now identify e_σ with σ and $p \in P$ with $p \cdot id_\Sigma$. Then we can write w.l.o.g. $P \subset S$ and obtain

$$f = \sum_{\sigma \in \Sigma} f(\sigma)\sigma,$$

together with

$$f \cdot g = \sum_{\sigma,\tau \in \Sigma} f(\sigma)g(\tau)^\sigma \sigma\tau.$$

If $p \in P$ and $\sigma \in \Sigma$, this implies

$$\sigma p = p^\sigma \sigma.$$

Hence, we can interpret an element of S as a finite sum $\sum\limits_{\sigma \in \Sigma} r_\sigma \sigma$ with $r_\sigma \in P$. Consequently, we obtain

$$\left(\sum_{\sigma \in \Sigma} r_\sigma \sigma\right) \cdot \left(\sum_{\tau \in \Sigma} s_\tau \tau\right) = \sum_{\sigma,\tau \in \Sigma} r_\sigma s_\tau^\sigma \sigma\tau$$

which is similar to the multiplication in skew polynomial rings.
Note that S is a non-commutative k-algebra if and only if $\Sigma \neq \{id\}$.

(2.1) Remark
We can interpret S as a free P-module with left basis Σ.

Since we will investigate the computations of Gröbner bases in S, we are interested in gradings of the ring S. The next lemma presents a natural grading induced by Σ.

(2.2) Lemma
Denote S_σ the additive abelian group $P\sigma$ of S for each $\sigma \in \Sigma$. Then, we can write $S = \bigoplus\limits_{\sigma \in \Sigma} S_\sigma$ as a graded ring. Note that $S_{id} = P$.

Proof
Since $S_\sigma \cap S_\tau = \emptyset$ for $\sigma \neq \tau$ and $S = \sum\limits_{\sigma \in \Sigma} S_\sigma$ follow directly from the definitions of S respectively S_σ, we can conclude that $S = \bigoplus\limits_{\sigma \in \Sigma} S_\sigma$ holds. According to Definition 1.23, we have to check $S_\sigma S_\tau \subset S_{\sigma\tau}$ for each $\sigma, \tau \in \Sigma$. For any $p = p_1 \sigma p_2 \tau \in S_\sigma S_\tau$ with $p_1, p_2 \in P$ we can conclude $p = p_1 p_2^\sigma \sigma\tau \in S_{\sigma\tau}$, and, therefore, we have $S_\sigma S_\tau \subset S_{\sigma\tau}$. $\square$

The following lemma reveals another interpretation of S in a special case.

(2.3) Lemma
Let Σ be freely generated by $\sigma_1, \ldots, \sigma_m$. If Σ is also abelian, S is isomorphic (as a k-algebra) to a multiple Ore extension of P, namely

$$S \cong P[s_1; \sigma_1][s_2; \sigma_2] \cdots [s_m; \sigma_m] =: \overline{P}.$$

Proof
Consider the P-linear mapping $\phi : S \to \overline{P}$ defined by

$$\phi\left(\sum_i a_i m_i \sigma_1^{\nu_1} \dots \sigma_m^{\nu_m}\right) = \sum_i a_i m_i s_1^{\nu_1} \dots s_m^{\nu_m}$$

This mapping is well-defined because Σ is freely generated by $\sigma_1, \dots, \sigma_m$ and also bijective. In addition,

$$\phi(\sigma_i p) = \phi(\sigma_i(p)\sigma_i) = \sigma_i(p)s_i = s_i p = \phi(\sigma_i)\phi(p) \ \forall p \in P, i \in \{1, ..., m\}$$

holds, so ϕ is a k-algebra isomorphism. □

From now on, we will assume that Σ suffices the assumptions of Lemma 2.3. Therefore, we will interpret S as $\overline{P}$ and we will identify Σ with $\langle s_1, ..., s_m\rangle$. In addition, we will also write $s^v = \prod_{i=1}^{m} s_i^{v_i}$ for any $v \in \mathbb{N}^m$.

(2.4) Definition
Fix any $v \in \mathbb{N}^m$ and consider the element $f = ns^v$ for any $n \in P$. We define the *s-degree* of f as $deg_s(f) = v$.

In the following example, we will introduce the prototype of the theory presented in this section.

(2.5) Example
Consider the k-algebra $P = k[X \times \mathbb{N}^m]$ with a finite or countably infinite set X. Let σ_i denote the shift in the i-th component, i.e.

$$\sigma_i : \ k[X \times \mathbb{N}^m] \to k[X \times \mathbb{N}^m], \ \ x_k \begin{pmatrix} j_1 \\ \vdots \\ j_i \\ \vdots \\ j_n \end{pmatrix} \mapsto x_k \begin{pmatrix} j_1 \\ \vdots \\ j_i + 1 \\ \vdots \\ j_n \end{pmatrix}$$

Then, $\Sigma = \langle \sigma_1, ..., \sigma_m\rangle$ is a commutative, freely generated submonoid of $End_k P$. Thus, S is isomorphic to $k[X \times \mathbb{N}^m][s_1; \sigma_1][s_2; \sigma_2] \cdots [s_n; \sigma_m]$.

2.1 Skew Monoid Rings of Commutative Polynomial Rings

We now assume that P is a commutative polynomial ring, i.e. $P = k[X]$ for a finite or countably infinite set $X = \{x_0, x_1, ...\}$. In addition, we assume that every σ_i is injective and monomial (that means $\sigma(p)$ is a monomial for any monomial p). Since P is a domain, this means that also S is domain. Note that the example of the last subsection (the Multi-Letterplace ring) suffices these assumptions. In what follows, we will make use of the following notations:

Symbol	**Meaning**
$mon(P)$	set of Monomials of P
$mon(S)$	set of Monomials of S
$Mon(S)$	$\{ms \mid m \in mon(P),\, s \in mon(S)\}$

In order to examine Gröbner bases of ideals, it is important to clarify divisibility in S. Note that, since s_i already represents the variable corresponding to σ_i, we will use the notations s^i, σ^i to denote arbitrary elements of $mon(S)$ and Σ respectively. It does not denote the power of s or σ.

(2.6) Definition
For $(k_i)_{i\in\mathbb{N}}, (g_i)_{i\in\mathbb{N}} \in \mathbb{N}^{(\mathbb{N})}$, $p_1 = \prod_{i=0}^{\infty} x_i^{k_i}, p_2 = \prod_{i=0}^{\infty} x_i^{g_i} \in mon(P)$, $s^1 = \prod_{i=0}^{m} s_i^{q_i}, s^2 = \prod_{i=0}^{m} s_i^{w_i} \in mon(S)$ and $f, g \in S$, we say that

1. p_1 divides p_2 if $k_i \leq g_i \,\forall i$.
2. s^1 divides s^2 if $q_i \leq w_i \,\forall i$.
3. f *left-divides* g if there is $h \in S$ fulfilling $g = hf$.

We use the notation $a|b$ to denote that a left-divides b.

(2.7) Remark
If $f, g \in Mon(S)$ and $g = hf$, we can conclude that $h \in Mon(S)$ holds because each σ_i is both monomial and injective.

The next proposition connects these concepts of divisibility.

(2.8) Proposition
Denote $v = ms^1, w = ns^2 \in Mon(S)$ with $m, n \in mon(P)$ and $s^1, s^2 \in mon(S)$. Then v left-divides w if and only if $s^1|s^2$ and $m^{\frac{s^2}{s^1}}|n$.

Proof
Assume v left-divides w. Then, there is $a = ks^3 \in Mon(S), k \in mon(P), s^3 \in mon(S)$ with $ns^2 = w = av = ks^3ms^1 = km^{s^3}s^3s^1$. Thus $s^3s^1 = s^2$, which implies both $s^1|s^2$ and $s^3 = \frac{s^2}{s^1}$. Furthermore, $km^{\frac{s^2}{s^1}} = n$ and, therefore, $m^{\frac{s^2}{s^1}}|n$.
Assume now that $s^3 \in mon(S)$ and $k \in mon(P)$ fulfill $s^3s^1 = s^2$ and $km^{s^3} = n$ respectively. Then, fix $a = ks^3 \in Mon(S)$ and we have $av = ks^3ms^1 = km^{s^3}s^3s^1 = ns^2 = w$. Thus, v left-divides w. □

The concept of divisibility in P also induces another concept of divisibility in $Mon(S)$.

(2.9) Definition
Denote $v = ms^1, w = ns^2 \in Mon(S)$ with $m, n \in mon(P)$ and $s^1, s^2 \in mon(S)$. Then v P-divides w if $s^1 = s^2$ and $m|n$.

It is obvious that P-divisibility implies left-divisibility. The next lemma investigates the connection between the two concepts closer.

(2.10) Lemma
Denote $v = ms^1, w = ns^2 \in Mon(S)$ with $m, n \in mon(P)$ and $s^1, s^2 \in mon(S)$. Then v left-divides w if and only if sv P-divides w for some $s \in mon(S)$.

Proof
If v left-divides w there is $a = ks \in Mon(S), k \in mon(P), s \in mon(S)$ such that $ns^2 = w = av = ksms^1 = km^s ss^1$ holds. Hence $ss^1 = s^2$ and $m^s|n$ which implies that sv P-divides w.
Choose $s \in mon(S)$ such that sv P-divides w. This implies $km^s = n$ for some $k \in mon(P)$ and $ss^1 = s^2$. Therefore, we have $ksv = km^s ss^1 = ns^2 = w$ and we conclude that v left-divides w. □

We say that w is *a multiple* of v, if there are $a, b \in Mon(S)$ fulfilling $w = avb$. These last results offer us a possibility to characterize when w is a multiple of v.

(2.11) Lemma
Let $v = ms^1, w = ns^2$ be in $Mon(S)$ with $m, n \in mon(P)$ and $s^1, s^2 \in mon(S)$. Then, the following are equivalent:

1. w is a multiple of v
2. w is a left-multiple of $v\widetilde{s}$ for some $\widetilde{s} \in mon(S)$
3. w is a P-multiple of $\widehat{s}v\widetilde{s}$ for some $\widetilde{s}, \widehat{s} \in mon(S)$.

Proof
1) $\Rightarrow$ 2)
Since w is a multiple of v, there are $a, b \in Mon(S)$ such that $w = avb$. Since $b \in Mon(S)$, there are $\widetilde{s} \in mon(S)$ and $p \in mon(P)$ with $b = p\widetilde{s}$. This implies $w = ap^{s^1}v\widetilde{s}$. Thus, w is a left multiple of $v\widetilde{s}$.
2) $\Rightarrow$ 3)
Since w is a left-multiple of $v\widetilde{s}$ there is $a = q\widehat{s}$ with $q \in mon(P), \widehat{s} \in mon(S)$ such that $w = av\widetilde{s}$. Therefore $w = q\widehat{s}v\widetilde{s}$, which means that w is a P-multiple of $\widehat{s}v\widetilde{s}$.
3) $\Rightarrow$ 1)
Since $mon(S), mon(P) \subset Mon(S)$ this implication is obvious. □

2.2 Monomial Orderings and Σ-Compatibility

In this subsection, we will take a closer look at monomial orderings on both $mon(P)$ and $Mon(S)$. From now on, we will use Y to denote $mon(P)$ or $Mon(S)$ whenever it is convenient. Recall the definition of a monomial order on Y.

(2.12) Remark
If $<$ is a monomial ordering on Y, we can conclude that $1 \leq w \; \forall w \in Y$: Assume $1 > w$ for some $w \in Y$. Since $<$ is compatible with multiplication, it follows that $w^n > w^{n+1} \; \forall n \in \mathbb{N}$. Therefore, the set $\{1, w, w^2, ...\}$ has no minimal element, in contradiction to the fact that $<$ is a well-ordering.

(2.13) Remark
Considering the definition of $<_{glex}$ and $<_{lex}$ it is easy to see that these orderings are compatible with the shift operation, i.e. $v <_{glex/lex} w$ if and only if $e_i + v <_{glex/lex} e_i + w$ for any $w, v \in \mathbb{N}^m$ and $i \in \{1, .., m\}$.

In this thesis, we are interested in monomial orderings on $mon(P)$ which are compatible with the action of Σ. The next definition specifies this concept of compatibility.

(2.14) Definition
Let $<$ be a monomial ordering on $mon(P)$. Then $\sigma \in \Sigma$ is *compatible* with $<$ if $m < n$ implies $m^\sigma < n^\sigma$ for all $m, n \in mon(P)$.
We call Σ compatible with $<$ if every $\sigma \in \Sigma$ is compatible with $<$.

(2.15) Remark
For any $\Sigma = \langle \sigma_1, .., \sigma_m \rangle$, it follows that Σ is compatible with $<$ if and only if σ_i is compatible with $<$ for all $i \in \{1, .., m\}$.

(2.16) Example
Consider $P = k[X \times \mathbb{N}^m]$ and recall the total well-ordering $<_{glex}$ on $\mathbb{N}^m$. We will define a monomial ordering $<_P$ on $mon(P)$, which we will call a *"letter over place ordering"*. We put $1 <_P q$ for all $q \in mon(P) \setminus \{1\}$ and start by defining an ordering on monomials of degree one:

$$x_k \begin{pmatrix} j_1 \\ \vdots \\ j_m \end{pmatrix} <_P x_l \begin{pmatrix} i_1 \\ \vdots \\ i_m \end{pmatrix} \;:\Leftrightarrow\; k < l \text{ or } (k = l \text{ and } \begin{pmatrix} j_1 \\ \vdots \\ j_m \end{pmatrix} <_{glex} \begin{pmatrix} i_1 \\ \vdots \\ i_m \end{pmatrix}).$$

We can write any $q, n \in mon(P)$ as $q = \prod_{i=1}^{q'} q_i$ with $q_i = x_{k_i} \begin{pmatrix} j_{i_1} \\ \vdots \\ j_{i_n} \end{pmatrix}$ and $n = \prod_{i=1}^{n'} n_i$ with $n_i = x_{l_i} \begin{pmatrix} h_{i_1} \\ \vdots \\ h_{i_n} \end{pmatrix}$. Since we have already ordered monomials of degree one, we can assume that both $q_i <_P q_{i+1}$ and $n_i <_P n_{i+1}$ hold for all i. We say that $q <_P n$ if and only if $q' < n'$ or $q' = n'$ and there is a $j \in \{1, ..., n'\}$ such that $q_i = n_i$ for all $i < j$ and $q_j <_P n_j$. This ordering is obviously total and compatible with multiplication. Since $<_{glex}$ is a well-ordering, it is also easy to see that $<_P$ is a well-ordering. Thus, $<_P$ is a monomial ordering.
The ordering is also compatible with Σ, since $<_{glex}$ is compatible with the shift operation (recall Remark 2.13): Since Σ only affects the multi-places and not the total degree nor the letters, it is sufficient to note that

$$\begin{pmatrix} j_1 \\ \vdots \\ j_n \end{pmatrix} <_{glex} \begin{pmatrix} i_1 \\ \vdots \\ i_n \end{pmatrix}$$

holds if and only if

$$e_i + \begin{pmatrix} j_1 \\ \vdots \\ j_n \end{pmatrix} <_{glex} e_i + \begin{pmatrix} i_1 \\ \vdots \\ i_n \end{pmatrix}$$

holds.

The following proposition demonstrates that the claim of Σ compatibility is not too restrictive.

(2.17) Proposition
Let $<$ be a monomial ordering on $Mon(S)$. Denote $\prec$ the restriction of $<$ on $mon(P)$. Then, Σ is compatible with $\prec$ and, for all $n, l \in mon(P)$, $s^1, s^2 \in Mon(S)$ we have:

$$ns^1 < ls^2 \;\Rightarrow\; n < l \,or\, s^1 < s^2. \,(*)$$

Proof
Consider $n \prec l$ with $n, l \in mon(P)$. By compatibility with multiplication, we can conclude $s_i n < s_i l$ and therefore $n^{s_i} s_i < l^{s_i} s_i$ for any $i \in \{1, .., m\}$.This implies $n^{s_i} \prec l^{s_i}$, since assuming $n^{s_i} \succeq l^{s_i}$ leads to the contradiction $n^{s_i} s_i \geq l^{s_i} s_i$. By recalling Remark 2.15, we have shown that Σ is compatible with $\prec$.
We shall proof $(*)$ by contraposition. Thus, assume $n \geq l$ and $s^1 \geq s^2$. It follows immediately that $ns^1 \geq ns^2 \geq ls^2$ holds. □

The following proposition reverses the last result.

(2.18) Proposition
Let $<_p$ and $<_s$ denote monomial orderings on $mon(P)$ and $mon(S)$, respectively. Consider the total ordering $<$ on $Mon(S)$ defined by

$$ms^1 < ns^2 \;\Leftrightarrow\; s^1 <_s s^2 \text{ or } \left(s^1 = s^2 \text{ and } m <_p n\right) \quad \forall m, n \in mon(P)\; s^1, s^2 \in mon(S).$$

If Σ is compatible with $<_p$, then $<$ is a monomial ordering on $Mon(S)$ extending $<_p$.

Proof
It is obvious, that the restriction of $<$ on $mon(P)$ is in fact $<_p$. So, we only have to prove that $<$ is a well-ordering which is compatible with multiplication.

1. $<$ is a well-ordering:
   Assume that an infinitely descending chain of the form

   $$n_1 s^1 > n_2 s^2 > n_3 s^3 > ...$$

   with $n_i \in mon(P)$ and $s^i \in mon(S)$ exists. By definition of $<$, it is obvious that $s_i \geq s_{i+1}$ holds for all i. This implies that the set $\{s^i \mid i \in \mathbb{N} \setminus \{0\}\}$ is finite. Otherwise, since the ordering is total, we get an infinitely descending chain of the form

   $$s^{\nu(1)} >_s s^{\nu(2)} >_s s^{\nu(3)} >_p ...$$

   whereby $\nu : \mathbb{N} \to \mathbb{N}$ is a strictly increasing mapping. This chain contradicts the fact that $<_s$ is a well-ordering.

Thus, the existence of first chain implies the existence of another infinitely descending chain. Since the set $\{s^i \mid i \in \mathbb{N} \setminus \{0\}\}$ is finite, there is some $t \in mon(S)$ which occurs infinitely often in the initial chain. Considering this sub-chain and the definition of $<$, we obtain

$$n_{\pi(1)} >_P n_{\pi(2)} >_P n_{\pi(3)} >_P \ldots$$

whereby $\pi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing mapping. But, this chain contradicts the assumption that $<_p$ is a well-ordering. Therefore, $<$ is also a well-ordering.

2. $<$ is compatible with multiplication
We will only prove that $<$ is compatible with left multiplication, since right multiplication is proven analogously. Fix $a = ms^1, b = ns^2, c = qs^3 \in Mon(S)$ with $m, n, q \in mon(P)$ and $s^1, s^2, s^3 \in mon(S)$ and assume $a < b$. This implies either $s^1 <_s s^2$ or $s^1 = s^2$. If $s^1 <_s s^2$, then $s^3s^1 <_s s^3s^2$ and, therefore, $ca = qs^3ms^1 = qm^{s^3}s^3s^1 < qn^{s^3}s^3s^2 = qs^3ns^2 = cb$.
If $s^1 = s^2$, it follows that $s^3s^1 = s^3s^2$ and $m <_p n$. Since Σ is compatible with $<_p$, we can conclude that $m^{s^3} <_p n^{s^3}$ and $qm^{s^3} <_p qn^{s^3}$ hold. This implies, by definition of $<$, $qm^{s^3}s^3s^1 < qn^{s^3}s^3s^1$, so we can conclude again that $ca = qs^3ms^1 = qm^{s^3}s^3s^1 < qn^{s^3}s^3s^2 = qs^3ns^2 = cb$ holds. □

These last propositions result in the following corollary.

