Linear control systems over Ore algebras: Effective algorithms for the computation of parametrizations

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Abstract

In this paper, we study linear control systems over Ore algebras. Within this mathematical framework, we can simultaneously deal with different classes of linear control systems such as time-varying ordinary differential systems (ODEs), differential time-delay systems, partial differential equations (PDEs), multidimensional discrete systems etc. We give effective algorithms which check whether or not a linear control system over some Ore algebra is controllable, parametrizable, flat or π-free.

Keywords Linear systems over Ore algebras, differential time-delay linear systems, controllability, parametrization, flatness, π-freeness, non-commutative Gröbner bases.

1 Introduction

Over the last thirty years, for practical and theoretical reasons, different new classes of linear control systems have been introduced such as differential time-delay systems, multidimensional systems, partial differential equations, hybrid systems, repetitive systems... All these classes of systems are characterized by the fact that they are governed by new types of mathematical equations and need new techniques in order to analyze their structural properties and to synthesize new control laws. With this growth of new types of control systems, we are led to generalize some previously known results and techniques so that they can be used for more general classes of systems. The main interest is to get similar concepts, techniques and algorithms for studying different classes of linear systems.

In this paper, we study linear control systems over *Ore algebras*. An Ore algebra is an algebra of non-commutative functional operators which satisfy certain commutation rules. For instance, differential/time-delay/advance/discrete shift/divided differences... operators are examples of elements of some Ore algebras. Within this mathematical framework, we can simultaneously deal with different classes of linear control systems such as time-varying ordinary differential systems (ODEs), differential time-delay systems, partial differential equations (PDEs), multidimensional discrete systems... Moreover, the recent extension of *Gröbner bases* to some non-commutative polynomial rings allows us to work effectively in some Ore algebras [4, 5].

The purpose of this paper is to give effective algorithms which check whether or not a linear control system over some Ore algebras is controllable, parametrizable, flat or π-free. These problems have been largely studied in [6, 13, 14, 15] for linear differential time-delay systems and, in [18, 19, 20, 27, 28], for linear multidimensional systems. The main novelty of this paper is to present some algorithms which work for both classes of systems as well as for new ones. In particular, this approach allows us to effectively obtain some parametrizations of a controllable plant and the flat outputs of the flat system. Let us notice that such algorithms were still missing for linear differential time-delay systems (see [6, 13, 14, 15] for more details). Some results developed in this paper seem to give new effective tools for the study of the structural properties of linear systems. They could play important roles for the study of motion planning or tracking [6, 13, 14, 15].

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In an appendix, we have illustrated all the main results and algorithms of the paper with explicit examples. All the computations were done using the Maple package *OreModules* based on the library *Mgfun* [4]. These libraries as well as all the example worksheets of the appendix are available on the web page `http://wwwb.math.rwth-aachen.de/OreModules`. Some results are new (e.g the parametrization of the electric transmission line [13, 25]) and we believe that these examples have some educational interest.

2 Ore algebras

2.1 Definition and examples

In order to deal with different classes of linear systems (ODEs, PDEs, differential time-delay systems, discrete systems, multidimensional systems ...) in a unified framework, we represent them by means of matrices with entries in some *Ore algebras* of functional operators.

Definition 2.1. 1. [10] Let A be an integral domain (i.e. $a\,b = 0,\ a \neq 0 \Rightarrow b = 0$). The *skew polynomial ring* $A[\partial; \sigma, \delta]$ is the non-commutative ring consisting of all polynomials in ∂ with coefficients in A obeying the commutation rule

$$\partial\, a = \sigma(a)\, \partial + \delta(a), \quad a \in A, \tag{1}$$

where σ is a *k-algebra endomorphism of* A, namely $\sigma : A \to A$ satisfies

$$\begin{cases} \sigma(1) = 1, \\ \sigma(a+b) = \sigma(a) + \sigma(b), \quad a, b \in A, \\ \sigma(a\,b) = \sigma(a)\,\sigma(b), \quad a, b \in A, \end{cases}$$

and δ *is a σ-derivation of* A, namely $\delta : A \to A$ satisfies:

$$\begin{cases} \delta(a+b) = \delta(a) + \delta(b), \quad a, b \in A, \\ \delta(a\,b) = \sigma(a)\,\delta(b) + \delta(a)\,b, \quad a, b \in A. \end{cases}$$

