Non-commutative Subsystems of SINGULAR

Viktor Levandovskyy
Lehrstuhl D für Mathematik, RWTH Aachen University, Germany
Viktor.Levandovskyy@math.rwth-aachen.de

RWTH AACHEN UNIVERSITY

On this poster, $\mathbb{K}$ stands for a field, **lm** for leading monomial function with respect to some monomial ordering on a monoid.

SINGULAR:PLURAL

SINGULAR plural

Status: SINGULAR:PLURAL by Greuel, Levandovskyy, Motsak and Schönemann is distributed together with SINGULAR as its integral part since version 3-0-0 from 2005 on.

Algebras supported: *GR*-algebras, that is factors of *G*-algebras by two-sided ideals. Note, that among other, common algebras of linear operators with *polynomial* coefficients can be realized as *GR*-algebras.

Definition

Let $A = \mathbb{K}\langle x_1, \ldots, x_n\rangle / I$, I is the the two-sided ideal, generated by $\{x_j x_i - c_{ij} x_i x_j - d_{ij} \mid 1 \leq i < j \leq n\}$, where $c_{ij} \in \mathbb{K}^*$ and $d_{ij} \in \mathbb{K}\langle x_1, \ldots, x_n\rangle$. WLOG d_{ij} are given in standard monomials $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. A is called a G-algebra, if

- $c_{ik} c_{jk} \cdot d_{ij} x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} \cdot x_i d_{jk}$ reduces to zero modulo I for all $1 \leq i < j < k \leq n$ and
- there exists a monomial ordering $\prec$ on $\mathbb{K}[x_1, \ldots, x_n]$, such that for each $i < j$, such that $d_{ij} \neq 0$, one has $\mathrm{lm}(d_{ij}) \prec x_i x_j$.

Functionality: all global monomial orderings are supported;

- fast arithmetics for left and two-sided ideals and submodules implemented;
- Gröbner bases, syzygies, transformation matrices and free resolutions;
- **all** Gröbner basics implemented; 14 libraries provided.

New: After redesign in 2007 it supports smart algebra backends (via specialized context-depending implementations) such as SINGULAR:SCA.

Highlights

Gelfand-Kirillov dimension of modules, slim Gröbner bases (by M. Brickenstein), centers and centralizers in *GR*-algebras, free resolutions of left modules.

Subpackages: *D*-modules suite ([`dmod, dmodapp, dmodvar, bfun`].`lib`), control theory package (`control.lib, involut.lib, jacobson.lib`).

Currently under development:

- further memory usage and reductions improvements
- non-commutative homological algebra `nchomalg.lib`
- studies of non-commutative factorization `ncfactor.lib`
- better syzygies and resolutions with the methods of Schreyer and of La Scala and Stillman, as well as other functionality.

SINGULAR:SCA

SINGULAR sca

Status: SINGULAR:SCA by Greuel, Motsak and Schönemann is distributed together with SINGULAR starting from version 3-0-3 (2007) on.

Algebras supported: factor algebras of $C \otimes_{\mathbb{K}} E$, where C is a central localization (at the complement of a prime ideal) of polynomial graded commutative algebra (where for two graded elements a, b one has $ba = (-1)^{|a||b|} ab$) and E is an exterior algebra.

Functionality: provides very fast polynomial arithmetic and intrinsic implementations of standard bases over $\mathbb{Z}_2$-graded commutative algebras (utilizing the presence of zero-divisors) and also fast Gröbner bases, syzygies, free resolutions, ideal and submodule arithmetics over them.

The support of mixed monomial orderings on the commutative part of a graded commutative algebra allows to work within its central localizations.

Highlights: `sheafCohBGG2` from `sheafcoh.lib`

provides much faster sheaf cohomology computation via Bernstein-Gel'fand-Gel'fand correspondence.

Currently under development: preimage of a graded two-sided ideal under a graded algebra homomorphism between graded commutative algebras.

SINGULAR:LETTERPLACE

SINGULAR letterplace

Status: SINGULAR:LETTERPLACE by Levandovskyy and Schönemann with contributions by La Scala: algorithms for homogeneous input are distributed 2009 with SINGULAR version 3-1-0 from 2009 on.
General inhomogeneous algorithms are under beta-testing.

Support: Joint project (KR 1907/3-1; LE 2697/2-1), funded by the German DFG for 2010-2013 with Prof. Dr. M. Kreuzer (Passau).

Algebras supported:

Associative finitely presented $\mathbb{K}$-algebras (including free algebras) over a field.

Functionality: Two-sided Gröbner basis of an ideal up to a given degree bound via Letterplace Gröbner basis algorithm of La Scala and Levandovskyy;

- `fpadim.lib` by G. Studzinski: computation of a $\mathbb{K}$-dimension and of a $\mathbb{K}$-basis of a module are provided;
- `freegb.lib`: letterplace arithmetics and tools.

Highlights

Fast new algorithms for Gröbner bases over associative finitely presented $\mathbb{K}$-algebras.

Currently under development:

- Generalized Letterplace Gröbner basis algorithms
  (a) for an inhomogeneous two-sided ideal (beta-testing),
  (b) for a left ideal in a finitely presented algebra.
- Enhancements of Letterplace Gröbner basis algoritm,
- SAGE-LETTERPLACE interconnection (B. Erocal et al.).

SINGULAR:LOCAPAL

SINGULAR locapal

Status: SINGULAR:PLURAL:LOCAPAL by Levandovskyy and Schönemann (with contributions by O. Motsak and C. Koutschan) is under beta-testing.

Algebras supported: OLGAs, that is an **O**re **L**ocalization of a **G**-**A**lgebra, where the Ore localization is taken with respect to a **commutative** integral subalgebra of a *G*-algebra without 0.
Among other, common algebras of linear operators with *rational* coefficients are supported.

Definition

Let A be a G-algebra in variables $x_1, \ldots, x_m, y_1, \ldots, y_n$. Suppose, that $B \subset A$ is a G-algebra in variables $x_1, \ldots, x_m$. If $S := B \setminus \{0\}$ is an Ore set in A and A admits an elimination ordering $Y \succ X$, then $S^{-1}A$ exists and is called an OLGA. Note, that computations in an OLGA $S^{-1}A$ are algorithmic (*but often complicated*).

Functionality: Provides two different methods for fraction-free computation of a Gröbner basis. Allows to compute:

- a left Gröbner basis of an ideal resp. a submodule of a free module, a left syzygy module, a left normal form,
- an $S^{-1}B$-dimension (**holonomic rank** in case $S^{-1}B \cong \mathbb{K}(X)$).

Connected: `jacobson.lib` provides fraction-free computation of a diagonal form for matrices over an Ore principal ideal domain.