(2.19) Corollary
S has a monomial ordering if and only if P has a monomial ordering compatible with Σ.

Proof
Note that the existence of monomial orderings on $mon(P)$ and $mon(S)$ is guaranteed by Higman's Lemma ([Hig52]). In Proposition 2.17 we have shown that a monomial ordering on S induces a monomial ordering on P which is compatible with Σ. If we have a compatible, monomial order on P, we can, by choosing an arbitrary monomial ordering on $mon(S)$, construct a monomial ordering on S by use of Proposition 2.18.□

(2.20) Example
Reconsider $P = k[X \times \mathbb{N}^m]$ and $\Sigma = \langle \sigma_1, .., \sigma_m \rangle$ whereby each σ_i denotes the shift in the i-th component. We have already defined a monomial ordering $<_P$ on P which is compatible with Σ in Remark 2.16. We can endow $mon(S)$ with the graded lexicographic order $<_S$, which is also a monomial order. These orderings induce now a Σ-compatible ordering on $Mon(S)$ as shown in Proposition 2.18.

2.3 Gröbner Bases in S and P

In this subsection, we assume that S is endowed with a monomial ordering $<$. As usual, we start with some new notations. Fix $f = \sum_{i=1}^{m} a_i m_i s^i \in S$ with $a_i \in k \setminus \{0\}$, $m_i \in mon(P)$, $s^i \in mon(S)$ and $G \subset S$. By choosing $g \in \{1, .., m\}$ such that $m_g s^g = \max_<\{m_i s^i\}$ holds we obtain:

Symbol		**Notation**
$lm(f)$		$m_g s^g$
$lsm(f)$	*"s-degree of f"*	s^g
$lc(f)$		a_g
$lt(f)$		$lc(f) \cdot lm(f)$
$lm(G)$		$\{lm(f) \mid f \in G \setminus \{0\}\}$
$LM(G)$		ideal generated by $lm(G)$
$LM_l(G)$		left ideal generated by $lm(G)$
$LM_P(G)$		P-module generated by $lm(G)$

Recall the definition of Gröbner bases.

(2.21) Definition
Let J be a (left) ideal in S. We call $G \subset J$ a *Gröbner basis* of J, if $LM(G) = LM(J)$ ($LM_l(G) = LM_l(J)$). If J is an ideal in P or a module over S or P, we obtain analogous definitions.

The next proposition illustrates a very important aspect of Gröbner bases, since a Gröbner basis G of an ideal I satisfies $LM(G) = LM(I)$.

(2.22) Proposition
Let J be an ideal in S. Then

$$\{w + J \mid w \in Mon(S) \setminus LM(J)\}$$

is a k-basis of S/J.
Let J be a left ideal in S. Then

$$\{w + J \mid w \in Mon(S) \setminus LM_l(J)\}$$

is a k-basis of S/J.
Let J be a P-submodule of S. Then

$$\{w + J \mid w \in Mon(S) \setminus LM_P(J)\}$$

is a k-basis of S/J.

Proof
Let J be an ideal in S. The case $J = S$ is trivial, so we will assume that J is a proper ideal. Denote $N = span\,\{Mon(S) \setminus LM(J)\}$ and fix a monomial $w \in Mon(S)$. We shall prove that there is $f \in N$ such that $w - f \in J$ holds by induction on the monomial ordering.
Induction base: It is $w = 1$, so we can conclude $w \in Mon(S) \setminus LM(J)$ since $J \neq S$. Thus, we can choose $f = w$.
Induction step: If $w \in N$, the statement is obvious. If $w \notin N$, there are $p, q \in Mon(S)$ and $g \in J$ such that $w = p \cdot lm(g) \cdot q$. Consider $f := w - \frac{1}{lc(g)}pgq$ and note that $lm(f) < w$. By use of the induction hypothesis, there is $h \in N$ fulfilling $f - h \in J$. Since $pgq \in J$, it follows that $w - h = \underbrace{f - h}_{\in J} + \underbrace{\frac{1}{lc(g)}pgq}_{\in J} \in J$. Note that if $f \in N \cap J$, then $f = 0$ and therefore $w = \frac{1}{lc(g)}pgq \in J$. □

The reduction process of the last proof yields an important characterization of Gröbner bases.

(2.23) Proposition
Let J be a (left) ideal in S or a submodule of a free S-module and $G \subset J$. Then, the following are equivalent:

1. G is a (left) Gröbner basis of J
2. For any $f \in J$, there is a Gröbner representation of f with respect to G. (i.e. $f = \sum_i f_i g_i h_i$ respectively $f = \sum_i f_i g_i$ with $g_i \in G$ and $lm(f) \geq lm(f_i) \cdot lm(g_i) \cdot lm(h_i)$).

Proof
Note that if G is a Gröbner basis, we have $LM(G) = LM(J)$ respectively $LM_l(G) = LM_l(J)$. Since $[f] = [0]$ in S/J, the reduction process in Proposition 2.22 yields the desired Gröbner representation of f with respect to G. By reversing the arguments one obtains the other implication. □

(2.24) Remark
A monomial ordering on S induces a natural multi-grading on S. By denoting $S_s = Ps$ for all $s \in mon(S)$ we can write $S = \bigoplus_{s \in mon(S)} S_s$. Furthermore, we have $S_{s^1} \cdot S_{s^2} \subset S_{s^1 s^2}$ (in general $S_{s^1} \cdot S_{s^2} \neq S_{s^1 s^2}$!) and $S_1 = P$. We call an element $f \in S_s$ *s-homogeneous*.

If the ideal J is graded with respect to $<$, we obtain a useful characterization of bases and Gröbner bases.

(2.25) Proposition
Let J be a graded ideal of S and let G be a subset of *s-homogeneous* elements of J. Then, the following are equivalent:

i) G is a basis of J

ii) $G\Sigma$ is a left basis of J

iii) $\Sigma G \Sigma$ is a P-basis of J

Proof
Note that, since G is a subset of s-homogeneous elements, every element of G has the form $g_i s^i$ with $g_i \in P$ and $s^i \in mon(S)$.
i) $\Rightarrow$ ii)
Fix $f \in J$. Since G is basis of J, we can write f as $f = \sum_i f_i g_i s^i h_i$ with $f_i, h_i \in S$ and $g_i s^i \in G$. By rewriting $h_i = \sum_k h_{ik} \overline{s}_k$ with $h_{ik} \in mon(P)$ and $\overline{s}_k \in mon(S)$ one obtains

$f = \sum\limits_i f_i g_i s^i \sum\limits_k h_{ik}\overline{s}_k = \sum\limits_i \sum\limits_k f_i h_{ik}^{s^i} g_i s^i \overline{s}_k$. Thus, $G\Sigma$ is a left basis of J.
ii) $\Rightarrow$ iii)
For any $f \in J$, there are $f_i \in S$, $g_i s^i \in G$ and $\widetilde{s}_i \in mon(S)$ such that $f = \sum\limits_i f_i g_i s^i \widetilde{s}_i$ holds. If we rewrite $f_i = \sum\limits_k f_{ik}\overline{s}_k$, then $f = \sum\limits_i \left(\sum\limits_k f_{ik}\overline{s}_k\right) g_i s^i \widetilde{s}_i = \sum\limits_i \sum\limits_k f_{ik}\overline{s}_k g_i s^i \widetilde{s}_i$. We can conclude that $\Sigma G \Sigma$ is a P-basis of J since $f_{ik} \in P$.
iii) $\Rightarrow$ i)
The ideal I generated by G is obviously a subset of J. In addition, the P-module generated by $\Sigma G \Sigma$ is on the one hand a subset of I, and, on the other hand, it is equal to J by assumption. Thus, we obtain both $J \subset I$ and $I \subset J$ and can conclude that G is a basis of J. □

(2.26) Proposition
Let J be a graded ideal of S and let G be a subset of *s-homogeneous* elements of J. Then, the following are equivalent:

i) G is a Gröbner basis of J

ii) $G\Sigma$ is a Gröbner left basis of J

iii) $\Sigma G \Sigma$ is a Gröbner P-basis of J

Proof
Note that, since G is a subset of s-homogeneous elements, every element of G has the form $g_i s^i$ with $g_i \in P$ and $s^i \in mon(S)$.
i) $\Rightarrow$ ii)
Fix $f \in J$. Since G is a Gröbner basis of J, f has a Gröbner representation with respect to G, i.e. $f = \sum\limits_i f_i g_i s^i h_i$ with $f_i, h_i \in S$ and $g_i s^i \in G$. By rewriting $h_i = \sum\limits_k h_{ik}\overline{s}_k$ with $h_{ik} \in mon(P)$ and $\overline{s}_k \in mon(S)$ one obtains $f = \sum\limits_i f_i g_i s^i \sum\limits_k h_{ik}\overline{s}_k = \sum\limits_i f_i \sum\limits_k h_{ik}^{s_i} g_i s^i \overline{s}_k$. Since $lm(f) \geq lm(f_i) \cdot lm(g_i)s^i \cdot lm(h_{ik})\overline{s}_k = lm(f_i) \cdot lm(h_{ik})^{s^i} \cdot lm(g_i)s^i \cdot \overline{s}_k$ we can conclude (since Σ is compatible $<$) $lm(f) \geq lm(f_i) \cdot lm(h_{ik}^{s^i}) \cdot lm(g_i)s^i \cdot \overline{s}_k$. Thus, f has a Gröbner representation with respect to $G\Sigma$ as a left basis.
ii) $\Rightarrow$ iii)
For any $f \in J$, we have a Gröbner representation with respect to $G\Sigma$, i.e. there are $f_i \in S$, $g_i s^i \in G$ and $\widetilde{s}_i \in mon(S)$ such that $f = \sum\limits_i f_i g_i s^i \widetilde{s}_i$. If we rewrite $f_i = \sum\limits_k f_{ik}\overline{s}_k$, then $f = \sum\limits_i \left(\sum\limits_k f_{ik}\overline{s}_k\right) g_i s^i \widetilde{s}_i = \sum\limits_i \sum\limits_k f_{ik}\overline{s}_k g_i s^i \widetilde{s}_i$. Since $lm(f) \geq lm(f_{ik}) \cdot \overline{s}_k \cdot lm(g_i) \cdot s^i \cdot \widetilde{s}_i$, this is in fact a Gröbner representation of f with respect to $\Sigma G \Sigma$ as a P-basis.

iii) $\Rightarrow$ i)
Since every Gröbner representation with respect to $\Sigma G\Sigma$ as a P-basis is also a Gröbner representation with respect to G as an S-basis, this implication is obvious. $\square$

One of our goals is the improvement of the computation of s-homogeneous Gröbner bases. Therefore, we have to investigate the influence of Σ on divisibility and the gcd of two elements.

(2.27) Remark
Since all $\sigma \in \Sigma$ are ring homomorphisms, we can conclude that $f|g$ implies $f^\sigma|g^\sigma$. Furthermore, the equation $\sigma\left(\frac{f}{g}\right) = \frac{\sigma(f)}{\sigma(g)}$ holds for any $\sigma \in \Sigma$.

The next proposition will motivate when to define Σ as compatible with divisibility in $mon(P)$. Recall that we identify each element $\sigma = \prod_{i=1}^{n} \sigma_i^{k_i}$ of Σ with the element $s = \prod_{i=1}^{m} s_i^{k_i} \in mon(S)$.

(2.28) Proposition
Let s be an arbitrary element of $mon(S)$. Then the following are equivalent:

i) $gcd(x_i^s, x_j^s) = 1$ for all $i \neq j$

ii) $gcd(m^s, n^s) = gcd(m,n)^s$ for all $m, n \in mon(P)$

Proof
i) $\Rightarrow$ ii)
Consider $m, n \in mon(P)$ with $gcd(m,n) = 1$. We rewrite $m = x_{i_1} \cdot ... \cdot x_{i_k}$ and $n = x_{j_1} \cdot ... \cdot x_{j_l}$ and conclude $\{x_{i_1}, ..., x_{i_k}\} \cap \{x_{j_1}, ..., x_{j_l}\} = \emptyset$. Therefore, by use of the assumption, it follows that $\{x_{i_1}^s, ..., x_{i_k}^s\} \cap \{x_{j_1}^s, ..., x_{j_l}^s\} = \emptyset$ holds. Thus, we can derive $gcd(m^s, n^s) = 1 = 1^s$.
If $gcd(m,n) = u$ for some $u \in mon(P)$, we obtain $gcd(\frac{m}{u}, \frac{n}{u}) = 1$. We have already shown that $gcd(\left(\frac{m}{u}\right)^s, \left(\frac{n}{u}\right)^s) = 1$ and by use of Remark 2.27 it follows that $gcd(\frac{m^s}{u^s}, \frac{n^s}{u^s}) = 1$ which yields $gcd(m^s, n^s) = u^s$.
ii) $\Rightarrow$ i)
Since Σ is monomial, we can conclude that both x_i and x_i^s are elements of $mon(P)$. Thus, this implication is trivial. $\square$

(2.29) Remark
If $s \in mon(S)$ fulfills these equivalent conditions, it follows immediately that the corresponding homomorphism σ is injective.

In the situation of the last proposition we easily obtain two useful results.

(2.30) Corollary
If $s \in mon(S)$ fulfills $gcd(x_i^s, x_j^s) = 1$ for all $i \neq j$, we can conclude that

$$m \mid n \quad \Leftrightarrow \quad m^s \mid n^s \;\; \forall m, n \in mon(P)$$

and

$$lcm(m^s, n^s) = lcm(m, n)^s \quad \forall m, n \in mon(P)$$

hold.

Proof
Considering Remark 2.27, we only have to prove $m^s | n^s \Rightarrow m | n$. If we assume $m^s | n^s$ for some $m, n \in mon(P)$, we obtain, by applying Proposition 2.28, $m^s = gcd(m^s, n^s) = gcd(m, n)^s$. Since the endomorphism $\sigma \in \Sigma$ corresponding to s is injective, we conclude $m = gcd(m, n)$, which implies $m | n$.
By applying Proposition 2.28 and Remark 2.27 we immediately obtain

$$lcm(m, n)^s = \left(\frac{mn}{gcd(m, n)}\right)^s = \frac{m^s n^s}{gcd(m^s, n^s)} = lcm(m^s, n^s).$$ □

These last results yield the following definition.

(2.31) Definition
We say that $\sigma \in \Sigma$ is *compatible with divisibility* in $mon(P)$ if $i \neq j$ implies $gcd(x_i^\sigma, x_j^\sigma) = 1$.
If every $\sigma \in \Sigma$ is compatible with divisibility in $mon(P)$, we call Σ compatible with divisibility in $mon(P)$.

(2.32) Remark
If $\Sigma = \langle \sigma_1, .., \sigma_m \rangle$, then Σ is compatible with divisibility in $mon(P)$ if and only if σ_i is compatible with divisibility in $mon(P)$ for all $i \in \{1, ..., n\}$.

2.4 The Buchberger Algorithm and the Σ-Criterion

In this subsection we will recall basic results concerning Gröbner bases computations of modules and apply it to S as a P-module. Furthermore, we will introduce a special criterion which is applicable for any s-homogeneous basis of an ideal $J \subset S$. We will call this criterion Σ-criterion.

(2.33) Definition
Fix $f, g \in S \setminus \{0\}$ such that $lm(f) = ms$ and $lm(g) = ns$ with $m, n \in mon(P)$ and $s \in mon(S)$. Denote $a = lc(f), b = lc(g)$ and $l = lcm(m, n)$.
Then the *s-polynomial* of f, g is defined as $spoly(f, g) = \frac{l}{am} f - \frac{l}{bn} g$. Note that the equation $spoly(f, g) = -spoly(g, f)$ holds.

The following proposition is one of the most famous criteria in Gröbner basis computations for modules. We have already stated it in the introduction. This version is just the adaption to our current setting.
For this purpose, we interpret S as a P-module and omit the action of Σ on P. Hence, we interpret S as a left P-module.

(2.34) Proposition (Buchberger criterion)
Let G be a generating set of a P-module $J \subset S$. Then G is a Gröbner basis of J if and only if for all $f, g \in G \setminus \{0\}$ fulfilling $lsm(f) = lsm(g)$ the s-polynomial $spoly(f, g)$ has a Gröbner representation with respect to G.

The following lemma will prove beneficial for finding a Σ-criterion.

(2.35) Lemma
Fix $f, g \in S$ and choose $s^1, s^2 \in mon(S)$ such that $lsm(fs^1) = lsm(s^1 f) = lsm(s^2 g) = lsm(gs^2)$. Denote $s = gcd(s^1, s^2)$ and fix $t_1, t_2 \in mon(S)$ fulfilling $st_i = s^i$. Then, $spoly(s^1 f, s^2 g) = s\,(spoly(t_1 f, t_2 g))$ and $spoly(fs^1, gs^2) = spoly(ft_1, gt_2)s$.

Proof
We shall prove the first claim, the other one is proven analogously. Note that $lsm(t_1 f) = lsm(t_2 g)$, so the s-polynomial is well defined. By use of the notations

Symbol		**Notation**
$lt(s^1 f)$		$am^{s^1} s^f s^1$
$lt(s^2 g)$		$bn^{s^2} s^g s^2$
$lt(t_1 f)$		$am^{t_1} s^f t_1$
$lt(t_2 g)$		$bn^{t_2} s^g t_2$
$lcm(m^{t_1}, n^{t_2})$		q
$lcm(m^{s^1}, n^{s^2})$	cf. Corollary 2.30	q^s

we obtain $h = spoly(t_1 f, t_2 g) = \frac{q}{am^{t_1}} t_1 f - \frac{q}{bn^{t_2}} t_2 g$. Therefore, we conclude

$$sh = s\left(\frac{q}{am^{t_1}} t_1 f - \frac{q}{bn^{t_2}} t_2 g\right) = \frac{q^s}{am^{s^1}} s^1 f - \frac{q^s}{bn^{s^2}} s^2 g = spoly(s^1 f, s^2 g). \qquad \square$$

We now introduce the $\mathbf{\Sigma}$-criterion for an s-homogeneous basis of an ideal $J \subset S$.

(2.36) Proposition ($\mathbf{\Sigma}$-criterion)
Let G be an s-homogeneous basis of a graded two-sided ideal $J \subset S$. Then, G is a Gröbner basis if and only if

$$\forall f, g \in G \setminus \{0\} \text{ and } s^1, s^2, s^3, s^4 \in mon(S) \text{ such that } lsm(s^1 f s^2) = lsm(s^3 g s^4)$$
$$\text{and } gcd(s^1, s^3) = 1 = gcd(s^2, s^4) \text{ hold, the s-poynomial}$$
$$spoly(s^1 f s^2, s^3 g s^4) \text{ has a Gröbner representation with respect to } G' = \Sigma G \Sigma.$$

Proof
If G is a Gröbner basis of J, the implication is obviously right, since every s-polynomial $spoly(s^1 f s^2, s^3 g s^4)$ is an element of J (cf. Proposition 2.23).
Assume now, that the second condition holds. By Proposition 2.26 it is sufficient to prove that $G' := \Sigma G \Sigma$ is a Gröbner basis of J as a P-module. Since G is a generating set of J, Proposition 2.25 yields that G' is a basis of J as a P-module. Considering the Buchberger criterion (cf. Proposition 2.34), it is sufficient to show that every s-polynomial $h = spoly(s^1 f s^2, s^3 g s^4)$ with $f, g \in G$ and $s^i \in mon(S)$ such that $lsm(s^1 f s^2) = lsm(s^3 g s^4)$ holds has a Gröbner representation with respect to G'.
We denote $s := gcd(s^1, s^3), \widetilde{s} = gcd(s^2, s^4)$ and choose $t_i \in mon(S)$ such that $s^{1,3} = st_{1,3}$ and $s^{2,4} = \widetilde{s}t_{2,4}$ hold. Furthermore, we define $\widehat{h} := spoly(t_1 f t_2, t_3 g t_4)$. By use of Lemma 2.35, we can conclude that $h = s\widehat{h}\widetilde{s}$ holds. By assumption, $\widehat{h}$ has a Gröbner representation with respect to G'.
Therefore, we will now show that if any $h \in S$ has a Gröbner representation with respect to G', the same holds true for sht for any $s, t \in mon(S)$. So, if we write $h = \sum_i p_i g_i$ with $p_i \in P$,$g_i \in G'$ and $lm(h) \geq lm(p_i g_i)$, we obtain immediately $sht = s\left(\sum_i p_i g_i\right) t = \sum_i p_i^s s g_i t$. Since $s g_i t \in \Sigma G' \Sigma = G'$ and $lm(sht) = slm(h)t \geq slm(p_i) \cdot lm(g_i)t = lm(p_i^s) slm(g_i)t = lm(p_i^s) \cdot lm(s g_i t)$, we can conclude that sht has a Gröbner representation with respect to G'. $\square$

The next criterion yields a similar result for left ideals in S. Note that this criterion holds without the assumption of an s-homogeneous basis.