2. [5] Let $A = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field k (if $n = 0$ then $A = k$). The skew polynomial ring $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ is called *Ore algebra* if the σ_i's and δ_j's commute for $1 \leq i, j \leq m$ and satisfy:

$$\sigma_i(\partial_j) = \partial_j, \quad \delta_i(\partial_j) = 0, \quad j < i.$$

Remark 2.2. [5, 10] Let $D = A[\partial; \sigma, \delta]$ be a skew polynomial ring. Every element P of D has a unique normal form which is given by $P = \sum_{i=1}^{n} a_i\, \partial^i$ for suitable $a_i \in A$ and $n \in \mathbb{N}$. If $a_n \neq 0$, then the degree of P is n. For every Ore algebra, we get a similar normal form of its elements by moving all products of $\partial_1, \ldots, \partial_m$ on the right in each summand.

Example 2.3. Our first example is the Weyl algebra $A_1 = k[t][\partial; \sigma, \delta]$, where:

$$\sigma = \mathrm{id}_{k[t]}, \quad \delta = \frac{d}{dt}.$$

Rule (1) expresses the commutation of the operator which acts as differentiation on t:

$$\partial\, a = a\, \partial + \frac{da}{dt}, \quad a \in k[t] \qquad (\partial\,(a\,y) = a\,\partial\, y + \left(\frac{da}{dt}\right) y).$$

For instance, we associate $R = [\partial\, I - A(t) : \ -B(t)] \in A_1^{n \times (n+m)}$ with the time-varying Kalman system $\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$, where A and B are polynomial matrices in t.

Similar to polynomial rings in $2\,n$ indeterminates, we can define the so-called Weyl algebra $A_n = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$, where σ_i and δ_i on $k[x_1, \ldots, x_n]$ are the maps

$$\sigma_i = \mathrm{id}_{k[x_1, \ldots, x_n]}, \quad \delta_i = \frac{\partial}{\partial\, x_i}, \quad i = 1, \ldots, n,$$

and every other commutation rule is prescribed by Definition 2.1. In particular, we have:

$$\partial_i\, x_j = x_j\, \partial_i + \delta_{ij}, \quad 1 \leq i, j \leq n, \qquad \text{where } \delta_{ij} = 1, \text{ if } i = j, \text{ and } 0 \text{ else}.$$

Example 2.4. The algebra of shift operators with polynomial coefficients is another special case of an Ore algebra. For $h \in \mathbb{R}$, we define $S_h = k[t][\delta_h; \sigma_h, \delta]$ by:

$$\sigma_h(a)(t) = a(t-h), \quad \delta(a) = 0, \quad a \in k[t].$$

Hence, the commutation rule $\delta_h\, t = (t-h)\, \delta_h$ actually represents the action of the shift operator on polynomials. For instance, the matrix of operators $R = [1 - \delta_h \,:\, -\delta_h^2] \in S_h^{1\times 2}$ is associated with the time-delay system $x(t) = x(t-h) + u(t - 2\,h)$. Let us notice that ∂_h is a *time-delay operator* if $h > 0$ and an *advance operator* if $h < 0$.

Example 2.5. In order to treat differential time-delay systems, we mix the constructions of the two preceding examples. We define the Ore algebra $D_h = k[t][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$, where:

$$\sigma_1 = \mathrm{id}_{k[t]}, \quad \delta_1 = \frac{d}{dt}, \quad \sigma_2(a)(t) = a(t-h), \quad \delta_2 = 0, \quad h \in \mathbb{R}, \quad a \in k[t].$$

For instance, we associate the matrix $R = [\partial\, I - A(t) \,:\, -B(t)\, \delta_h] \in D_h^{n\times(n+m)}$ with the system $\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t-h)$, where A and B are polynomial matrices in t.

If the considered system also involves the advance operator, which is the inverse operator of the time-delay operator, then we may work with the Ore algebra

$$H_h = k[t][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2][\tau_h; \sigma_3, \delta_3],$$

where σ_i, δ_i, $i = 1, 2$, are as above and:

$$\sigma_3(a)(t) = a(t+h), \quad \delta_3 = 0, \quad a \in k[t].$$

Ore algebras with other functional operators can also be defined (discrete shifts, divided differences, q-shift, Eulerian operators ...). We refer to [5, 10] for more examples.

2.2 Properties & Gröbner bases

We summarize the most important properties of Ore algebras that will enable us to computationally deal with modules over Ore algebras.

