Computer Algebra meets Ring Theory

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In modern applications, more and more ring-theoretic results and techniques are used, in particular when these become implemented and available in computer algebra systems. Evolution of a typical algorithmic story can be sketched as follows:
theory → procedure/algorithm → prototype implementation → powerful implementation → applications → enhancements → ...
Some relevant systems: SINGULAR, GAP, MACAULAY2, MAGMA, SAGE, MAS, JAS, BERGMAN, MAPLE, MATHEMATICA ...

GR-algebras and SINGULAR:PLURAL

Definition

Let $A = K\langle x_1, \dots, x_n\rangle / \langle \{x_j x_i - c_{ij} x_i x_j - d_{ij} \mid 1 \le i < j \le n\}\rangle$, where $c_{ij} \in K^*$ and $d_{ij} \in K\langle x_1, \dots, x_n\rangle$. A is called a **G-algebra** (or an algebra of solvable type or a PBW algebra), if

- standard monomials $\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \mid \alpha \in \mathbb{N}_0^n\}$ form a K-basis of A
- there exists a monomial total well-ordering $\prec$ on $K\langle X\rangle$, such that for each $i < j$ either $d_{ij} = 0$ or $\mathrm{lm}(d_{ij}) \prec x_i x_j$ holds.

A **GR-algebra** is a factor of a G-algebra by a two-sided ideal.

A nice category of GR-algebras includes many algebras, ubiquitous in both theory and in applications.

SINGULAR plural

GR-algebras are **completely** implemented in SINGULAR:PLURAL, feat.

- fast arithmetics for left, right and two-sided ideals and submodules
- Gröbner bases, syzygies, transformation matrices
- homological algebra: free resolutions, some ***Ext*** and ***Tor*** modules
- kernel of a module homomorphism
- preimage of ideal under a ring homomorphism
- dimensions of fin. pres. modules:
  - Gel'fand-Kirillov, projective, homological grade $j(M)$
- subcentral character decomposition
- equidimensional (or purity) filtration
- factorization and factorizing Gröbner algorithm (see just below)

Highlights

- Rich functionality for D-modules and for systems and control theory.
- Toolbox `olga.lib` for explicit manipulations with left (right) fractions in an Ore localization of a G-algebra A at a m.c. Ore set $S \subset A$.

Freely available within SINGULAR at `www.singular.uni-kl.de`

Noncommutative Factorization up to central units

Definition (cf. Bell-Heinle-L. 2017)

Let A be a (not necessarily commutative) domain. We say that A is a finite factorization domain (FFD, for short), if every nonzero, non-unit element of A has at least one factorization into irreducible elements and there are at most finitely many distinct factorizations into irreducible elements up to multiplication of the irreducible factors by central units in A.

Theorem (Bell-Heinle-L. 2017)

Let K be a field and let A be a K-algebra. If there exists a finite-dimensional filtration $\{V_n \colon n \in \mathbb{N}_0\}$ on A such that the associated graded algebra $B = \mathrm{gr}_V(A)$ has the property that $B \otimes_K \overline{K}$ is a (not necessarily commutative) domain, then A is a finite factorization domain.

Corollary: $K\langle x_1, \dots, x_n\rangle$ is an FFD over any field K.

Factorizing Gröbner algorithm

Over an FFD A, we can formulate the factorizing Gröbner algorithm:
For a (left) ideal $L \subset A$, compute (left) ideals $\{B_i\}$ with $L \subseteq \bigcap B_i$.
Note, that this is one of very few possibilities for partially decomposing a given fin. pres. module!

Some References

- J. P. Bell, A. Heinle and V. Levandovskyy. On noncommutative finite factorization domains. *Transactions of the American Mathematical Society*, 369:2675–2695, 2017.
- A. Heinle and V. Levandovskyy. A Factorization Algorithm for G-Algebras and its Applications. *J. Symb. Comput.*, 2017, to appear.

Finitely Presented Algebras with SINGULAR:LETTERPLACE

Letterplace embedding of the free associative algebra $K\langle X\rangle := K\langle x_1, \dots, x_n\rangle$ into the Letterplace ring $K[X \mid \mathbb{N}] := K[x_1(1), \dots, x_n(1), x_1(2), \dots, x_n(2), \dots]$ (commutative but infinitely generated) has been studied by La Scala and L. By invoking the Letterplace Gröbner basis algorithm over $K[X \mid \mathbb{N}]$, one can compute Gröbner bases for ideals in $K\langle X\rangle$.

SINGULAR letterplace

Freely available within SINGULAR at `www.singular.uni-kl.de`

Functionality:

- Two-sided Gröbner basis of an ideal up to a given degree bound
- ideal membership and triviality of the algebra
- Gel'fand-Kirillov of an algebra
- upper bound for the global dimension of an algebra

We aim at similar broad functionality as SINGULAR:PLURAL has reached. Also, we wish to address matrix theory and computations over the free field (Amitsur, Cohn).

Currently under development:

- Recognizable properties of fin.pres. algebras (Noetherianity, semiprimeness etc.)
- Factorization of polynomials over an FFD
- Two-sided factorizing Gröbner algorithm over an FFD
- Bisyzygies and free resolutions

Some References

- R. La Scala and V. Levandovskyy. Skew polynomial rings, Gröbner bases and the letterplace embedding of the free associative algebra. *Journal of Symb. Computation*, 48:1, 2013.

Open Questions (to Ring Theorists)

- Is being a domain a recognizable property?
- Can the computation of the uniform dimension and/or Krull-Rentschler-Gabriel dimension be turned into algorithms?
- Are there other useful and algorithmizable dimensions?
- Which approaches to the (partial) decomposition of modules can be algorithmized?
- Is there an algorithmizable analog of primary decomposition?

Selected desperate needs

Let A be a K-algebra and $S \subset A$ an Ore set. For a left ideal $L \subset A$, denote $L^S := \{a \in A \mid \exists s \in S : sa \in L\}$ (S-closure of L or the contraction of $S^{-1}A \cdot L$ to A).

1. Need new results for computing the Gel'fand-Kirillov dimension of $S^{-1}A$ even for A being a G-algebra. Can this be assisted by computations?
2. Algorithm for L^S is known for A being the nth Weyl algebra and $S = K[X] \setminus \{0\} \subset A$. Can one go beyond this case?

Synergy

Ring and Modules Theory on one side and Computer Algebra on the other have a great potential of mutual enrichment and even enlightening. Let us reveal this potential together!