(2.37) Proposition (Left Σ-criterion)
Let G be a basis of a left ideal $J \subset S$. Then, G is a Gröbner basis if and only if

$$\forall f, g \in G \setminus \{0\} \text{ and } s^1, s^2 \in mon(S) \text{ such that } lsm(s^1 f) = lsm(s^2 g)$$
$$\text{and } gcd(s^1, s^2) = 1 \text{ the s-poynomial}$$
$$spoly(s^1 f, s^2 gs) \text{ has a Gröbner representation with respect to } G' = \Sigma G.$$

Proof
The proof is similar to the last proof:
If G is a Gröbner basis of J, the implication is obviously right, since every s-polynomial $spoly(s^1 f, s^2 g)$ is an element of J (cf. Proposition 2.23).
Assume now, that the second condition holds. It is sufficient to prove that $G' := \Sigma G$ is a Gröbner basis of J as a P-module. Since G is a generating set of J,we can conclude that G' is a basis of J as a P-module. Note that G does not need to be an s-homogeneous subset for this implication to hold. Considering the Buchberger criterion (cf. Proposition 2.34), it is sufficient to show that every s-polynomial $h = spoly(s^1 f, s^2 g)$ with $f, g \in G$ and $s^i \in mon(S)$ such that $lsm(s^1 f) = lsm(s^2 g)$ holds has a Gröbner representation with respect to G'.
We denote $s := gcd(s^1, s^2)$ and choose $t_i \in mon(S)$ such that $s^{1,2} = st_{1,2}$ holds. Furthermore, we define $\widehat{h} := spoly(t_1 f, t_2 g)$. By use of Lemma 2.35, we can conclude $h = s\widehat{h}$. By assumption, $\widehat{h}$ has a Gröbner representation with respect to G'. Therefore, it only remains to show that if any $h \in S$ has a Gröbner representation with respect to G', the same holds true for sh for any $s \in mon(S)$. But, we have already proven a stronger result in the proof of Proposition 2.36. □

We will now use this Proposition to establish an algorithm to compute an s-homogeneous Gröbner basis in a more efficient way than the ordinary Buchberger algorithm would. For this purpose we recall a standard procedure to determine the normal form of a polynomial.

Algorithm 1: Reduce Procedure

Data: $f \in S$, $G \subset S$
Result: $h \in S$ such that $f - h \in \langle G \rangle_P$ with $h = 0$ or $h \notin (LM(G))_P$

$h := f$;
while $h \neq 0$ *and* $h \in (LM(G))_P$ **do**
    find $g \in G \setminus \{0\}$ such that $lm(g)$ P-divides $lm(h)$;
    $h := h - \frac{lt(h)}{lt(g)} g$
end
return h;

The next algorithm is an adaption of the classic Buchberger algorithm taking the last proposition into consideration. We did not include the product criterion for brevity, but this is of course another possible improvement.

Algorithm 2: SkewGBasis

Data: H, an s-homogeneous basis of a graded two sided ideal J in S
Result: G, an s-homogeneous Gröbner basis of J

$G := H$;
$B := \{(f, g) \mid f, g \in G\}$;
while $B \neq \emptyset$ **do**
  choose $(f, g) \in B$; ;
  $B := B \setminus \{(f, g)\}$;
  for $s^1, s^2, s^3, s^4 \in mon(S)$ *with* $gcd(s^1, s^3) = 1 = gcd(s^2, s^4)$ *such that* $lsm(s^1 f s^2) = lsm(s^3 g s^4)$ **do**
    $h :=$ Reduce$(spoly(s^1 f s^2, s^3 g s^4), \Sigma G \Sigma)$;
    if $h \neq 0$ **then**
      $B := B \cup \{(h, h), (h, k), (k, h) \mid k \in G\}$;
      $G := G \cup \{h\}$;
    end
  end
end
return G;

The following proposition states that we can compute truncated bases of some graded two-sided ideal of S in finite time.

(2.38) Proposition
Let $J \subset S$ be a graded two-sided ideal and $v \in \mathbb{N}^m$. We assume that there are only finitely many $w \in \mathbb{N}^m$ fulfilling $w < v$. If H is an s-homogeneous basis of J and $H_v = \{f \in H \mid deg_s(f) \leq v\}$ is finite, then there is an s-homogeneous Gröbner basis G of J such that G_v is also finite. In addition, the algorithm SkewGBasis can compute this basis in a finite number of steps.

Proof
Since H_v is finite, the set $L = \{s^w f s^l \mid f \in H_v,\ w + deg_s(f) + l \leq v\}$ is also finite. Thus, the set $X' \subset X$ containing all variables occurring in L is also finite. By use of the notations $P^v = k[X']$ and $S^v = \bigoplus_{w \leq v} P^v s^w$, we can conclude that a v-truncated version (degree boundary v in the for-loop) of SkewGBasis actually computes a Gröbner basis of J^v as a P^v module. Here, J^v is the P^v submodule of S^v generated by L. Obviously, S^v

has finite rank and therefore both P^v and S^v are noetherian. Thus, all strictly increasing chains of submodules of S^v are finite, which implies that there can only be finitely many loop runs: The sets named G in the algorithm , which are updated in every loop run, induce a a strictly increasing subset of modules. Consequently, we obtain termination.□

(2.39) Remark
Note that there are total orderings on $\mathbb{N}^m$ such that there are only finitely many $w \in \mathbb{N}^m$ fulfilling $w < v$ for any $v \in \mathbb{N}^m$. One important example is the graded lexicographic order.

In this section we have studied graded ideals of S and their Gröbner bases. This theory was presented in [LSL13] for a cyclic monoid $\Sigma = \langle\sigma\rangle$ whereby σ has infinite order. We have extended this theory for a commutative, freely generated $\Sigma = \langle\sigma_1, .., \sigma_m\rangle$. While some proofs had to be adapted to the new environment, the main ideas were taken over. In the next section, we will examine a special class of ideals in P, which are invariant under the action of Σ. We will see a close connection between these ideals and the graded ideals of S, and, therefore, we will be able to apply the results of this section to obtain useful results considering the ideals in P.

§3 Σ-Invariant Ideals of P

In this section, we will take a close look at certain ideals of P. By defining a new kind of basis which generates the ideal up to the action of Σ, we will be able to use the results of the last section to obtain new results. In the sixth sections, we will see that the ideals we consider are very important in the application of the Multi-Letterplace ring.
Similarly to the last section, our approach is based on [LSL13] and generalizes the results obtained in this paper.
In this section, Σ will be an arbitrary submonoid of $End(P)$ freely generated by $\sigma_1, \ldots, \sigma_m$.
We start by defining, when we call a module/ideal Σ-invariant.

(3.1) Definition
Let M be a P-module. We say that M is Σ-*invariant* if there is a monoid homomorphism $\rho : \Sigma \to End_k(M)$ such that $\rho(\sigma)(fx) = f^\sigma \rho(\sigma)(x)$ for all $f \in P, \sigma \in \Sigma$ and $x \in M$. If this is the case, we write $\sigma \cdot x$ to denote $\rho(\sigma)(x)$.
A module homomorphism $f : M \to M'$ of two Σ-invariant modules is called a homomorphism of Σ-invariant modules if $f(\sigma \cdot m) = \sigma \cdot f(m)$ for all $\sigma \in \Sigma$ and $m \in M$.

(3.2) Remark
Note that if I is an ideal in P, I is Σ-invariant if $\sigma(I) \subset I$ for all $\sigma \in \Sigma$. This fact illustrates the term Σ-*invariant* and allows us to interpret I as an S-module by use of the scalar multiplication $\sigma \cdot f = f^\sigma$.

The next proposition links Σ-invariant P-modules and left S-modules.

(3.3) Proposition
The category of Σ-invariant P-modules is equal to the category of left S-modules.

Proof
Let M be a Σ-invariant P-module. We have to define an S-module structure on M. By use of the endomorphism ρ, we can define $(\sum_i p_i\sigma_i) \cdot x = \sum_i p_i\rho(\sigma_i)(x)$ for all $x \in M$ and $f = \sum_i p_i\sigma_i \in S$.
If M is a left S-module , it is also a P-module since $P \subset S$. Furthermore, the mapping $\rho : \Sigma \to End_k(M)$ with $\rho(\sigma)(x) = \sigma \cdot x$ is a suitable monoid homomorphism, whereby $\sigma \cdot x$ is the scalar multiplication inherited by the left S-module structure.
We still have to prove that any homomorphism $f : M \to M'$ of Σ-invariant P-modules M and M' is also a homomorphism of M and M' as left S-modules. Thus, we have to prove $f((\sum_i p_i\sigma_i) \cdot x) = (\sum_i p_i\sigma_i) \cdot f(x)$. In fact, since f is P-linear, we obtain $f((\sum_i p_i\sigma_i) \cdot x) = \sum_i p_i f(\sigma_i \cdot x) = \sum_i p_i(\sigma_i \cdot f(x)) = (\sum_i p_i\sigma_i) \cdot f(x)$. □

We will now define a new kind of basis for Σ-invariant ideals.

(3.4) Definition
Let I be a Σ-invariant ideal in P and let G be a subset of I. We say that G is a Σ-*basis* of I if G is a basis of I as a left S-module.

(3.5) Remark
It is obvious that G is a Σ-basis if and only if $\Sigma \cdot G$ is a basis of I as a P-ideal.

From now on, we will assume that Σ is compatible with our monomial ordering and divisibility. Then, we immediately obtain the following proposition.

(3.6) Proposition
Let G be a subset of P. Then, we have $lm(\Sigma \cdot G) = \Sigma \cdot lm(G)$. Therefore, if I is a Σ-invariant ideal in P, $LM_P(I)$ is also Σ-invariant.

Proof
The compatibility of Σ with the monomial ordering on P yields $lm(\sigma \cdot f) = \sigma \cdot lm(f)$ for any $\sigma \in \Sigma$ and $f \in P$. $\square$

The following definition is a natural combination of the two concepts of Σ-bases and Gröbner bases.

(3.7) Definition
Let $I \subset P$ be a Σ-invariant ideal and let G be a subset of I. Then, G is a *Gröbner* Σ-*basis* of I if $lm(G)$ is a basis of $LM_P(I)$ as a left S-module, or, equivalently, $\Sigma \cdot G$ is a Gröbner basis of I as a P-ideal.

We will now introduce a similar criterion to the one presented in Proposition 2.36. Before we can prove this criterion, we need a lemma similar to Lemma 2.35.

(3.8) Lemma
Fix $f, g \in P$ and $s^1, s^2 \in \Sigma$. Then, the equation

$$spoly(s^1 \cdot f, s^2 \cdot g) = gcd(s^1, s^2) \cdot spoly(\frac{s^1}{gcd(s^1, s^2)} \cdot f, \frac{s^2}{gcd(s^1, s^2)} \cdot g)$$

holds.

Proof
Denote $s = gcd(s^1, s^2)$ and choose $t_1, t_2 \in \Sigma$ such that $s \cdot t_i = s^i$ holds. By use of the notations

Symbol		**Notation**
$lt(f)$		am
$lt(g)$		bn
$lt(t_1 \cdot f)$		am^{t_1}
$lt(t_2 \cdot g)$		bn^{t_2}
$lt(s^1 \cdot f)$		am^{s^1}
$lt(s^2 \cdot g)$		bn^{s^2}
$lcm(m^{t_1}, n^{t_2})$		q
$lcm(m^{s^1}, n^{s^2})$	cf. corollary 2.30	q^s

we immediately obtain $s \cdot spoly(t_1 \cdot f, t_2 \cdot g) = s \cdot \left(\frac{q}{am^{t_1}} t_1 \cdot f - \frac{q}{bn^{t_2}} t_2 \cdot g\right) = \frac{q^s}{am^{s^1}} s^1 \cdot f - \frac{q^s}{bn^{s^2}} s^2 \cdot g = spoly(s^1 \cdot f, s^2 \cdot g)$. □

We are now able to prove the Σ-criterion in P.

(3.9) Proposition (Σ-criterion in P)
Let G be a Σ-basis of a Σ-invariant ideal $I \subset P$. Then, G is a Gröbner Σ-basis if and only if for all $s^1, s^2 \in mon(S)$ with $gcd(s^1, s^2) = 1$ and for all $f, g \in G$, the s-polynomial $spoly(s^1 \cdot f, s^2 \cdot g)$ has a Gröbner representation with respect to G.

Proof
If G is a Gröbner Σ-basis, we can conclude that $G' = \Sigma \cdot G$ is a Gröbner basis of I. Thus, by considering Proposition 1.16, we immediately obtain the first implication.
Assume now that the second condition holds. We have to prove that for any $f, g \in G$ and $s^1, s^2 \in \Sigma$ the s-polynomial $h = spoly(s^1 \cdot f, s^2 \cdot g)$ has a Gröbner representation with respect to G'. By use of Lemma 3.8 we obtain $h = s \cdot \widetilde{h}$, whereby $s = gcd(s^1, s^2)$ and $\widetilde{h} = spoly(\frac{s^1}{s} \cdot f, \frac{s^2}{s} \cdot g)$. By assumption, we know that $\widetilde{h}$ has a Gröbner representation with respect to G'. Therefore, it is sufficient to prove that if any $h \in P$ has a Gröbner representation, the same holds for $s \cdot h$ for any $s \in \Sigma$.
Assume that $h = \sum\limits_l f_l\,(s_l \cdot g_l)$ with $f_l \in P, s_l \in \Sigma$ and $g_l \in G$ holds. Then, for any $s \in \Sigma$ we can write $s \cdot h = s \cdot \left(\sum\limits_l f_l\,(s_l \cdot g_l)\right) = \sum\limits_l (s \cdot f_l)\,((ss_l) \cdot g_l)$. Since $s \cdot f_l \in P$ and $(ss_l) \cdot g_l \in G' = \Sigma G$, we can conclude that $s \cdot h$ has a Gröbner representation with respect to G'. Note that $lm(h) \geq lm(f_l(s_l \cdot g_l))$ implies $lm(s \cdot h) \geq lm((s \cdot f_l)(ss_l \cdot g_l))$. Thus G' is a Gröbner basis and this implies by definition of G' that G is a Gröbner Σ-basis of I. □

We are now able to establish a similar algorithm to the one presented in the last section for ideals in P.

Algorithm 3: SigmaGBasis

Data: H, a Σ-basis of a Σ-invariant ideal J in P
Result: G, a Gröbner Σ-basis of J

$G := H$;
$B := \{(f,g) \mid f,g \in G\}$;
while $B \neq \emptyset$ **do**
  choose $(f,g) \in B$;
  $B := B \setminus \{(f,g)\}$;
  for $s^1, s^2 \in mon(S)$ *with* $gcd(s^1,s^2) = 1$ **do**
    $h :=$ REDUCE$(spoly(s^1 \cdot f, s^2 \cdot g), \Sigma \cdot G)$;
    if $h \neq 0$ **then**
      $B := B \cup \{(h,h),(h,k),(k,h) \mid k \in G\}$;
      $G := G \cup \{h\}$;
    end
  end
end
return G;

We will now introduce a new P-module homomorphism $\pi : S \to P$ with $s \mapsto 1$ for all $s \in mon(S)$. This mapping will help us applying the results of the last section in order to obtain useful results concerning Σ-invariant ideals of P. Note that π is a left S-module epimorphism: Fix any $as \in S$ with $a \in k$ and $s \in mon(S)$. Consider $f = \sum_{i=1}^n a_i m_i s^i$ with $a_i \in k$, $m_i \in mon(P)$ and $s^i \in mon(S)$. We obtain

$$\pi\left(a(s \cdot f)\right) = \pi(af^s) = \sum_{i=1}^n a a_i m_i^s = a\left(s \cdot \left(\sum_{i=1}^n a_i m_i\right)\right) = a\left(s \cdot \pi(f)\right).$$

Since π is a homomorphism, we can conclude that it is in fact a left S-module homomorphism.

(3.10) Definition
Let J be a graded ideal of S. By use of the mapping π we obtain a corresponding ideal in P, namely $J^P = \pi(J)$.

Note that the ideal J^P is in fact Σ-invariant.

(3.11) Lemma
Let J be a graded ideal of S. Then, J^P is a Σ-invariant ideal of P.

Proof
We have to prove that $a^s \in J^P$ for any $a \in J^P$ and $s \in mon(S)$. Since $a \in J^P$, there is some $t \in mon(S)$ such that $at \in J$, Thus, we can conclude $sat = a^s st \in J$. This implies that $a^s = \pi(a^s st) \in J^P$ holds. □

We will now examine if the mapping π is compatible with bases of ideals.

(3.12) Proposition
Let J be a graded ideal of S and let G be an s-homogeneous basis of J. Then, $G^P = \pi(G)$ is a Σ-basis of J^P.

Proof
Recall that π is a left S-module epimorphism. Since G is s-homogeneous, every element of G has the form $g_i s^i$ with $g_i \in P$ and $s^i \in mon(S)$. Thus, we can conclude that $G\Sigma$ is a left basis of J: For any $h \in J$ we have $h = \sum_i l_i g_i s^i r_i = \sum_i l_i g_i s^i \sum_j a_{ij} m_{ij} s_{ij} = \sum_i \sum_j l_i g_i s^i a_{ij} m_{ij} s_{ij} = \sum_i \sum_j l_i a_{ij} m_{ij}^{s^i} g_i s^i s_{ij}$ whereby $l_i, r_i \in S$ and $r_i = \sum_j a_{ij} m_{ij} s_{ij}$ with $m_{ij} \in mon(P), s_{ij} \in mon(S)$ and $a_{ij} \in k \setminus \{0\}$.
Therefore $G^P = \pi(G) = \pi(G\Sigma)$ is a basis of I^P as a left S-module, which implies that G^P is a Σ-basis. □

3.1 Σ-Compatible Multi-Gradings on P

In the last subsection we have established a mapping from S to P preserving bases. Before we are able to provide a corresponding mapping from P to S, we have to establish a new special multi-grading on P. For this purpose, we will introduce a new monoid extending $(\mathbb{N}^m, \max(\cdot,\cdot))$, whereby max depends on the chosen total ordering on $\mathbb{N}^m$. Denote $\widehat{\mathbb{N}}^m = \mathbb{N}^m \cup \{-\infty\}$ and consider the monoid $(\widehat{\mathbb{N}}^m, \max(\cdot,\cdot))$ with $\max(v, -\infty) = -\infty$ for all $v \in \mathbb{N}^m$. Thus, $-\infty$ is the new identity and the monoid is both commutative and idempotent.