Proposition 2.6. *[5] If A has the* left Ore property*, namely, for each pair $(a_1, a_2) \in A \times A$, there is a pair $(0,0) \neq (b_1, b_2) \in A \times A$ such that $b_1\, a_1 = b_2\, a_2$, then $A[\partial; \sigma, \delta]$ has the left Ore property, too.*

Proposition 2.7. *[10] If A is an integral domain and σ is injective, then the skew polynomial ring $A[\partial, \sigma, \delta]$ is an integral domain.*

Proposition 2.8. *[10] If A is a left Noetherian ring and σ is an automorphism (e.g. A_n, S_h, D_h, H_h), then the skew polynomial ring $A[\partial; \sigma, \delta]$ is a left Noetherian ring.*

In order to study effectively systems over (non-commutative) polynomial rings, we need to introduce some algorithmic methods based on Gröbner bases. We first need for term orders in order to compare (non-commutative) polynomials.

Definition 2.9. 1. Let D be an Ore algebra. A *term order* $<$ on D is defined as an order on the set of monomials $\mathrm{Mon}(D)$ satisfying $1 < m$ for all monomials $m \in \mathrm{Mon}(D)$ and, if $m_1 < m_2$ holds for two monomials $m_1, m_2 \in \mathrm{Mon}(D)$, then $m_1 \cdot n < m_2 \cdot n$ for all $n \in \mathrm{Mon}(D)$.

2. Given a polynomial $P \in D$ and a term order $<$, one can compare the monomials with non-zero coefficient in P w.r.t. $<$. The greatest of these monomials is the *leading monomial* $\mathrm{lm}(P)$ of P.

Definition 2.10. [1] Let A be a polynomial ring and I be an ideal of A. A set of non-zero polynomials $G = \{g_1, \ldots, g_t\}$ is called a *Gröbner basis* for I if for all $0 \neq f \in I$, there exists $1 \leq i \leq t$ such that $\mathrm{lm}(g_i)$ divides $\mathrm{lm}(f)$.

Remark 2.11. A consequence of the condition that defines Gröbner bases is that every polynomial in I is *reduced* to 0 modulo G, i.e., by iterative division of the leading monomial of f by suitable $g_i \in G$ one obtains the zero polynomial.

For the case of commutative polynomial rings, Buchberger's algorithm ([1], [2]) computes Gröbner bases of polynomial ideals. The next theorem states that this algorithm can be applied for certain Ore algebras. Every Ore algebra within our scope is of this kind.

Theorem 2.12. *[5, 7] Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring with n indeterminates over the field k and $A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ be an Ore algebra with*

$$\sigma_i(x_j) = a_{ij} x_j + b_{ij}, \quad \delta_i(x_j) = c_{ij} \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \tag{2}$$

for certain $a_{ij} \in k \setminus \{0\}$, $b_{ij} \in k$, $c_{ij} \in A$. If the c_{ij} are of total degree at most 1 in the x_i's, then a non-commutative version of Buchberger's algorithm terminates for any term order on $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_m$, and the result of this algorithm is a Gröbner basis w.r.t. the given term order.

An important technique that uses Gröbner bases is *elimination of variables.*

Definition 2.13. [1] Let D be the polynomial ring over k with variables $x_1, \ldots, x_n$, $y_1, \ldots, y_m$. Assume that term orders $<_x$ and $<_y$ on the monomials that contain only the x_i's or y_i's resp. are given. An *elimination order* is the term order defined by

$$m_1 \cdot n_1 < m_2 \cdot n_2 \iff m_1 <_x m_2 \quad \text{or} \quad m_1 = m_2 \text{ and } n_1 <_y n_2,$$

where m_1, m_2 (resp. n_1, n_2) are monomials containing only the x_i's (resp. y_i's).

The elimination order that we shall use in this paper is the one induced by the degree reverse lexicographical orders on $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ resp. This is a very common order called *lexdeg* in the Maple package *Groebner.*

Example 2.14. Given an ideal I of D, we obtain a Gröbner basis of the ideal $I \cap k[y_1, \ldots, y_m]$ by computing the Gröbner basis G of I w.r.t. an elimination order and intersecting G with $k[y_1, \ldots, y_m]$ (which amounts only to omit all polynomials in G that contain any x_i).