(3.13) Remark
If we denote $\sigma^{-\infty} \equiv 0$ for all $\sigma \in \Sigma$, we can extend Σ in a similar way: Define $\widehat{\Sigma} = \Sigma \cup \{0\}$ (with $0\sigma = 0$) and we can conclude that $\widehat{\Sigma} \cong \widehat{\mathbb{N}}^m$ holds. (Consider the homomorphism $f : \widehat{\Sigma} \to \widehat{\mathbb{N}}^m$ with $f(s_i^k) = ke_i$ for all i.) Furthermore, we denote $\widehat{S} = \widehat{\Sigma} * P$ and $\widehat{\pi} : \widehat{S} \to P$, whereby $\widehat{\pi}$ is the left $\widehat{S}$-module homomorphism with $s_j^i \mapsto 1$ for all $i \in \widehat{\mathbb{N}}$ and $j \in \{1, ..n\}$.

We will now declare $-\infty + v = -\infty$ for all $v \in \mathbb{N}^m$, which leads us to the following definition.

(3.14) Definition
A mapping $w : mon(P) \to \widehat{\mathbb{N}}^m$ fulfilling

i) $w(1) = -\infty$

ii) $w(mn) = \max\{w(m), w(n)\}$ for all $m, n \in mon(P)$

iii) $w(s_i \cdot m) = e_i + w(m)$ for all $i \in \{1, .., m\}$

is called a multi-weight function.

(3.15) Remark
If we denote $P_v = span\{m \in mon(P) \mid w(m) = v\}$ for any $v \in \widehat{\mathbb{N}}^m$, it follows immediately that $P = \bigoplus_{v \in \widehat{\mathbb{N}}^m} P_v$. We will call an element $f \in P_v$ *w-homogeneous* of weight v and we will write $w(f) = v$, even if f is not monomial.

(3.16) Remark
Note that property iii) implies $w(s^v \cdot m) = v + w(m)$ for any $v \in \mathbb{N}^m$. This property could also be generalized by replacing it with the condition $w(s_i \cdot m) = v_i + w(m)$ for arbitrary $v_i \in \mathbb{N}^m$.

(3.17) Example
Reconsider the Multi-Letterplace ring $P = k[X \times \mathbb{N}^m]$. In this case, the construction of a multi-weight function w is natural:

$$\begin{aligned}
&w : mon(k[X \times \mathbb{N}^m]) \to \widehat{\mathbb{N}}^m \\
&x_i \begin{pmatrix} j_1 \\ \vdots \\ j_n \end{pmatrix} \mapsto \begin{pmatrix} j_1 \\ \vdots \\ j_n \end{pmatrix}, \quad 1 \mapsto -\infty \\
&w\left(\prod_{i=1}^{n} x_{k_i} \begin{pmatrix} j_{i_1} \\ \vdots \\ j_{i_n} \end{pmatrix} \right) = \max_i \left\{ w\left(x_{k_i} \begin{pmatrix} j_{i_1} \\ \vdots \\ j_{i_n} \end{pmatrix} \right) \right\}
\end{aligned}$$

By definition of w, both property i) and ii) are fulfilled. So, only the last one remains to be proven. Remember that σ_i is the shift in the i-th component, which implies

$$w\left(s_i \cdot x_k \begin{pmatrix} j_1 \\ \vdots \\ j_i \\ \vdots \\ j_n \end{pmatrix} \right) = w\left(x_k \begin{pmatrix} j_1 \\ \vdots \\ j_i + 1 \\ \vdots \\ j_n \end{pmatrix} \right) = \begin{pmatrix} j_1 \\ \vdots \\ j_i + 1 \\ \vdots \\ j_n \end{pmatrix} = e_i + \begin{pmatrix} j_1 \\ \vdots \\ j_i \\ \vdots \\ j_n \end{pmatrix} = e_i + w\left(x_k \begin{pmatrix} j_1 \\ \vdots \\ j_i \\ \vdots \\ j_n \end{pmatrix} \right).$$

Similar to graded ideals with respect to a certain grading, we will now define when we call an ideal in P *w-graded*.

(3.18) Definition
Let I be an ideal in P. We call I *w-graded* if $I = \sum\limits_{v \in \widehat{\mathbb{N}}^m} I_v$ with $I_v = I \cap P_v$.

Recall the notation $s^v = \prod\limits_{i=1}^{m} s_i^{v_i}$ for any $v \in \mathbb{N}^m$. From now on, we will also use $s^{-\infty} \equiv 0$.

(3.19) Definition
For any $f \in P$, consider the (finite) sum decomposition $f = \sum_v f_v$ with $f_v \in P_v$. We define the mapping $\xi : P \to \widehat{S}$ via $f \mapsto \sum_v f_v s^v$. Then, ξ is obviously both injective and homogeneous, i.e. a w-homogeneous element of P is mapped onto an s-homogeneous element of S.

We will now investigate the influence of $\widehat{\Sigma}$ on ξ.

(3.20) Proposition
The map ξ is $\widehat{\Sigma}$-equivariant.

Proof
We have to prove $\xi(s^v \cdot f) = s^v \xi(f)$ for all $f \in P$ and $v \in \widehat{\mathbb{N}}^m$. Since ξ is a linear map, it is sufficient to prove the equation for $f \in P_w$ for an arbitrary $w \in \widehat{\mathbb{N}}^m$.
So, for any $v \in \widehat{\mathbb{N}}^m$, we obtain $s^v \cdot f \in P_{v+w}$. Thus, we can conclude that the equation

$$\xi(s^v \cdot f) = (s^v \cdot f)\, s^{v+w} = s^v f s^w = s^v \xi(f)$$

holds. □

Note that $\xi(I) \subset S$ holds for any proper ideal I in P, because $I \neq P$ implies $I_{-\infty} = \{0\}$. Thus, we will only consider proper ideals from now on. By this restriction, we avoid the use of $-\infty$ and, since only proper ideals are interesting concerning Gröbner bases and membership problems, this restriction is in fact none.

(3.21) Definition
Let $I \subsetneq P$ be a w-graded Σ-invariant ideal. Then, the ideal $I^S = \langle \xi(I) \rangle$ in S is called the *skew analogue* of I.

(3.22) Lemma
Let $I \subsetneq P$ be a w-graded Σ-invariant ideal. Then,

i) I^S is generated by $G^S := \xi(\bigcup_{v\in\mathbb{N}^m} I_v) = \bigcup_{v\in\mathbb{N}^m} \{fs^v \mid f \in I_v\}$

ii) I^S is left generated by $G^S\Sigma$

iii) I^S is generated as a P-module by $\Sigma G^S\Sigma$.

Proof

i)

Note that, due to the linearity of ξ, $\xi(I) = \xi(\sum_v I_v) = \sum_v \xi(I_v) = \sum_v I_v s^v$. Thus, we can conclude $\langle\xi(I)\rangle = \langle G^S\rangle$.

ii)+iii)

Follow immediately from the fact that G^S is an s-homogeneous basis of I as an ideal in S (cf. Proposition 2.26). □

The next proposition will clarify the connection between the mappings π and ξ.

(3.23) Proposition

Let $I \subsetneq P$ be a w-graded Σ-invariant ideal. Then, the projection of π reverses the action of ξ, i.e

$$I^{SP} = I.$$

Proof

Denote $J = I^{SP} = \pi(I^S)$ and note that the elements $f_v s^w$ with $f_v \in I_v$ and $w \geq v$ form a left basis of I^S (cf. Lemma 3.22). For any fixed f_v and $w \geq v$ we obtain $\pi(f_v s^w) = f_v \in J$. Since π is a left S-module homomorphism, the elements $\pi(f_v s^w) = f_v$ form a left basis of J. Since they are all contained in I, we can conclude that $J \subset I$ holds.

On the other hand, the elements f_v form a basis of I, since we can write $I = \sum_v I_v$. Thus, it follows immediately that $I \subset J$ holds too which implies $I^{SP} = J = I$. □

(3.24) Remark

The last proposition revealed a one-to-one correspondence between all w-graded, Σ-invariant proper ideals in P and their skew analogues. Our next goal is to find a one-to-one correspondence between certain (Gröbner) bases as well. For this purpose, the following lemmata will prove useful.

(3.25) Lemma

If any $m, n \in mon(P)$ and $s^v \in mon(S)$ fulfill $(s^v \cdot m)\,|n$, we can conclude that the inequation $w(n) - v \geq w(m)$ holds.

Proof
By assumption, there is $q \in mon(P)$ with $n = q(s^v \cdot m)$. We obtain, by recalling Definition 3.14(ii) and iii), that $w(n) \underset{ii)}{\geq} w(s^v \cdot m) \underset{iii)}{=} v + w(m)$ holds, which implies the desired equation. □

The last lemma yields another important equation helping understand the connection between the least common multiple and the weight function w.

(3.26) Lemma
Fix $m, n \in mon(P)$ and $l = lcm(m, n)$. Then, the weight of l is the maximum weight of m and n, i.e. $w(l) = \max\{w(m), w(n)\}$.

Proof
Note that we both have $m, n|l$ and $l|mn$. By use of Lemma 3.25 with $v = (0, .., 0) \in \mathbb{N}^m$, we can conclude that the inequality

$$\left.\begin{matrix} w(m) \\ w(n) \end{matrix}\right\} \leq w(l) \leq w(mn) = \max\{w(m), w(n)\}$$

holds, which directly implies $w(l) = \max\{w(m), w(n)\}$. □

(3.27) Remark
Fix $f \in P_v$ and $g \in P_w$ and denote $l = lcm(lm(f), lm(g))$. We have just shown that $w(l) = \max\{w(lm(f)), w(lm(g))\} = \max\{w(f), w(g)\}$ holds. Since $h := spoly(f, g) = \frac{l}{lt(f)}f - \frac{l}{lt(g)}g$ it follows that $w(h) = w(l)$ if $h \neq 0$: For every summand h_i of $\frac{l}{lt(f)}f$ we have $w(h_i) = \max\{w(\frac{l}{lt(f)}), w(f)\}$ since $f \in P_v$. An analogous argument holds for g and therefore $w(\frac{l}{lt(f)}f)$ and $w(\frac{l}{lt(g)}g)$ are both well defined. One summand of $\frac{l}{lt(f)}f$ is obviously l, so $w(\frac{l}{lt(f)}f) = w(l)$, the same holds for $\frac{l}{lt(g)}g$. Thus, $w(h)$ is both well defined and equal to $w(l)$.

We will now see that certain **Σ**-bases of ideals $I \subset P$ are mapped onto ordinary bases of $I^S \subset S$. This property is remarkable because a **Σ**-basis is in general a proper subset of an ordinary basis.

(3.28) Proposition
Let $I \subsetneq P$ be a w-graded **Σ**-invariant ideal. If $G = \bigcup_v G_v$ is a w-homogeneous **Σ**-basis of I (with $G_v \subset I_v$), then $G^S := \xi(G) = \{fs^v \mid f \in G_v\}$ is an s-homogeneous basis of I^S.

Proof
We have already seen that the elements fs^w with $f \in I_v$ and $w \geq v$ form a left basis of I^S. Therefore, it is sufficient to show that all fs^w are contained in the ideal generated by G^S. Since G^S is obviously s-homogeneous, we can apply Lemma 2.25 and conclude that it is also sufficient to show that all fs^w are contained in the P-module generated by $\Sigma G^S \Sigma$.
So, fix $f \in I_v$ and choose an arbitrary $w \geq v$. By assumption, we can write f as $f = \sum_i a_i f_i(s^{w_i} \cdot g_{v_i})$ with $a_i \in k, f_i \in mon(P), w_i \in \mathbb{N}^m$ and $g_{v_i} \in G_{v_i}$. Since $w(f) = v$, we can conclude $w(f_i(s^{w_i} \cdot g_{v_i})) = v$ for all i. By use of Lemma 3.25, it follows that $v - w_i \geq v_i$ holds. Therefore, it follows immediately that $w - w_i \geq v_i$ holds and consequently $g_{v_i} s^{w-w_i} \in G^S\Sigma$. This implies $fs^w = \sum_i a_i f_i (s^{w_i} \cdot g_{v_i}) s^w = \sum_i a_i f_i s^{w_i} g_{v_i} s^{w-w_i}$ and, by taking into consideration that $s^{w_i} g_{v_i} s^{w-w_i} \in \Sigma G^S \Sigma$ holds, we have proven that fs^w is in fact contained in the P-module generated by $\Sigma G^S \Sigma$. □

(3.29) Remark
Recall that the compatibility of the monomial ordering with the multiplication in S implies that $ms < ns$ if and only if $m < n$ for all $m, n \in mon(P)$ and $s \in mon(S)$. Thus, for any $f \in P$ we immediately obtain $lm(fs) = lm(f)s \in S$.

(3.30) Proposition
Let $I \subsetneq P$ be a w-graded Σ-invariant ideal and denote $G = \bigcup_v I_v$. Then, $G^S = \xi(G)$ is an s-homogeneous Gröbner basis of I^S.

Proof
By definition of I^S, G^S is an s-homogeneous basis of I^S. In addition, G is a Gröbner Σ-basis of I, since $I = \sum_v I_v$ ($f = \sum_v f_v$ implies $lm(f) = lm(f_w)$ for some w and hence $lm(f) \in LM(G)$). Considering Proposition 2.26, we will prove that $\Sigma G^S \Sigma$ is a Gröbner basis of I as a P-module. Buchbergers criterion (cf. Proposition 2.34) now states that this holds true if $spoly(s^{k'} f s^v s^k, s^{l'} g s^w s^l)$ has a Gröbner representation with respect to $\Sigma G^S \Sigma$ for any $f \in G_v, g \in G_w$ and $k, k', l, l' \in \mathbb{N}^m$ fulfilling $k' + v + k = l' + w + l =: c$. Obviously, $h = spoly(s^{k'} \cdot f, s^{l'} \cdot g)$ has a Gröbner representation with respect to $\Sigma \cdot G$. Thus, we can write $h = \sum_q h_q (s^{v_q} \cdot g_q)$ with $h_q \in P$ and $g_q \in G$ and we obtain:

$$\begin{aligned} &spoly(s^{k'} f s^v s^k, s^{l'} g s^w s^l) = spoly((s^{k'} \cdot f) s^{k'+v+k}, (s^{l'} \cdot g) s^{l'+w+l}) \\ &\underset{Prop.2.35}{=} spoly(s^{k'} \cdot f, s^{l'} \cdot g) s^c = hs^c = \sum_q h_q (s^{v_q} \cdot g_q) s^c = \sum_q h_q s^{v_q} g_q s^{c-v_q} \end{aligned}$$

Due to the definition of ξ, we have to check that $c \geq v_q + w(g_q)$ holds. Recall Remark 3.27 which implies $w(h) = \max\{w(s^{k'} \cdot f\}, w(s^{l'} \cdot g)\} = \max\{k' + w(f), l' + w(g)\}$.

In addition, we know that $w(h) = \max\{w(h_q), w(s^{v_q} \cdot g_q)\} \geq v_q + w(g_q)$ holds. The definition of c yields:

$$c = \begin{cases} k' + v + k \geq k' + v = k' + w(f) = w(s^k \cdot f) \\ l' + w + l \geq l' + w = l' + w(g) = w(s^{l'} \cdot g) \end{cases}$$

which implies $c \geq \max(w(s^{k'} \cdot f), w(s^{l'} \cdot g)) = w(h) \geq v_q + w(g_q)$.
Furthermore, note that $lm(h) \geq lm(h_q(s^{v_q} \cdot g_q))$ implies $lm(hs^c) \geq lm(h_q(s^{v_q} \cdot g_q)s^c)$.□

We will now prepare the analysis of the behavior of Gröbner bases under the action of ξ. The following lemma will start by proving that the leading monomial is invariant under ξ.

(3.31) Lemma
Let G be a subset of $\bigcup_{v\in\mathbb{N}^m} P_v$ and let $I \subsetneq P$ be a w-graded Σ-invariant ideal. Then, both $lm(G)^S = lm(G^S)$ and $LM_P(I)^S = LM(I^S)$ hold.

Proof
Fix any f in P_v for any $v \in \mathbb{N}^m$. Since f is w-homogeneous, we can conclude that $w(lm(f)) = w(f) = v$ holds. Thus, $\xi(f) = fs^v$ and $\xi(lm(f)) = lm(f)s^v$. Recalling Remark 3.29 yields $lm(fs^v) = lm(f)s^v$ which immediately implies $lm(G)^S = lm(G^S)$. Denote now $G = \bigcup_v I_v$. Then, I^S is the ideal generated by G^S. Since $I = \sum_v I_v$, it also follows that $lm(G) = lm(I)$. Note that $G \subset \bigcup_v P_v$, so we have $lm(G)^S = lm(G^S)$. Therefore, $LM_P(I)^S$ is generated by $lm(G^S)$ and, by taking Proposition 3.30 into consideration, we obtain that $LM(I^S)$ is also generated by $lm(G^S)$ which obviously implies $LM_P(I)^S = LM(I^S)$. □

We are now finally able to prove that certain Gröbner bases in P are mapped onto Gröbner bases in S.

(3.32) Proposition
Let $I \subsetneq P$ be a w-graded Σ-invariant ideal and let $G = \bigcup_v G_v$ be a w-homogeneous Gröbner Σ-basis of I.Then, $G^S = \xi(G)$ is an s-homogeneous Gröbner basis of I^S.

Proof
By definition of ξ and since G is w-homogeneous, it is obvious that G^S is s-homogeneous. Note that $lm(G)$ is a w-homogeneous Σ-basis of $LM_P(I)$. By use of Proposition 3.28 it follows that $lm(G)^S$ is a basis of $LM_P(I)^S$. So, considering Lemma 3.31, we can conclude that $lm(G^S)$ is a basis of $LM(I^S)$. Thus, by definition, G^S is a Gröbner basis of I^S. □

Of course, we are also interested in the other direction and we will now see that we get a similar result by use of the mapping π.

(3.33) Proposition
Let $I \subsetneq P$ be a w-graded Σ-invariant ideal and let G be an s-homogeneous Gröbner basis of I^S. Then, $G^P = \pi(G)$ is a Gröbner Σ-basis of I.

Proof
Note that it is sufficient to prove that $lm(f)$ is contained in the ideal generated by $lm(G^P)$ for $f \in I_v$ for any $v \in \mathbb{N}^m$. Thus, fix any $f \in I_v$ and consider $\xi(f) = fs^v \in I^S$. Since G is an s-homogeneous Gröbner basis of I^S, there is $gs^k \in G$ with $g \in P$ and $k \in \mathbb{N}^m$ fulfilling $lm(fs^v) = qs^w lm(gs^k)s^r$ with $w, r \in \mathbb{N}^m$ and $q \in mon(P)$. We conclude that

$$lm(f)s^v = lm(fs^v) = qs^w lm(gs^k)s^r = qs^w lm(g)s^{k+r} = qlm(s^w \cdot g)s^{w+k+r}$$

holds, which implies $lm(f) = qlm(s^w \cdot g)$. Considering $g = \pi(gs^k) \in G^P$, we have proven that G^P is in fact a Gröbner Σ-basis of I. $\square$

(3.34) Remark
These last two propositions complete the result described in Remark 3.24 since we have now an equivalence of certain bases of corresponding ideals. From a computational point of view, the Gröbner bases computation of w-graded Σ-invariant ideals can be transferred to the Gröbner bases computation of their skew analogues and vice versa.

(3.35) Proposition
Let $J \subsetneq P$ be a w-graded Σ-invariant ideal and fix $v \in \mathbb{N}^m$ such that there are only finitely many $w \in \mathbb{N}^m$ with $w < v$. If H is a w-homogeneous basis of J and $H_v = \{f \in H \mid w(f) \leq v\}$ is finite, then there is w-homogeneous Gröbner basis G of J such that G_v is also finite. In addition, the algorithm SigmaGBasis can compute this basis in a finite number of steps.

Proof
The algorithm SigmaGBasis computes a set $G \subset P$ such that $\Sigma \cdot G$ is a Gröbner basis of J. Considering Lemma 3.26 and property (iii) of Definition 3.14, we conclude that the elements of both $\Sigma \cdot H$ and $\Sigma \cdot G$ are w-homogeneous. Since H_v is finite, the set $L = \{s^q \cdot f \mid q + w(f) \leq v\}$ is also finite. Thus, the set $X' \subset X$ containing all variables occurring in L is also finite. By use of the notation $P^v = k[X']$ we can conclude that a v-truncated version (multidegree boundary v in the for-loop) of SigmaGBasis actually computes a Gröbner basis of J^v as an ideal in P^v. Here, J^v is the ideal generated by

L. Note that P^v is noetherian. Hence, all strictly increasing chains of ideals in P^v are finite, which implies that there can only be finitely many loop runs: The sets named G in the algorithm, which are updated in every loop run, induce a a strictly increasing chain of ideals. Consequently, we obtain termination. □

(3.36) Remark

Note that there are orderings (like the graded lexicographic ordering) on $\mathbb{N}^m$ such that any $v \in \mathbb{N}^m$ has only finitely many predecessors. Then, this assumption is always satisfied.

In this section, we have seen that Σ-invariant w-graded ideals of P allow us to improve Gröbner bases computations and solve the membership problem. The ideas in this chapter are based on [LSL13] with some changes due to the more general setting. One important difference is the transition from weight functions to multi-weight functions, which allows us to obtain multi-graded ideals in S.

§4 Multi-Weight Functions in the Multi-Letterplace Ring

In this thesis, we have already introduced multi-weight functions in the third section. We will now take a closer look at multi-weight functions for the Multi-Letterplace ring. Our main goal is to determine wether there is a suitable multi-weight function w to make a fixed ideal w-graded. If this is the case, we can avoid the homogenization presented in the next section which is important in terms of computational effectivity. However, we will see that it is not always possible to endow the Multi-Letterplace ring with a multi-weight function suitable for a fixed ideal.

Throughout this section, we will consider P to be a Multi-Letterplace ring with $X = \{x_i \mid i \in I\}$ for a finite or countably infinite set I. Furthermore, let $\Sigma = \langle \sigma_1, .., \sigma_m \rangle$ be the monoid generated by the shifts. In addition, $\widetilde{w}$ will denote the standard multi-weight function presented in Example 3.17.

First of all, recall the definition of a multi-weight function:

(4.1) Definition
A mapping $w : mon(P) \to \widehat{\mathbb{N}}^m$ fulfilling

i) $w(1) = -\infty$

ii) $w(mn) = \max\{w(m), w(n)\}$ for all $m, n \in mon(P)$

iii) $w(s_i \cdot m) = e_i + w(m)$ for all $i \in \{1, .., m\}$

is called a multi-weight function.

The following definition will simplify notations throughout this section.

(4.2) Definition
Let $v = (v_1, .., v_m)^T \in \mathbb{N}^m$ be an arbitrary vector. We will use the notation

$$x_i(v) = x_i \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}.$$

Then, $\sigma_v \in \Sigma$ denotes the shift fulfilling $\sigma_v \cdot x_i(0) = x_i(v)$ for any i, i.e.

$$\sigma_v = \prod_{j=1}^{m} \sigma_j^{v_j}.$$

(4.3) Definition
Fix $v, w \in \mathbb{N}^m$. Then we define $\max(v, w) \in N^m$ via $\max(v, w)_i = \max\{v_i, w_i\}$ for all $i \in \{1, .., m\}$.

Before we will try to find multi-weight functions adjusted to ensure that a certain ideal is w-graded, we will investigate when two multi-weight functions are equal.

(4.4) Proposition
Let $w, v : mon(P) \to \widehat{\mathbb{N}}^m$ be two multi-weight functions.
Then

$$w \equiv v \;\Leftrightarrow\; w(x_i(0)) = v(x_i(0)) \;\forall i \in I.$$

Proof
The first implication is obvious, so assume that the second condition holds. Let $n = \prod_i x_{k_i}(a_i)$ be an arbitrary monomial with $a_i \in \mathbb{N}^m$ and $k_i \in I$. By use of property $ii)$ and $iii)$ of the multi-weight functions, we obtain:

$$\begin{aligned} w(n) &= \max_i\{w(x_{k_i}(a_i))\} = \max_i\{w(\sigma_{a_i} \cdot x_{k_i}(0))\} = \max_i\{a_i + w(x_{k_i}(0))\} \\ &= \max_i\{a_i + v(x_{k_i}(0))\} = \max_i\{v(\sigma_{a_i} \cdot x_{k_i}(0))\} = \max_i\{v(x_{k_i}(a_i))\} = v(n) \end{aligned} \qquad \square$$

By use of the last proposition, the next corollary follows immediately.

(4.5) Corollary
Let w be an arbitrary multi-weight function. Then, $w(x_i(v)) = w(x_i(0)) + \widetilde{w}(x_i(v)) = w(x_i(0)) + v$ holds for any $x_i(v) \in P$.

The last proposition gives us an idea of the possibilities we have (or do not have) to construct a multi-weight function. Fix any $f \in P$ and denote $F = \{f\}$. We will now try to find a multi-weight function which makes the ideal $J = \langle \Sigma \cdot F \rangle$ w-graded. In order to illustrate the next proposition, consider the following two examples.
In these examples, we choose the ring $P = \mathbb{R}[X \times \mathbb{N}^2]$ and we endow $\mathbb{N}^2$ with the lexicographical order.

(4.6) Example
Fix $X = \{x_1, x_2, x_3\}$ and $f = x_2 \begin{pmatrix}1\\2\end{pmatrix} x_3 \begin{pmatrix}1\\0\end{pmatrix} - x_1 \begin{pmatrix}0\\1\end{pmatrix}$ and note that

$$\widetilde{w}\left(x_2 \begin{pmatrix}1\\2\end{pmatrix} x_3 \begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\2\end{pmatrix} \neq \begin{pmatrix}0\\1\end{pmatrix} = \widetilde{w}\left(x_1 \begin{pmatrix}0\\1\end{pmatrix}\right)$$

holds. Consequently, f is not $\widetilde{w}$-homogeneous. However, we can use the fact that the multi-places $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ belong to different letters. Hence, we define a multi-weight function w via

$$w\left(x_2\begin{pmatrix}0\\0\end{pmatrix}\right) = w\left(x_3\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}0\\0\end{pmatrix},\quad w\left(x_1\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}.$$

By Proposition 4.4 w is uniquely determined by these three function values. We can conclude that

$$w\left(x_2\begin{pmatrix}1\\2\end{pmatrix}x_3\begin{pmatrix}1\\0\end{pmatrix}\right) = \max\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\} = \begin{pmatrix}1\\2\end{pmatrix}$$
$$= \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} + w\left(x_1\begin{pmatrix}0\\0\end{pmatrix}\right) = w\left(\sigma_2\cdot x_1\begin{pmatrix}0\\0\end{pmatrix}\right) = w\left(x_1\begin{pmatrix}0\\1\end{pmatrix}\right)$$

holds. Thus, f is w-homogeneous and, since f is a Σ-basis of J, J is w-graded.

(4.7) Example
Consider now $f = x_1\begin{pmatrix}1\\0\end{pmatrix} + x_1\begin{pmatrix}0\\0\end{pmatrix}$ and $X = \{x_1\}$. We will show that there is no non-trivial multi-weight function w such that f is w-homogeneous. Hence, let w be an arbitrary multi-weight function. Note that

$$w\left(x_1\begin{pmatrix}1\\0\end{pmatrix}\right) = w\left(\sigma_1\cdot x_1\begin{pmatrix}0\\0\end{pmatrix}\right) = e_1 + w\left(x_1\begin{pmatrix}0\\0\end{pmatrix}\right) \neq w\left(x_1\begin{pmatrix}0\\0\end{pmatrix}\right)$$

holds as long as $w\left(x_1\begin{pmatrix}0\\0\end{pmatrix}\right) \neq -\infty$. Hence, only the multi-weight function $v \equiv -\infty$ makes f homogeneous.

We will now describe when there is a multi-weight function which makes a fixed $f \in P$ homogeneous. Consider first the following definition.

(4.8) Definition
Let $f = \sum_i a_i m_i$ with $a_i \neq 0$ be a polynomial in P. Then, X_f denotes the set of all letters occurring in f, i.e.

$$X_f = \{x_i \mid \exists v \in \mathbb{N}^m \text{ such that } x_i(v) \text{ divides } m_j \text{ for some j}\}.$$

The maximum Multi-place vector v for a fixed letter x_i will be denoted by $v_{f,i}$.
We denote the set of all vectors occurring in f as V_f, i.e.

$$V_f = \{v \in \mathbb{N}^m \mid x_j(v) \text{ divides } m_i \text{ for some } i \in \mathbb{N} \text{ and } x_j \in X_f\}.$$

Furthermore, we define the *maximum vector* v_f of f as

$$v_f = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \text{ with } v_i = \min\{k \in \mathbb{N} \mid k \geq w_i \, \forall w \in V_f\}.$$

This proposition solves the problem of finding a suitable multi-weight function.

(4.9) Proposition
Fix $n \in \mathbb{N}$ and let $f = \sum_{i=1}^{n} a_i m_i$ with $a_i \in k \setminus \{0\}$ and $m_i \in mon(P)$ be an arbitrary element of P. There is a non-trivial multi-weight function w such that f is w-homogeneous if and only if for all $i \in \{1, ..n\}$ there are $x_{j_i} \in X_f$ and $v_i \in \mathbb{N}^m$ such that

$$x_{j_i}(v_i)|m_i \text{ and } x_{j_i}(v')|m_k \text{ implies } v' \leq v \; \forall k \in \{1, .., n\}, v' \in \mathbb{N}^m.$$

Proof
Assume that the second condition holds and choose $j_i \in X_f$ and $v_i \in \mathbb{N}^m$ such that they fulfill the condition for any $i \in \{1, ..n\}$. We define a multi-weight function w via

$$\begin{aligned} w\left(x_{j_i}(0)\right) &= v_f - v_i && \text{for all } i \in \{1, .., n\} \\ w\left(x_k(0)\right) &= 0 && \text{for all } k \neq j_i \; \forall i \in \{1, .., n\}. \end{aligned}$$

Note that $v_f - v_i \in \mathbb{N}^m$. According to Proposition 4.4, w is uniquely determined by these values. It follows now that $w\left(x_{j_i}(v_i)\right) = v_f$ and $w\left(x_{j_i}(v')\right) \leq v_f$ (since $v' \leq v_f$ and Corollary 4.5) for any $v' \in \mathbb{N}^m$ such that $x_{i_j}(v')|m_k$ for some $k \in \{1, .., n\}$. Furthermore, by definition of q, we can conclude $w\left(x_l(u)\right) = \widetilde{w}\left(x_l(u)\right) \leq v_f$ for any $l \neq i_j$ and $u \in \mathbb{N}^m$ fulfilling $x_l(u)|m_k$ for some $k \in \{1, .., n\}$. Hence, we conclude $w(m_i) = v_f$ for all $i \in \{1, .., n\}$ which implies that f is w-homogeneous.
Later on, we will need minimal multi-weight functions (this will be defined precisely later). For this purpose, we define a multi-weight function w_1 via

$$\begin{aligned} w_1\left(x_{j_i}(0)\right) &= w\left(x_{j_i}(0)\right) - re_1 && \text{for all } i \in \{1, .., n\} \\ w_1\left(x_k(0)\right) &= 0 && \text{for all } k \neq j_i \; \forall i \in \{1, .., n\}. \end{aligned}$$

whereby $r \in \mathbb{N}$ is the maximal number such that w_1 is still positive (i.e. $w(x_i(0)) \in \mathbb{N}^m$) and f is still w_1-homogeneous. We continues this procedure for all $p \in \{2, \ldots, m\}$.
We will prove the other implication by proving its contraposition. Hence, assume that the second condition does not hold. Therefore, without loss of generality, there is no $x_j \in X_f$ and $u \in \mathbb{N}^m$ such that $x_j(u)|m_1$ and $x_j(u')|m_k$ implies $u \geq u'$. Let now w

be an arbitrary multi-weight function. We have to prove that f is not w-homogeneous. Choose $x_l(u)$ such that $x_l(u)|m_1$ and $w\left(x_l(u)\right) = w(m_1)$ hold. By assumption, there is $x_l(u')$ fulfilling $x_l(u')|m_k$ for $k \neq 1$ and $u' > u$. It follows

$$w(m_1) = w\left(x_l(u)\right) = w\left(x_l(0)\right) + u < w\left(x_l(0)\right) + u' = w\left(x_l(u')\right) \leq w(m_k),$$

so f is not w-homogeneous. □

This proposition also shows that the problem of finding a multi-weight function for fixed f can be solved by an algorithm in finite time. For this purpose, we will first introduce an algorithm 4 which determines the maximal vector v for any letter occurring in a polynomial $f \in P$. Note that this algorithm obviously depends on the chosen total ordering on $\mathbb{N}^m$.

Algorithm 4: MaxVectors

Data: $f = \sum_{i=1}^{n} a_i m_i \in P$, $m_i = \prod_{l=1}^{n_i} x_{i_l}(v_{i_l})$, $X_f \subset I$ contains all letters occurring in f

Result: $M \subset \mathbb{N}^m \times H \times \mathcal{P}\left(\{1,..,n\}\right)$ a set containing the maximal vector v for any $i \in H$ and the monomials inheriting the maximal element $x_i(v)$.

$M := \emptyset$;
for $x_i \in X_f$ **do**
    $max := 0 \in \mathbb{N}^m$; $K := \emptyset$;
    for j *from* 1 *to* n **do**
        for l *from* 1 *to* n_i **do**
            if $j_l = i$ **then**
                if $v_{j_l} > max$ **then**
                    $max := v_{j_l}$;
                    $K := \{j\}$;
                end
                if $v_{j_l} = max$ **then**
                    $K := K \cup \{j\}$;
                end
            end
        end
    end
    $M := M \cup \{(max, i, K)\}$;
end
return M;

The next algorithm 5 now constructs a multi-weight function w if there is any. It returns a set W which consists of all tuples (i, v) with $i \in I$ and $v \in \mathbb{N}^m \setminus \{0\}$ such that $w(x_i(0)) = v$. If there is no tuple with $j \in I$ then we define $w(x_j(0)) = 0$. Note, by considering the proof of Proposition 4.9, that the set W is finite. Remark that this algorithm does not consider the reduction process in order to slightly simplify the algorithm.

Algorithm 5: MWeight

Data: $f = \sum_{i=1}^{n} a_i m_i \in P$, X_f contains all letters occurring in f, v_f

Result: $W \subset I \times \mathbb{N}^m$, a finite set which defines a multi-weight function w such that f is w-homogeneous. If there is no such function, the set $\{-1\}$ will be returned.

M :=MaxVectors(f,X_f);
for j *from* 1 *to* n **do**
    $ismax := false$;
    for $(v', i', K') \in M$ **do**
        if $j \in K'$ **then**
            $W := W \cup \{(i', v_f - v')\}$;
            $ismax := true$;
            $break$;
        end
    end
    if $\neg ismax$ **then**
        return $\{-1\}$;
    end
end
return W;

The next two examples will illustrate these algorithms. We fix $k = \mathbb{R}$ and $m = 2$. Furthermore, we consider $X = \{x_1, x_2, x_3\}$ and endow $\mathbb{N}^2$ with the graded lexicographic order.

(4.10) Example

We want to apply the algorithms on

$$f = x_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We write $f = m_1 + m_2$ with

$$m_1 = x_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad m_2 = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

First, note that $v_f = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ holds. The first algorithm looks for the maximum vector of each letter and stores this combination and the corresponding monomials. Thus, we obtain

$$M_f = \left\{ \left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, 1, \{1\} \right), \left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}, 2, \{2\} \right), \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 3, \{2\} \right) \right\}.$$

Hence, for any monomial m_j of f there is a letter x_i such that there is an element $(v, i, K) \in M_f$ with $j \in K$. In this case, we even have two indices for the monomial m_2. Thus, there is no unique output of our algorithm.

For m_1 we pick the letter x_1 with $v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and for m_2 we choose x_2 and consequently $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Therefore, we obtain the multi-weight function w defined by

$$\begin{aligned} w(x_1(0)) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ w(x_2(0)) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ w(x_3(0)) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

In fact, f is w-homogeneous with $w(f) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = v_f$.

(4.11) Example

In this example, consider

$$f = x_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_1 \begin{pmatrix} 0 \\ 2 \end{pmatrix} x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We write $f = m_1 + m_2 + m_3$ with

$$m_1 = x_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad m_2 = x_1 \begin{pmatrix} 0 \\ 2 \end{pmatrix} x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad m_3 = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We obtain $v_f = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. The first algorithm yields

$$M_f = \left\{ \left(\begin{pmatrix} 2 \\ 4 \end{pmatrix}, 1, \{1\} \right), \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, 2, \{2\} \right) \right\}.$$

We note that there is no $q = (v_q, i_q, K_q) \in M_f$ such that $3 \in K_q$ holds. Thus, the third monomial does not satisfy the condition of the proposition and there is no multi-weight function w such that f is w-graded.
You can also obtain this result by use of this inequality

$$w(m_3) = w\left(x_2\begin{pmatrix}1\\1\end{pmatrix}\right) = w\left(x_2\begin{pmatrix}0\\0\end{pmatrix}\right) + \begin{pmatrix}1\\1\end{pmatrix} < w\left(x_2\begin{pmatrix}0\\0\end{pmatrix}\right) + \begin{pmatrix}2\\1\end{pmatrix} = w\left(x_2\begin{pmatrix}2\\1\end{pmatrix}\right) \leq w(m_2).$$

Before we are able to solve the same question for more than one f, we need some definitions. Note that if f is w-graded, then f is also w'-graded if $w'(m) = w(m) + v$ for all $m \in mon(P)$ and for some fixed $v \in \mathbb{N}^m$. Of course, these additional solutions are not really interesting, since they only yield more complex gradings in $S = P * \Sigma$.

(4.12) Definition
Let w be a multi-weight function such that $f = \sum_i a_i m_i \in P$ is w-homogeneous. Consider the set $Q = \{x_i \mid \exists v \in \mathbb{N}^m : \ x_i(v)|m_j \text{ and } w(x_i(v))) = w(f)\}$. We say that w is minimal (w.r.t. f) if $w(x_i(0)) = 0$ for all $x_i \notin Q$ and the function defined by

$$\widetilde{w}(x_i(0)) = \begin{cases} w(x_i(0)) - v & , x_i \in Q \\ w(x_i(0)) & , x_i \notin Q \end{cases}$$

is a non-positive (i.e. $w(x_i(0)) \in \mathbb{Z}^m \setminus \mathbb{N}^m$ for some $x_i \in Q$) function for any $v \in \mathbb{N}^m \setminus \{0\}$.

(4.13) Remark
Note that multi-weight functions obtained by the construction in Proposition 4.9 are minimal. We have to show that $w - e_k$ is non-positive. After the reduction process, we can conclude that there is an index $s \in Q$ and a vector $v \in \mathbb{N}^m$ such that $w(x_s(v)) = w(f), w(x_s(0))_k = 0$. Hence, $w(x_k(0)) - e_i \notin \mathbb{N}^m$. (We know that $w - e_k$ is non-positive, this means that $w(x_k(0))_i = 0$ for some i_j defined in the proof, or f is not $\widetilde{w}$-homogeneous. In this case, there must be some $x_l \in X_f$ such that $l \neq j_i$ and $x_l(w(f))$ occurs in f. Hence, $w(x_l(0)) = 0$ and $x_l \in Q$, so $\widetilde{w}$ is non-positive.)

The next definition combines the results of Proposition 4.9 with the last definition.

(4.14) Definition
Let $f = \sum\limits_{i=1}^{n} a_i m_i$ with $a_i \in k \setminus \{0\}$ and $m_i \in mon(P)$ be an arbitrary element of P. Then we define a set of all possibilities to construct a multi-weight function w such that f is w-graded:

$$Com_f = \Big\{(j_1, v_1) \times ... \times (j_n, v_n) \in (I \times \mathbb{N}^m)^n \ \mid x_{j_i}(v_i)|m_i \text{ and } x_{j_i}(v')|m_k \Rightarrow v' \leq v \ \forall k \in \{1, .., n\}, v' \in \mathbb{N}^m\Big\}.$$

For any $q_f = (j_1, v_1) \times ... \times (j_n, v_n) \in Com_f$, we define the *set of dependencies* as

$$D_{q_f} = \{(j_i, j_k, v_i - v_k) \in I \times I \times \mathbb{Z}^m \mid 1 \le i < k \le n\}.$$

We will write, by abuse of notation, $j_i \in q$ to denote that the index j_i occurs in q and we will also write $q_1 \cap q_2$ for $q_i \in Com_f$.
From Proposition 4.9 it follows that every q_f induces a multi-weight function w_{q_f} such that f is w_{q_f}-homogeneous.

(4.15) Remark
Note that the multi-weight functions w_{q_f} are minimal. Thus, the set $\{w_{q_f} \mid q_f \in Com_f\}$ is the set of all minimal multi-weight functions such that f is homogeneous. In addition, Com_f is always finite.

From now on, we will always consider two different polynomials f and g in P. We want to characterize when there is a multi-weight function w such that f and g are w-homogeneous. For this purpose, we analyze when $q_f \in Com_f$ and $q_g \in Com_g$ are compatible with each other. This compatibility is necessary to combine two multi-weight functions w_f and w_g, which make f, g $w_{f,g}$-homogeneous, to a multi-weight function w suitable for both f and g.

(4.16) Definition
Fix $f, g \in P$ and choose $q_f \in Com_f$ and $q_g \in Com_g$. We say that D_{q_f} and D_{q_g} are compatible if $w = v$ holds for all $(i, j, w) \in D_{q_f}$ and $(i, j, v) \in D_{q_g}$.
If there is an index $i \in q_f \cap q_g$, then we put $d(q_f, q_g) = \max(w_{q_f}(x_i(0)), w_{q_g}(x_i(0))) - w_{q_f}(x_i(0))$ and $d(q_g, q_f) = \max(w_{q_g}(x_i(0)), w_{q_f}(x_i(0))) - w_{q_g}(x_i(0))$.
Otherwise, we set $d(q_f, q_g) = d(q_g, q_f) = 0$.

(4.17) Remark
Note that $d(q_f, q_g)$ is well-defined, since it does not depend on the choice of the index i: If $i, j \in q_f \cap q_g$, then we can conclude $w_{q_f}(x_i(0)) - w_{q_f}(x_j(0)) = w_{q_g}(x_i(0)) - w_{q_g}(x_j(0))$ since D_{q_f} and D_{q_g} are compatible. Hence, it follows $w_{q_f}(x_i(0)) - w_{q_g}(x_i(0)) = w_{q_f}(x_j(0)) - w_{q_g}(x_j(0))$ which immediately implies the statement.

These definitions now yield the following proposition. Note that this result can naturally be extended to finitely many functions. In this proof, we will use the notation $i \in X_f$ do denote $x_i \in X_f$ if it is convenient.

(4.18) Proposition
Fix $f, g \in P$. Then, there is a non-trivial multi-weight function w such that f and g are w-homogeneous if and only if there are $q_f \in Com_f$ and $q_g \in Com_g$ such that D_{q_f} and D_{q_g} are compatible and, for any $i \in X_f$ with $i \notin q_f$ but $i \in q_g$ the inequality

$$w_{q_g}(x_i(0)) + v_{i,f} + d(q_g, q_f) \leq w_{q_f}(f) + d(q_f, q_g)$$

holds and an analogous inequality holds with f and g switched.

Proof
" $\Leftarrow$ "
We write $q_f = (j_1, v_1) \times ... \times (j_n, v_n)$ and $q_g = (j'_1, v'_1) \times ... \times (j'_{n'}, v'_{n'})$. The proof consists of two steps.
Step 1: Assume $j_k = j'_{k'}$ for some k, k'. Define $v \in \mathbb{N}^m$ as $\max(w_{q_g}(x_{j_k}(0)), w_{q_f}(x_{j_k}(0)))$. Then, consider $\widehat{w}_{q_f}$ and $\widehat{w}_{q_g}$ defined via

$$\begin{aligned}
&\widehat{w}_{q_g}(x_i(0)) = w_{q_g}(x_i(0)) + v - w_{q_g}(x_{j_k}(0)) = w_{q_g}(x_i(0)) + d(q_g, q_f) \ \forall i \in q_g \\
&\widehat{w}_{q_g}(x_i(0)) = w_{q_g}(x_i(0)) \ \forall i \notin q_g \\
&\widehat{w}_{q_f}(x_i(0)) = w_{q_f}(x_i(0)) + v - w_{q_f}(x_{j_k}(0)) = w_{q_g}(x_i(0)) + d(q_f, q_g) \ \forall i \in q_f \\
&\widehat{w}_{q_f}(x_i(0)) = w_{q_f}(x_i(0)) \ \forall i \notin q_f.
\end{aligned}$$

This implies $\widehat{w}_{q_g}(x_{j_k}(0)) = v = \widehat{w}_{q_f}(x_{j_k}(0))$. Note that f is $\widehat{w}_{q_f}$-homogeneous and g is $\widehat{w}_{q_g}$-homogeneous. In addition, the inequality of the assumption yields

$$\widehat{w}_{q_g}(x_i(0)) + v_{f,i} \leq \widehat{w}_{q_f}(f).$$

Step 2: We will now assume that step 1 is done. We rename $w_{g_f} := \widehat{w}_{q_f}$ and $w_{q_g} := \widehat{w}_{q_g}$. Fix now any $j_i \in q_g \cap q_f$ if $q_g \cap q_f \neq \emptyset$. Then, after step 1, we can conclude that there is an index $j_k \in q_g \cap q_f$ such that $w_{q_f}(x_{j_k})(0)) = w_{q_g}(x_{j_k}(0))$ holds ($j_i = j_k$ is possible). This implies

$$w_{q_f}(x_{j_i}(0)) = w_{q_f}(x_{j_i}(v_{j_i})) - v_{j_i} = w_{q_f}(x_{j_k}(v_{j_k})) - v_{j_i} = w_{q_f}(x_j(0)) + v_{j_k} - v_{j_i}$$

and

$$w_{q_g}(x_{j_i}(0)) = w_{q_g}(x_{j_i}(v'_{j_i})) - v'_{j_i} = w_{q_g}(x_{j_k}(v'_{j_k})) - v'_{j_i} = w_{q_g}(x_j(0)) + v'_{j_k} - v'_{j_i}.$$

Since D_{q_f} and D_{q_g} are compatible, it follows that

$$w_{q_f}(x_{j_i}(0)) = w_{q_f}(x_{j_k}(0)) + v_{j_k} - v_{j_i} = w_{q_g}(x_{j_k}(0)) + v'_{j_k} - v'_{j_i} = w_{q_g}(x_{j_i}(0))$$

holds.
Hence, we define a multi-weight function as follows

$$\begin{aligned} w(x_i(0)) &= w_{q_g}(x_i(0)) = w_{q_f}(x_i(0))\ \forall i \in q_g \cap q_f \\ w(x_i(0)) &= w_{q_g}(x_i(0))\ \forall i \in q_g \setminus q_f \\ w(x_i(0)) &= w_{q_f}(x_i(0))\ \forall i \in q_f \setminus q_g \end{aligned}$$

For all other indices i, we obtain, by definition of w_{q_f} and w_{q_g}, that $w_{q_f}(x_i(0) = w_{q_g}(x_i(0)) = 0$ holds. Therefore we also put $w(x_i(0)) = 0$. In addition, the inequality of the assumption yields for all $i \in (X_f \cap q_g) \setminus q_f$

$$w(x_i(v_{i,f})) = w_{q_g}(x_i(0)) + v_{i,f} \leq w_{q_f}(f)$$

and vice versa. This implies $w(f) = w_{q_f}$ and $w(g) = w_{q_g}(g)$. Thus, both f and g are w-homogeneous.
" $\Rightarrow$ "
Assume that both f and g are w-homogeneous. We write $f = \sum_j a_j m_j$ with $a_j \in k \setminus \{0\}$ and $m_j \in mon(P)$.
Consider the set $H_f = \{i \in I \mid w(x_i(v_i)) = w(f) \text{ and } x_i(v_i)|m_j \text{ for some j } \}$. If it contains more than n elements, then we can omit these additional indices such that there is still an index for any monomial m_j. Thus, we may assume that $\#H_f \leq n$ holds. We define H_q analogously.

Before we continue, we show that we may assume that $H_f \cap H_g$ is not empty if $X_f \cap H_q$ is not empty (or $X_g \cap H_f \neq \emptyset$): If $H_f \cap H_g = \emptyset$, let $i_1, .., i_l$ be the indices in $X_f \cap H_g$. W.l.o.g. i_1 is the index such that

$$w(f) - w(x_{i_j}(v_{i_j,f}))$$

is minimal. Then we define a new multi-weight function w' via

$$\begin{aligned} w'(x_i(0)) &= w(x_i(0)) + w(f) && \forall i \in H_g \\ w'(x_j(0)) &= w(x_j(0)) + w(x_{i_1}(v_{i_1,f})) && \forall j \in H_f \\ w'(x_k(0)) &= w(x_j(0)) && \forall k \notin H_g \cup H_f \end{aligned}$$

Then, f and g are w'-homogeneous, since

$$w'(x_{i_j}(v_{i_j,f})) = w(x_{i_j}(v_{i_j,f})) + w(f) \leq w(x_{i_1}(v_{i_1,f})) + w(f) = w'(x_j(v_j)) = w'(f)$$

and

$$w'(x_{k_j}(v_{k_j,g})) \leq w(x_{k_j}(v_{k_j,g})) + w(x_{i_1}(v_{i_1,f})) \leq w(g) + w(f) = w'(g)$$

hold for all $k_j \in X_g$. In addition, $w'(x_{i_1}(v_{i_1,f})) = w'(f)$, and hence, $i_1 \in H'_g \cap H'_f$ for w'. Hence, we can continue with w' and rename $w := w'$.

We will now define a minimal multi-weight function w_f. For this purpose, we define the multi-weight function w'_f via

$$\begin{aligned} w'_f(x_i(0)) &= w(x_i(0)) \quad \forall i \in H_f \\ w'_f(x_i(0)) &= 0 \qquad\qquad \forall i \notin H_f. \end{aligned}$$

Then, we apply the reduction process of Proposition 4.9 on w'_f to obtain w_f. This reduction process yields an element $n_f \in \mathbb{N}^m$ such that

$$\begin{aligned} w_f(x_i(0)) &= w(x_i(0)) - n_f \quad \forall i \in H_f \\ w_f(x_i(0)) &= 0 \qquad\qquad\qquad \forall i \notin H_f. \end{aligned}$$

holds.

Then, w_f is minimal and thus corresponds to a $q_f \in Com_f$. In addition f is w-graded. Define w_g and q_g analogously. We will now prove that q_f, q_g fulfill the conditions.
We have to show that D_{q_g} and D_{q_f} are compatible. Hence, assume $(i, v_i), (j, v_j) \in q_f$ and $(i, u_i), (j, u_j) \in q_g$. Without loss of generality, we may assume $i = 1$ and $j = 2$. We need to show $v_1 - v_2 = u_1 - u_2$. By definition of w_f and w_g we obtain

$$w(x_1(0)) = w_f(x_1(0)) + n_f = w_g(x_1(0)) + n_g$$

and

$$w(x_2(0)) = w_f(x_2(0)) + n_f = w_g(x_2(0)) + n_g$$

hold. This implies

$$w_f(x_1(0)) - w_f(x_2(0)) = w_g(x_1(0)) - w_g(x_2(0))$$

and, hence,

$$\begin{aligned} & v_1 - v_2 + w_f(x_1(0)) - w_f(x_2(0)) = w_f(x_1(v_1)) - w_f(x_2(v_2)) \\ &= w_f(f) - w_f(f) = 0 = w_g(g) - w_g(g) \\ &= w_g(x_1(u_1)) - w_g(x_2(u_2)) = u_1 - u_2 + w_g(x_1(0)) - w_g(x_2(0)) \end{aligned}$$

implies $v_1 - v_2 = u_1 - u_2$.
We now prove the second condition. Without loss of generality, we only prove one inequality. If $(X_f \setminus q_f) \cap q_g = \emptyset$, the second condition is an empty condition. Hence, we assume that there is an $i \in (X_f \setminus q_f) \cap q_g$. We have already seen that this implies that

$q_f \cap q_g$ is not empty.
Fix now $j \in q_f \cap q_g$. Then, we obtain

$$w_f(x_j(0)) = w(x_j(0)) - n_f$$
$$w_g(x_j(0)) = w(x_j(0)) - n_g$$
$$p = max(w_f(x_j(0)), w_g(x_j(0)))$$

and can conclude

$$d(q_f, q_g) = p - w(x_j(0)) + n_f$$
$$d(q_g, q_f) = p - w(x_j(0)) + n_g$$

Thus, fix any $i \in X_f$ with $i \notin q_f$ but $i \in q_g$. We obtain

$$w_{q_g}(x_i(0)) + v_{f,i} + d(q_g, q_f) = w(x_i(v_{f,i})) + p - w(x_j(0)) \leq w(f) + p - w(x_j(0))$$
$$= w(f) - n_f + p - w(x_j(0)) + n_f = w_{q_f}(f) + d(w, w_{q_f}). \qquad \square$$

(4.19) Remark
Note that this result can be extended to finitely many elements $f_i \in P$. There is Multi-weight function w such that all f_i are w-homogeneous if there are q_{f_i} such that $D_{q_{f_i}}$ are pairwise compatible and the inequalities also holds for any pair of elements.

In the next two examples, we put $X = \{x_1, x_2, x_3, x_4\}, k = \mathbb{R}, m = 2$ and endow $\mathbb{N}^2$ with the graded lexicographic order.

(4.20) Example
We consider

$$f = x_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

and

$$g = x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We pick the elements $q_f \in Com_f$ and $q_g \in Com_g$ defined by

$$q_f = \left(1, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) \times \left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \times \left(4, \begin{pmatrix} 3 \\ 3 \end{pmatrix}\right)$$

and

$$q_g = \left(1, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) \times \left(2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \times \left(3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

These elements correspond to the two multi-weight functions w_f and w_g defined by

index i	$w_f(x_i(0))$	$w_g(x_i(0))$
1	$\begin{pmatrix}1\\3\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$
2	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$
3	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$
4	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$

Note that D_q and D_f are compatible, since the only pair of indices $1, 2 \in q_f \cap q_g$ yields $w_f(x_1(0)) - w_f(x_2(0)) = w_g(x_1(0)) - w_g(x_2(0))$. Furthermore, we have to check the inequality for the index 3, since $3 \in X_f \setminus q_f$ and $3 \in q_f$. First, we note

$$d(q_f, q_g) = \begin{pmatrix}0\\0\end{pmatrix}, \quad d(q_g, q_f) = \begin{pmatrix}1\\2\end{pmatrix}.$$

Thus, we obtain

$$w_{q_g}(x_3(0)) + v_{3,f} + d(q_g, q_f) = \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

and

$$w_{q_f}(f) + d(q_f, q_g) = \begin{pmatrix}3\\3\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}.$$

Consequently, we can construct a multi-function w such that both f and g are w-homogeneous. Following the proof of the last proposition (" $\Leftarrow$ ") yields

index i	$\widehat{w}_f(x_i(0))$	$\widehat{w}_g(x_i(0))$	$w(x_i(0))$
1	$\begin{pmatrix}1\\3\end{pmatrix}$	$\begin{pmatrix}1\\3\end{pmatrix}$	$\begin{pmatrix}1\\3\end{pmatrix}$
2	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$
3	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}1\\3\end{pmatrix}$	$\begin{pmatrix}1\\3\end{pmatrix}$
4	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$

In fact, $w(f) = \begin{pmatrix}3\\3\end{pmatrix}$ and $w(g) = \begin{pmatrix}3\\4\end{pmatrix}$ hold.

(4.21) Example
We will now take a closer look at

$$f = x_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$g = x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There are only one $q_f \in Com_f$ and one $q_g \in Com_g$ which are defined by

$$q_f = \left(1, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) \times \left(2, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

and

$$q_g = \left(1, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) \times \left(2, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

These elements correspond to the two multi-weight functions w_f and w_g defined by

index i	$w_f(x_i(0))$	$w_g(x_i(0))$
1	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
3	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
4	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Note that D_q and D_f are not compatible, because we obtain for the indices 1 and 2

$$w_f(x_1(0)) - w_f(x_2(0)) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \neq \begin{pmatrix} -2 \\ -1 \end{pmatrix} = w_g(x_1(0)) - w_g(x_2(0)).$$

Hence, there is no non-trivial multi-weight function w such that both f and g are w-homogeneous.

§ 5 w-Homogenization in the Multi-Letterplace Ring

In section two and three, we have found many useful results concerning w-graded ideals in P. We were able to improve the computation of Gröbner bases and, in special cases, we were even able to guarantee a finite computation time of truncated Gröbner bases. These results allow us, for instance, to solve the ideal membership problem in w-graded Σ-invariant ideals. However, whenever we embed a certain structure into the Multi-Letterplace ring, we might not obtain w-graded ideals. While the assumption that an ideal is Σ-invariant is natural (especially considering that we can choose Σ), it is rather restrictive to assume the ideals to be w-graded. Therefore, we will develop a *w-homogenization* in this section to extend the applicability of our theory.
In the last section we have instead tried to construct a multi-weight function w such that an ideal is w-graded. While this solution is more effective if it works, we have seen that it is not always possible to construct such a function.

Contrary to the second and third sections, we will now only focus on the Multi-Letterplace ring in order to improve the efficiency of the homogenization.

5.1 Motivation

Consider the following system of difference equations in several variables:

$$\begin{aligned} 0 &= f_1(n+1,m)^2 f_2(n,m) - f_3(n,m+2) \quad && \forall (n,m) \in \mathbb{N}^2 \\ 0 &= f_3(n+1,m+1) f_1(n+1,m) - f_2(n,m) && \forall (n,m) \in \mathbb{N}^2 \end{aligned}$$

whereby each f_i is a function from $\mathbb{N}^m$ to $\mathbb{R}$. We are now interested in determining if an equation $q = 0$ is a consequence of the system and we are also interested in finding an equivalent system which is probably easier to analyze.

In the last section, we will define an embedding of these problems in a more general way. By use of the identification

$$x_i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \leftrightarrow f_i(n+v_1, m+v_2)$$

we obtain the ideal

$$I = \left\langle \Sigma \cdot \left\{ x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^2 x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} - x_3 \begin{pmatrix} 0 \\ 2 \end{pmatrix}, x_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \right\rangle$$

in the Multi-Letterplace ring $P = \mathbb{R}[\{x_1, x_2, x_3\} \times \mathbb{N}^2]$. Hereby, we choose the monoid $\Sigma = \langle \sigma_1, \sigma_2 \rangle$, whereby σ_i denotes the shift in the i-th component. We clearly see that

$$F = \left\{ \left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)^2 x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} - x_3 \begin{pmatrix} 0 \\ 2 \end{pmatrix}, x_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - x_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

is a Σ-basis.
Hence, I is a Σ-invariant ideal, but, we also note that the ideal is not w-graded with respect to the standard multi-weight function or any other multi-weight function. In the next subsection a possible solution of this problem is being presented.

5.2 w-Homogenization

Before we are able to establish a homogenization, we need to introduce some new notations. We will denote $P = k[X \times \mathbb{N}^m]$ and $P' = k[(X \cup \{y\}) \times \mathbb{N}^m]$.
In this subsection, Σ will always denote the submonoid of $End_k(P)$ generated by the shifts $\sigma_1, \ldots, \sigma_m$, while Σ' will analogously be the submonoid of $End_k(P)$ also generated by the shifts $\sigma'_1, \ldots, \sigma'_m$. In addition, w and w' will be the multi-weight functions introduced in Example 3.17.

(5.1) Definition
Let $f \in P$ be an arbitrary element. We can write f as $f = \sum_{i=1}^{n} a_i m_i$ with $a_i \in k \setminus \{0\}$ and $m_i \in mon(P)$. Then, we define the leading w-vector of f as

$$lwv(f) = \max\{w(m_i) \mid i \in \{1, .., n\}\}$$

for any fixed ordering on $\mathbb{N}^m$.
If $v = \begin{pmatrix} i_1 \\ \vdots \\ i_m \end{pmatrix}$, recall the notation

$$y(v) = y \begin{pmatrix} i_1 \\ \vdots \\ i_m \end{pmatrix}.$$

By use of this definition, we obtain an embedding of P in P'.

(5.2) Definition
Fix any ordering $<$ on $\mathbb{N}^m$. Then, we call

$$\tau : \; P \to P', \; f \mapsto y(lwv(f))f$$

the homogenization function of P with respect to $<$.

(5.3) Remark
Note that, by definition of $lwv(f)$, $\tau(f)$ is in fact in $P'_{lwv(f)}$ and hence w'-homogeneous.
Note that τ is in general not multiplicative. However, the equation $\tau(hf) = y(\max\{lwv(h), lwv(f)\})hf$ holds.

(5.4) Example
Consider $X = \{x_1, x_2, x_3\}$ and $m = 2$ and endow $\mathbb{N}^2$ with the graded lexicographic order. Then, we can conclude for $f = x_1 \binom{1}{0} x_2 \binom{0}{0} - x_3 \binom{0}{2}$ that

$$lwv(f) = \binom{0}{2}$$

and thus

$$\tau(f) = x_1 \binom{1}{0} x_2 \binom{0}{0} y \binom{0}{2} - x_3 \binom{0}{2} y \binom{0}{2}$$

hold.

The next lemma will show that τ is compatible with (Σ, Σ'), which will be crucial to prove a one-to-one correspondence between Σ-invariant ideals in P and a special class of ideals in P'.

(5.5) Lemma
The homogenization function commutes with (Σ, Σ'), i.e.

$$\tau \circ \sigma = \sigma' \circ \tau$$

whereby $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ fulfill $\sigma'|_P \equiv \sigma$.

Proof
Note that it is sufficient to prove the statement for σ_i and σ'_i. By definition of w, we have $w(\sigma_i \cdot m) = e_i + w(m)$ for any $m \in mon(P)$ which implies $lwv(\sigma_i \cdot f) = e_i + lwv(f)$. Hence, we can conclude

$$(\tau \circ \sigma_i)(f) = \tau(\sigma_i \cdot f) = y(e_i + lwv(f))(\sigma_i \cdot f) = \sigma'_i \cdot (y(lwv(f))f) = (\sigma'_i \circ \tau)(f)$$

for any $f \in P$. $\square$

After having established an embedding from P to P', we are now interested in an projection from P' onto P.

(5.6) Definition
Consider the ring homomorphism defined by

$$\pi : \; P \to P', \;\; y\begin{pmatrix} j_1 \\ \vdots \\ j_m \end{pmatrix} \mapsto 1$$

and $\pi\mid_P = id$. Then, this mapping is surjective and fulfills $\pi \circ \pi = \pi$.

(5.7) Remark
It is obvious that π is a left inverse of τ, i.e. $\pi \circ \tau = id|_P$.

It is useful to notice that π, similar to τ, commutes with (Σ, Σ').

(5.8) Lemma
The homomorphism π commutes with (Σ, Σ'), i.e.

$$\pi \circ \sigma' = \sigma \circ \pi$$

whereby $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ fulfill $\sigma'|_P = \sigma$.

Proof
Once again, it is sufficient to prove the statement for σ_i and σ'_i. In addition, we only have to consider monomials of total degree 1, since π, σ_i and σ'_i are ring homomorphisms. Hence, fix any $i \in \{1, .., m\}$ and $f \in \{x_j(v) \mid x_j \in X, \, v \in \mathbb{N}^m\} \cup \{y(v) \mid v \in \mathbb{N}^m\}$.
If $f = x_j(v)$, we obtain

$$(\pi \circ \sigma'_i)(x_j(v)) = \pi(x_j(v + e_i)) = x_j(v + e_i) = \sigma(x_j(v)) = (\sigma \circ \pi)(x_j(v))$$

and, if $f = y(v)$, we conclude

$$(\pi \circ \sigma'_i)(y(v)) = \pi(y(v + e_i)) = 1 = \sigma(1) = (\sigma \circ \pi)(y(v)). \qquad \square$$

In order to obtain information about Σ-invariant ideals in P, we have to define a corresponding ideal in P'.

(5.9) Definition
Let $I \subset P$ be a Σ-invariant ideal with Σ-basis $F \subset I$. Then, consider the ideal $I' \subset P'$ generated by $\Sigma' \cdot \tau(F)$. We will call this ideal the *homogeneous analogue of* I (with respect to F). Note that $\tau(F)$ is a Σ'-basis of I'.
Let now J be an ideal in P'. We denote the ideal $J^\pi = \pi(J)$ and call it the *projection of* J.

(5.10) Remark
Note that I' strongly depends on the choice of the Σ-basis F. Hence, since a Σ-basis is not unique, I' is also not unique.

(5.11) Remark
Since τ commutes with (Σ, Σ'), it is easy to see that $\Sigma' \cdot \tau(F) = \tau(\Sigma \cdot F)$ holds.

The fact that $\pi \circ \tau = id|_P$ holds leads us to the assumption that also $I = (I')^\pi$ holds.

(5.12) Proposition
Let I be a Σ-invariant ideal in P with Σ-basis F. Then, $I = (I')^\pi$ holds.

Proof
" $\subseteq$ "
Fix any $\sigma \cdot f \in \Sigma \cdot F$. Note that $\tau(\sigma \cdot f)$ is contained in I' and, hence, we conclude $\pi(\tau(\sigma \cdot f)) \in (I')^\pi$. Recalling Remark 5.7 yields $\pi(\tau(\sigma \cdot f)) = \sigma \cdot f$, which implies $\Sigma \cdot F \subseteq (I')^\pi$. We have just shown that a basis of I is contained in $(I')^\pi$, which implies $I \subseteq (I')^\pi$.
" $\supseteq$ "
Fix any $f \in I'$. By definition of I', there are finitely many $(\sigma^i)' \cdot f_i' \in \Sigma' \cdot \tau(F)$ and $g_i \in P'$ such that $f = \sum_i g_i((\sigma^i)' \cdot f_i')$ holds. By recalling Lemma 5.8, it follows immediately that $\pi(f) = \sum_i \pi(g_i)(\sigma^i \cdot \pi(f_i')) = \sum_i \pi(g_i)(\sigma^i \cdot f_i)$ is contained in I. $\square$

We will now verify that the *homogeneous* analogue of I is in fact w-graded.

(5.13) Proposition
Let $I \subset P$ be a Σ-invariant ideal with Σ-basis F. Then, I' is both Σ'-invariant and w-graded.

Proof
Since $F' := \tau(F)$ is a Σ-basis of I', I' is obviously Σ-invariant.
Let f be an arbitrary element of I'. Then, there are $(\sigma^i)' \cdot f_i' \in \Sigma' \cdot F'$, $a_i \in k$ and $g_i \in mon(P')$ such that $f = \sum_i a_i g_i((\sigma^i)' \cdot f_i')$ holds. Recall that all f_i' are w-homogeneous, so $w'((\sigma^i)' \cdot f_i')$ is well-defined. Denote $v = \max\{w(g_i), w((\sigma^i)' \cdot f_i')\}$ and it follows immediately that $a_i g_i((\sigma^i)' \cdot f_i')$ is contained in P_v'. Since $a_i g_i((\sigma^i)' \cdot f_i') \in I'$, we can conclude that $a_i g_i((\sigma^i)' \cdot f_i') \in I_v'$ holds.
Hence, $f \in \sum_v I_v'$ which implies $I \subset \sum_v I_v$ and thus $I = \sum_v I_v$. $\square$

One of our original goals was finding a way to transfer the membership problem from arbitrary Σ-invariant ideals to w-graded ideals. Assume now that the set $F \subset P$, which is a Σ-basis of I, is finite. For this purpose, we have to introduce a new mapping. But, first of all, we need to recall the following definitions for $f = \sum_i a_i m_i \in P$ with $a_i \neq 0$. The set of all vectors occurring in f is denoted by V_f.

$$V_f = \{v \in \mathbb{N}^m \mid x_j(v) \text{ divides } m_i \text{ for some } i \in \mathbb{N} \text{ and } x_j \in X_f\}.$$

We also need the *maximum vector* v_f of f.

$$v_f = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \text{ with } v_i = \min\{k \in \mathbb{N} \mid k \geq w_i \, \forall w \in V_f\}.$$

(5.14) Remark
Note that $V_{fg} = V_f \cup V_g$ holds for any $f, g \in P$.

From now on, we assume that $\mathbb{N}^m$ is endowed with an ordering satisfying the following condition: For any $v \in \mathbb{N}^m$ there are only finitely many $w \in N^m$ such that $w < v$ holds. One important ordering satisfying this condition is the graded lexicographic order.
In addition, we will make use of the following definition.

(5.15) Definition
Let $F \subset P$ be a finite set. Then we define

$$v_F = \max\{lwv(f_i) \;\mid\; f_i \in F\}.$$

Fix now any g in P. We denote by $y_{f,F}$ a monomial in P' defined by

$$y_{f,F} = \prod_{v \leq v_f + v_F} y(v).$$

This definition now induces a new embedding from P into P'.

(5.16) Definition
Fix any finite subset F of P. Then, τ_F denotes the injective mapping

$$\tau_F : \; P \to P', \; f \mapsto y_{f,F} \cdot f.$$

(5.17) Remark
Once again, we notice that τ_F has the left inverse π like τ. The proof is completely analogous.

In plain words, the function τ_F multiplies the argument with a product of all $y(v)$, whereby v are the vectors less than or equal the sum of the maximum vectors of f and F.

(5.18) Example
Reconsider the case when $X = \{x_1, x_2, x_3\}$ and $m = 2$ and endow $\mathbb{N}^2$ with the graded lexicographic order. Then, we can conclude for $f = x_1 \begin{pmatrix}1\\0\end{pmatrix} x_2 \begin{pmatrix}0\\0\end{pmatrix} - x_3 \begin{pmatrix}0\\1\end{pmatrix}$ and $g = \left\{x_2 \begin{pmatrix}0\\0\end{pmatrix} - x_3 \begin{pmatrix}0\\1\end{pmatrix}\right\}$ that

$$V_f = \left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\}$$

and thus

$$v_f = \begin{pmatrix}1\\1\end{pmatrix}$$

holds. Furthermore, we notice

$$v_g = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{and, consequently,} \quad v_f + v_g = \begin{pmatrix}1\\2\end{pmatrix}$$

which yields

$$\tau_g(f) = y\begin{pmatrix}0\\0\end{pmatrix} y\begin{pmatrix}1\\0\end{pmatrix} y\begin{pmatrix}0\\1\end{pmatrix} y\begin{pmatrix}1\\1\end{pmatrix} y\begin{pmatrix}1\\2\end{pmatrix} \left(x_1 \begin{pmatrix}1\\0\end{pmatrix} x_2 \begin{pmatrix}0\\0\end{pmatrix} - x_3 \begin{pmatrix}0\\1\end{pmatrix}\right).$$

This technical lemma will simplify the next important proposition.

(5.19) Lemma
Fix any finite set F as a Σ-basis of an ideal I in P. Consider now $f = \sum_i a_i h_i(\sigma^i \cdot f_i) \in I$ with $a_i \in k \setminus \{0\}$, $h_i \in mon(P)$ and $\sigma^i \cdot f_i \in \Sigma \cdot F$. Then, we can conclude

$$y\left(lwv\left(\sigma^i \cdot f_i\right)\right) \ \text{divides}\ y_{f,F} \ \ \forall i.$$

Proof
Assume that the statement is wrong. This implies that there is an index j such that $lwv\left(\sigma^j \cdot f_j\right) > v_f + v_F$ holds. Since $f_j \in F$ implies $lwv\left(f_j\right) \leq v_F$ we can conclude with a slight abuse of notation that

$$\sigma^j \left(\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\right) > v_f$$

holds. This immediately implies that the intersection of V_f and $V_{\sigma^j \cdot f_j}$ is empty. If we rewrite $a_j h_j \left(\sigma^j \cdot f^j\right) = \sum_k b_k n_k$ with $n_k \in mon(P)$ and $b_k \in k \setminus \{0\}$, it follows that $m_i \neq n_k$ for all i, k. Hence, all summands of $h_j \left(\sigma^j \cdot f_j\right)$ must be eliminated by other summands.

We will now assume that $h_e \left(\sigma^e \cdot f_e\right) = \sum_l c_l q_l$ with $q_l \in mon(P)$ and $c_l \in k \setminus \{0\}$ satisfies $q_{l'} = n_{k'}$ for some l', k'. By showing that this implies that also $h_e \left(\sigma^e \cdot f_e\right)$ is completely eliminated we can conclude that all these summands can be omitted which implies the statement.

By assumption, it follows that there is a $v \in V_{h_e(\sigma^e \cdot f_e)}$ with $v > v_f + v_F$. Recall that $V_{h_e(\sigma^e \cdot f_e)} = V_{h_e} \cup V_{\sigma^e \cdot f_e}$ holds (cf. Remark 5.14). If $v \in V_{h_e}$, then, since h_e is monomial, $q_l \neq m_i$ for all l, i. Thus, $h_e(\sigma^e \cdot f_e)$ is completely eliminated. If $v \in V_{\sigma^e \cdot f_e}$ we can once again conclude that $lwv(\sigma^e \cdot f_e) > v_f + v_F$ holds, which implies again that $h_e(\sigma^e \cdot f_e)$ is completely eliminated. $\square$

By use of the last definitions and the last lemma, the next proposition realizes the transfer of the membership problem.

(5.20) Proposition

Let I be a Σ-invariant ideal with a finite Σ-basis F in P and fix any $f \in P$. Then,

$$f \in I \quad \Leftrightarrow \quad \tau_F(f) \in I'.$$

Proof

" $\Rightarrow$ "

Let f be an arbitrary element of I. We can write $f = \sum_i a_i h_i(\sigma^i \cdot f_i)$ with $a_i \in k \setminus \{0\}$, $h_i \in mon(P)$ and $\sigma^i \cdot f_i \in \Sigma \cdot F$. Then, we obtain, by considering Lemma 5.19:

$$\tau_F(f) = y_{f,F} \sum_i a_i h_i(\sigma^i \cdot f_i) = \sum_i a_i \frac{y_{f,F}}{y(lwv(\sigma^i \cdot f_i))} h_i y(lwv(\sigma^i \cdot f_i))(\sigma^i \cdot f_i)$$

Since we have already proven that τ commutes with (Σ, Σ'), we can conclude that

$$y(lwv(\sigma^i \cdot f_i))(\sigma^i \cdot f) = \tau(\sigma^i \cdot f_i) = (\sigma^i)' \cdot \tau(f_i) = (\sigma^i)' \cdot f_i'$$

holds. Hereby, f_i' is in $\tau(F)$ and $(\sigma^i)' \in \Sigma'$ is chosen such that $(\sigma^i)' \mid_P = \sigma^i$ holds. This implies, by renaming $g_i := a_i h_i \frac{y_{f,F}}{y(lwv(\sigma^i \cdot f_i))} \in P'$,

$$\tau_F(f) = \sum_i g_i((\sigma^i)' \cdot f_i')$$

which is obviously in I'.

" $\Leftarrow$ "

Denote $F' = \tau(F)$ and recall that π is a ring homomorphism, which commutes with (Σ, Σ'). Assume now $\tau_F(f) \in I'$, this implies the existence of $h_i \in P'$ and $(\sigma^i)' \cdot f_i' \in \Sigma' \cdot F'$ fulfilling $\tau_F(f) = \sum_i h_i((\sigma^i)' \cdot f_i')$. We immediately obtain

$$f = \pi(\tau_F(f)) = \sum_i \pi(h_i)(\sigma^i \cdot f_i).$$

which implies $f \in I$. $\square$

(5.21) Remark
This proposition is wrong if we substitute τ_F with τ:
We endow $\mathbb{N}^2$ with the graded lexicographic order and consider the ideal $I \subset \mathbb{R}[\{x\} \times \mathbb{N}^2]$ with the Σ-basis $\{f\} = \left\{ x \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$. Then, the element $g = x \begin{pmatrix} 2 \\ 1 \end{pmatrix} x \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is contained in I. By definition,

$$\{\tau(f)\} = \left\{ \tau \left(x \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) \right\} = \left\{ y \begin{pmatrix} 1 \\ 2 \end{pmatrix} x \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

is a Σ-basis of I'. Since

$$\tau(g) = \tau \left(x \begin{pmatrix} 2 \\ 1 \end{pmatrix} x \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = x \begin{pmatrix} 2 \\ 1 \end{pmatrix} x \begin{pmatrix} 1 \\ 2 \end{pmatrix} y \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and

$$\sigma' \cdot y \begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq y \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{for all } \sigma' \in \Sigma'$$

we see that $\tau(g) \notin I'$.
However, the implication $\tau(g) \in I' \Rightarrow g \in I$ is correct.

So, if you want to determine, if an element g is contained in a Σ-invariant ideal of P with finite Σ-basis F, we have just proven that you can instead determine if $\tau_F(g)$ is contained in its homogeneous analogue. Since I' is w'-graded, you can use the results of the last sections to solve this problem.

We obtain a weaker results if F is not finite. However, it will still prove beneficial in this section. From now on, we will no longer assume that the order on $\mathbb{N}^m$ satisfies the condition that any $v \in \mathbb{N}^m$ only has finitely many predecessors.

(5.22) Proposition
Let I be an ideal in P with Σ-basis F and I' its homogeneous analogue with respect to F. Then, for any $f \in I$ there is an m in $mon\,(k[\{y\} \times \mathbb{N}^m])$ such that $mf \in I'$.

Proof
If $f \in I$, there are $h_i \in P$ such that $f = \sum_i h_i \left(\sigma^i \cdot f_i\right)$ with $\sigma^i \cdot f_i \in \Sigma \cdot F$ holds. Since this sum is finite, we can define $m = \prod_i y \left(lwv \left(\sigma^i \cdot f_i\right)\right)$. We will now continue similarly to the proof of the finite version of this proposition. Note that

$$mf = m \sum_i h_i(\sigma^i \cdot f_i) = \sum_i \frac{m}{y(lwv(\sigma^i \cdot f_i))} h_i y(lwv(\sigma^i \cdot f_i))(\sigma^i \cdot f_i)$$

holds. Since we have already proven that τ commutes with (Σ, Σ'), we can conclude that

$$y(lwv(\sigma^i \cdot f_i))(\sigma^i \cdot f) = \tau(\sigma^i \cdot f_i) = (\sigma^i)' \cdot \tau(f_i) = (\sigma^i)' \cdot f_i'$$

holds. Hereby, f_i' is in $\tau(F)$ and $(\sigma^i)' \in \Sigma'$ is chosen such that $(\sigma^i)' \mid_P = \sigma^i$ holds. This implies, by renaming $g_i := h_i \frac{m}{y(lwv(\sigma^i \cdot f_i))} \in P'$,

$$mf = \sum_i g_i((\sigma^i)' \cdot f_i')$$

which is obviously in I'. $\square$

We will now see that we can also transfer Gröbner bases from I' to I.

For this purpose, we assume that there is a monomial ordering on P. We extend this ordering in a way similar to the definition of the lexicographic order.

(5.23) Definition
Let $<$ be a monomial order on P and consider the monomial order $<_y$ defined by

$$y(v) <_y y(v') \;\Leftrightarrow\; v <_{gradlex} v'$$

on the set of all monomials in $k[\{y\} \times \mathbb{N}^m]$ (cf. Example 2.16).
We can now define a monomial order on P' via

$$m \prec n \;\Leftrightarrow\; \pi(m) < \pi(n) \text{ or } \left(\pi(m) = \pi(n) \text{ and } \frac{m}{\pi(m)} <_y \frac{n}{\pi(n)}\right)$$

for all $m, n \in mon(P')$. We call $\prec$ the *extension order* of $<$.

From now on, we assume that P' is endowed with an extension order. This allows us to prove the next lemma.

(5.24) Lemma
Let $<$ be a monomial order on P and denote $\prec$ the extension order. Then, we can conclude that

$$lm(f) = \pi\left(lm(mf)\right)$$

holds for any $f \in P$ and $m \in mon\left(k[\{y\} \times \mathbb{N}^m]\right)$.

Proof
Let f, m be arbitrary elements of P and $mon\,(k[\{y\} \times \mathbb{N}^m])$ respectively. It is obviously sufficient to show that $lm(mf) = m \cdot lm(f)$ holds. By considering the definition of the extension order it is clear that $n_1 < n_2$ for $n_i \in mon(P)$ implies $mn_1 \prec mn_2$. Consequently, $lm(mf) = m \cdot lm(f)$ holds as well. $\square$

We are now finally able to prove that we can transfer Gröbner Σ-bases from P' to P.

(5.25) Proposition
Let I be a Σ-invariant ideal in P with Σ-basis F. If $G' \subset P'$ is a Gröbner Σ'-basis of I', then $\pi(G')$ is a Gröbner Σ-basis of I.

Proof
Let f be an arbitrary element in I. We have to prove that $lm(f)$ is contained in the ideal $LM(\Sigma \cdot \pi(G'))$. We have already proven in Proposition 5.22 that there is $f' \in I'$ with $\pi(f') = f$ and $f' = mf$ for some $m \in k[\{y\} \times \mathbb{N}^m]$. Thus, f' has a Gröbner representation with respect to $\Sigma \cdot G'$. This means that we can write

$$f' = \sum_i h_i(s^i \cdot g_i)$$

with $h_i \in P'$ and $s^i \cdot g_i \in G'$. This implies

$$f = \pi(mf) = \pi(f') = \sum_i \pi(h_i)(s^i \cdot \pi(g_i))$$

and in addition $lm(f') \geq lm(h_i(s^i \cdot g_i))$ implies $lm(f) \geq lm(\pi(h_i)(s^i \cdot \pi(g_i)))$: Recalling Lemma 5.24 yields $\pi(lm(f')) = lm(f)$ and it is easy to see that $\pi(lm(g)) \geq lm(\pi(g))$ holds in general. $\square$

§ 6 Multi-Letterplace and Difference Equations

In this section we will show an application of the Multi-Letterplace ring. We will start with a system of difference equations with possibly more than one function $f_i : \mathbb{N}^m \to k$ involved. Note that the analysis of these systems is very important when solving PDE numerically. Hence, these systems not only play an important role in mathematics, but also in physics, chemistry and natural sciences in general. Consider, for instance, the system (without any inital values or boundary conditions)

$$\begin{aligned} 0 &= f_1(n,m)f_2(n,m) - f_1(n+1,m) \\ 0 &= f_2(n+1,m)f_2(n,m+2) + f_1(n+1,m+2). \end{aligned}$$

for all $(n,m) \in \mathbb{N}^2$.

The identification

$$f_i(n+v_1, m+v_2) = x_i\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$$

yields

$$\begin{aligned} 0 &= x_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} x_2\begin{pmatrix} 0 \\ 0 \end{pmatrix} - x_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 &= x_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} x_2\begin{pmatrix} 0 \\ 2 \end{pmatrix} + x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \end{aligned}$$

Then, the Σ-invariant ideal, corresponding to the initial system of difference equations, is Σ-generated by

$$F = \left\{x_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} x_2\begin{pmatrix} 0 \\ 0 \end{pmatrix} - x_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}, x_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} x_2\begin{pmatrix} 0 \\ 2 \end{pmatrix} + x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\},$$

whereby Σ is the monoid freely generated by the shifts σ_1 and σ_2.
More generally, we start with a system of p equations with q algebraically independent functions $f_j : \mathbb{N}^m \to k$. We only consider systems of difference equations with constant coefficients and we do not consider initial values and boundary conditions. Every equation is a finite sum of finite products of $f_j(\alpha)$ with constant coefficients. Thus, for any $1 \leq i \leq q$, there are $l_i \in \mathbb{N}$ multi-places

$$\alpha = \begin{pmatrix} n_1 + v_1 \\ \vdots \\ n_m + v_m \end{pmatrix}$$

such that $f_i(\alpha)$ occurs in the system of difference equations. In our previous example, the set of multi-places corresponding to f_1 would be

$$\left\{ \begin{pmatrix} n \\ m \end{pmatrix}, \begin{pmatrix} n+1 \\ m \end{pmatrix}, \begin{pmatrix} n+1 \\ m+2 \end{pmatrix} \right\}.$$

We will now collect the different function/multi-place combinations in the ordered set M:

$$\begin{aligned} M &= \{f_1(\alpha_{11}), \ldots, f_1(\alpha_{1l_1}) \\ &\quad \vdots \\ &\quad f_q(\alpha_{q1}), \ldots, f_q(\alpha_{ql_q})\}. \end{aligned}$$

We write M_i to denote the i-th element of the set and denote $t = \#M$ There are now p polynomials $G_1, \ldots, G_p \in k[y_1, \ldots, y_t]$ such that the system of difference equations has the form

$$0 = G_j(M_1, \ldots, M_t) \ \ \forall 1 \leq j \leq p.$$

By use of the identification

$$f_i(n_1 + v_1, \ldots, n_m + v_m) = x_i \left(\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \right)$$

we obtain the set $M' \subset P = k[\{x_1, \ldots, x_q\} \times \mathbb{N}^m]$ which results from the set M under the identification above. We can now interpret $G_j(M'_1, \ldots, M'_t)$ as an element of P, which yields the ideal

$$I = \langle \Sigma \cdot \{G_j(M'_1, \ldots, M'_t) \mid 1 \leq j \leq p\} \rangle.$$

This constructions immediately yields the next proposition.

(6.1) Proposition
Every finite system of difference equations with constant coefficients corresponds to an ideal $I \subset P$ with a finite Σ-basis.

Unfortunately, we can not expect I to be w-graded for an arbitrary multi-weight function. Of course, we always have the possibility to homogenize I as shown in the last section. But, we have also obtained some interesting results concerning the existence of a suitable multi-weight function in section four.

(6.2) Definition
Let A, B be two systems of difference equations with constant coefficients. We say that the difference equation q is a consequence of A if every solution of A is also a solution of q.
We say that A and B are equivalent if every consequence of A is also a consequence of B and vice versa.

This definition now yields an interesting implication.

(6.3) Proposition
Let A, B be two systems of difference equations with constant coefficients. Then, A and B are equivalent if the corresponding ideals I_A and I_B are equal.

Proof
According to the construction of I_A, we can conclude that $g \in I_A$ if $g = 0$ is a consequence of A (by interpreting $x_i(v) = f_i(n+v)$ with $n \in \mathbb{N}^m$). Since $I_A = I_B$ implies that the Σ-basis of I_A is contained in B, this means that the original difference equations of A are consequences of B and vice versa. Thus, A and B are equivalent. $\square$

(6.4) Remark
The other implication does not hold. Consider system A

$$0 = f(n) \ \ \forall n \in \mathbb{N}$$

and system B

$$0 = f(n)f(n) \ \ \forall n \in \mathbb{N}.$$

which are obivously equivalent. However,

$$I_A = \langle \{\sigma \cdot \{x(0)\}\} \rangle \supsetneq \langle \{\sigma \cdot \{x(0)x(0)\}\} \rangle = I_B$$

holds.

We will now see that the results of section four become very handy when the system of difference equations is linear.

6.1 Linear Systems

We will now investigate when P can be equipped with a multi-weight function such that the ideal $I \subset P$, corresponding to a system of linear difference equations, is w-graded. For this purpose, we try to find a multi-weight function w which makes F_i w-homogeneous.
Note that our F_i can be written as

$$F_i = a_1 x_{j_1}(v_1) + ... + a_n x_{j_n}(v_n)$$

with $a_l \in k \setminus \{0\}$ and $v_l \in \mathbb{N}^m$.
Assume now that $p = 1$ holds. Then, Proposition 4.9 immediately yields that there is a multi-weight function w so that F_1 is w-homogeneous if and only if

$$j_l = j_{l'} \quad \Rightarrow \quad v_l = v_{l'}$$

holds for all $l, l' \in \{1, .., n\}$.
When p is greater than one, we have to apply Proposition 4.18 and the following remark. Note that in the linear case $X_{F_i} \setminus q_{F_i} = \emptyset$ holds for any $q_{F_i} \in Com_{F_i}$. Hence, we do not have to worry about the inequation. Instead, we can conclude that we find a suitable w for $F_1, .., F_p$ if and only if there are q_{F_i} which are pairwise compatible.

(6.5) Remark
Linear difference equations very often occur in numerical analysis. Especially when solving differential equations numerically, the discretization process often results in difference quotients which yield difference equations. However, these equations mostly contain terms like $u(n+1, m) - u(n, m)$ which correspond to

$$f = x\begin{pmatrix}1\\0\end{pmatrix} - x\begin{pmatrix}0\\0\end{pmatrix}$$

which is obviously never w-homogeneous. However, one can still use homogenization to obtain a w-graded ideal.

(6.6) Example
Consider the system

$$\begin{aligned} 0 &= f_1(n+1, m) - f_2(n+2, m+1) + f_3(n+1, m+1) \\ 0 &= -f_1(n+2, m) + f_2(n+3, m+1) + f_4(n, m). \end{aligned}$$

which corresponds to the ideal

$$I = \left\langle \Sigma \cdot \left\{ g_1 := x_1\begin{pmatrix}1\\0\end{pmatrix} - x_2\begin{pmatrix}2\\1\end{pmatrix} + x_3\begin{pmatrix}1\\1\end{pmatrix} g_2 := -x_1\begin{pmatrix}2\\0\end{pmatrix} + x_2\begin{pmatrix}3\\1\end{pmatrix} + x_4\begin{pmatrix}0\\0\end{pmatrix} \right\} \right\rangle$$

We start to determine a Gröbner basis by searching the first overlap. Note that there is a multi-weight function w such that both g_i are w-homogeneous. In addition, we endow $\mathbb{N}^2$ with the graded lexicographic order and put $x_1 > x_2 > x_3$ in order to endow the Multi-Letterplace ring with the "letter over place" ordering. The first s-polynomial yields:

$$\sigma_1 \cdot g_1 + g_2 = x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, we can conclude $f_4(n, m) = -f_3(n + 2, m + 1)$ for all $(n, m) \in \mathbb{N}^2$. Therefore, we consider the ideal

$$I' = \left\langle \Sigma \cdot \left\{ g_1 := x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} g_3 := -x_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} - x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \right\rangle$$

and obtain

$$g_3 = -\sigma_1 \cdot g_1$$

i.e. the original system of equations can be replaced by

$$0 = f_1(n + 1, m) - f_2(n + 2, m + 1) + f_3(n + 1, m + 1).$$

6.2 A Non-Linear Example

In the nonlinear case, we can not simplify the conditions of Propositions 4.9 and 4.18. Consider the system

$$\begin{aligned} 0 &= -f_2(n)f_1(n + 1) + f_3(n)f_1(n + 1) \\ 0 &= f_1(n)f_2(n + 1) - f_3(n)f_3(n + 1) \end{aligned}$$

corresponding to the w-homogeneous system (we choose $X = \{x := x_1, y := x_2, z := x_3\}$)

$$\begin{aligned} 0 &= -y(0)x(1) + z(0)x(1) =: g_1 \\ 0 &= x(0)y(1) - z(0)z(1) =: g_2 \end{aligned}$$

in the Letterplace ring $\mathbb{R}[X \times \mathbb{N}]$. We will now start the computation of a Gröbner basis of $I = \langle \{\Sigma \cdot \{g_1, g_2\}\} \rangle$. We will endow the Letterplace ring with the graded lexicographic ordering with

$$x(1) > y(1) > z(3) > x(2) > y(2) > \ldots$$

Note that this ordering is compatible with $\Sigma = \langle \sigma \rangle$ whereby σ denotes the shift. Our Σ-basis is w-homogeneous (hereby w denotes the standard weight-function) and we have

$$lm(g_1) = y(0)x(1) \quad lm(g_2) = x(0)y(1).$$

We will now compute the first two s-polynomials. We obtain

$$spoly(\sigma \cdot g_2, g_1) = y(0)(\sigma \cdot g_2) + y(2)g_1 = z(0)x(1)y(2) - y(0)z(1)z(2) =: g_3'$$

which can be reduced by substracting $z(1)(\sigma \cdot g_2)$:

$$g_3 = -\left(g_3' - z(0)(\sigma \cdot g_2)\right) = y(0)z(1)z(2) - z(0)z(1)z(2).$$

Note that g_3 can not be further reduced. We continue with

$$spoly(g_2, \sigma \cdot g_1) = x(2)g_2 + x(0)(\sigma \cdot g_1) = x(0)z(1)x(2) - z(0)z(1)x(2) =: g_4,$$

which is already reduced. The next s-polynomial is also already reduced:

$$\begin{aligned} &spoly(g_2, \sigma \cdot g_3) = z(2)z(3)g_2 - x(0)(\sigma \cdot g_3) \\ &= x(0)z(1)z(2)z(3) - z(0)z(1)z(2)z(3) =: g_5. \end{aligned}$$

We complete our computation with two s-polynomials: After the reduction, $spoly(\sigma \cdot g_4, g_1)$ and $spoly(g_4, \sigma^2 \cdot g_2)$ yield

$$g_6 = z(0)x(1)z(2)x(3) - z(0)z(1)z(2)x(3), \quad g_7 = -z(0)z(1)x(2)y(3) + z(0)z(1)z(2)z(3).$$

We reinterpret g_3 and g_5 as difference equations and obtain

$$\begin{aligned} 0 &= (f_2(n) - f_3(n))f_3(n+1)f_3(n+2) \\ 0 &= (f_1(n) - f_3(n))f_3(n+1)f_3(n+2)f_3(n+3) \end{aligned}$$

for all $n \in \mathbb{N}$. Consequently, we see that if $f_3(n) \neq 0$ for all $n \in \mathbb{N}$ holds, f_1 and f_2 are determined by f_3. Hence, if f_3 is given and $f_3(n) \neq 0$, there is a solution if and only if f_1, f_2, f_3 with $f_1(n) = f_2(n) = f_3(n)$ satisfy the inital equations. In this case, the solution is unique.

§ 7 Conclusions and Future Directions

We have seen in this thesis that many results already proven for the Letterplace ring still hold true for the Multi-Letterplace ring. In the second and third section, the results were more generally obtained for a polynomial ring in combination with a monoid of endomorphisms. While this theory was mainly designed for the Multi-Letterplace ring

and the monoid generated by the shifts, it might still prove beneficial in a different context.
But, there is still more space for generalizations. In this thesis, we have assumed that the monoid Σ is finitely, freely generated and that it is commutative. The assumption that Σ is finitely generated does not play an important role for most of the theoretical results obtained in sections two and three. Hence, it is possible to prove most results of the theory without this assumption. However, it is very important in terms of the feasibility of the presented algorithms. If it is not freely generated, say $\sigma_1 = \sigma_2^3\sigma_3$ or $\sigma_1\sigma_3 = \sigma_2\sigma_4$, then S is not isomorphic to $P' := P[s_1;\sigma_1]...[s_m;\sigma_m]$ since the kernel is not trivial. However, we obtain, for example, $S \cong P'/\langle s_1 - s_2^3 s_3\rangle$ or $S \cong P'/\langle s_1 s_3 - s_2 s_4\rangle$. Consequently, many proofs do not work analogously, but one can still expect useful results. The last assumption is by far the most important one. Basically every result in section two strongly benefits from the fact that Σ is commutative. Hence, I do not expect that a similar approach will lead to significant results without this assumption. However, it may still be possible to obtain similar results from a different approach.
In the fourth section, we have constructed multi-weight functions w which allow us to treat an ideal as w-graded. But, we have also seen that many ideals are not w-graded for any multi-weight function. If the Multi-Letterplace ring will be supported in a computer algebra system, the algorithms presented in this section could be easily implemented and tested for feasibility. I have implemented a rudimentary version in Singular, but, since Singular does not directly support the Multi-Letterplace ring at the moment, its input method is rather inconvenient. In addition, one can still develop an algorithm based on Proposition 4.18.
In section five, we have discussed a w-homogenization which allows us to use some results of section two and three even if the original ideal itself is not w-graded. Contrary to the classical homogenization, the homogenization function multiplies the original element with one common factor. The function τ_F which transfers the membership problem works similarly. Hence, one could try to improve Gröbner basis computation for these elements by taking advantage of this special structure.
Since this thesis focused on the theoretical background of the Multi-Letterplace ring, its application is unfortunately underrepresented. Nevertheless, the last section gives an idea of the possibilities. Furthermore, we have seen a very interesting application in [LSL09] and [LSL13]. It might also be possible to combine the Multi-Letterplace approach with the concept of difference rings to improve the insight. In addition, one could find new applications. Whenever a certain monoid of endomorphisms plays an important role, an embedding into the Multi-Letterplace ring might prove useful.

References

[Eis95] David Eisenbud. *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics. 150. Berlin: Springer-Verlag. xvi, 1995.

[Fey51] Richard Feynman. An operator calculus having applications in quantum electrodynamics. *Phys. Rev., II. Ser.*, 84:108–128, 1951.

[Hig52] Graham Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. (3)*, pages 326–336, 1952.

[Lan02] Serge Lang. *Algebra. 3rd revised ed.* Graduate Texts in Mathematics. 211. New York, NY: Springer. xv, 2002.

[LS13] Roberto La Scala. Groebner bases and gradings for partial difference ideals. *preprint*, 2011-2013.

[LSL09] Roberto La Scala and Viktor Levandovskyy. Letterplace ideals and non-commutative Gröbner bases. *J. Symb. Comput.*, 44(10):1374–1393, 2009.

[LSL13] Roberto La Scala and Viktor Levandovskyy. Skew polynomial rings, Gröbner bases and the letterplace embedding of the free associative algebra. *J. Symb. Comput.*, 48:110–131, 2013.