3 Module theory over Ore algebras

Let us give a motivation for the use of modules. Let D be an integral domain and let us consider a system of equations

$$\sum_{j=1}^{p} R_{ij}\, y_j = 0, \quad 1 \leq i \leq q, \tag{3}$$

where $R_{ij} \in D$, $p, q \in \mathbb{N}$. By collecting the coefficients R_{ij}, we obtain a matrix $R \in D^{q \times p}$ which, multiplied by the column vector $y = (y_1 : \ldots : y_p)^T$, yields the system (3) again.

We set up the convention that D^r is always considered as the D-module of *row vectors* of length r ($r \in \mathbb{N}$). Let us consider the following left D-morphism (D-linear map):

$$\begin{array}{ccc} D^q & \xrightarrow{.R} & D^p, \\ (P_1 : \ldots : P_q) & \longmapsto & (P_1 : \ldots : P_q)\, R. \end{array}$$

Then, $\operatorname{im} .R = D^q\, R$ is the left D-module generated by the left D-linear combinations of the rows of R (namely, the ring D acts on the elements of $\operatorname{im} .R = D^q\, R$ from the left).

Let us show that the system (3) corresponds to the left D-module $M = D^p / D^q\, R$. If $\{e_i\}_{1 \leq i \leq p}$ (resp. $\{f_j\}_{1 \leq j \leq q}$) is the canonical basis of D^p (resp. D^q), namely e_i is the row vector with 1 in ith position and 0 elsewhere. Let us denote by $\pi : D^p \to M = D^p / D^q\, R$ the left D-morphism which maps every element of D^p to its residue class in M, i.e. modulo $D^q\, R$ ($\pi(\lambda_1) = \pi(\lambda_2) \Leftrightarrow \exists\, \mu \in D^q$ such that $\lambda_1 - \lambda_2 = \mu\, R$). Then, for $i = 1, \ldots, q$, we have

$$f_j\, R = (R_{j1} : \ldots : R_{jp}) = \sum_{i=1}^{p} R_{ji}\, e_i \in D^q\, R \Rightarrow \pi(f_j\, R) = \pi\left(\sum_{i=1}^{p} R_{ji}\, e_i\right) = \sum_{i=1}^{p} R_{ji}\, \pi(e_i) = 0,$$

and thus, if we denote by $y_i = \pi(e_i)$ the residue class of e_i in M, then M is defined by

$$\sum_{i=1}^{p} R_{ji}\, y_i = 0, \quad 1 \leq j \leq q, \quad \Leftrightarrow \quad R\, y = 0,$$

as well as by the left D-linear combinations of its equations, where $y = (y_1 : \ldots : y_p)^T$. The left D-module M is *finitely generated* because every element $m \in M$ can be written as $m = \sum_{i=1}^{p} P_i\, y_i$, where $P_i \in D$.

Definition 3.1. The finitely generated left D-module $M = D^p / D^q\, R$ is *associated with* (3).

Example 3.2. Let us reconsider the Ore algebra $D_h = \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ defined in Example 2.5 and the following wind tunnel model defined in [12]

$$\left\{\begin{array}{l} \dot{x}_1(t) = -a\, x_1(t) + k\, a\, x_2(t-h), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -\omega^2\, x_2(t) - 2\,\zeta\,\omega\, x_3(t) + \omega^2\, u(t), \end{array}\right. \tag{4}$$

where a, k, ζ and ω are real constants. The system (4) gives rise to the following matrix

$$R = \left(\begin{array}{cccc} \partial + a & -k\, a\, \delta_h & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{array}\right) \in D_h^{3\times 4}, \tag{5}$$

and thus, the system (4) corresponds to the left D_h-module $M = D_h^4 / D_h^3\, R$.

Definition 3.3. [24] A family $(M_i)_{i\in\mathbb{Z}}$ of D-modules together with a family $(d_i)_{i\in\mathbb{Z}}$ of D-module morphisms $d_i : M_i \to M_{i-1}$ is a *complex*, if $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. We write:

$$\ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots \tag{6}$$

The complex (6) is called *exact at position* i if the *defect of exactness of (6) at position* i,

$$H(M_i) = \ker d_i / \operatorname{im} d_{i+1},$$

is equal to 0 or, equivalently, if $\ker d_i = \operatorname{im} d_{i+1}$. The complex (6) is called *exact* if it is exact at every position. Finally, the exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$, i.e. f is injective, g is surjective and $\ker g = \operatorname{im} f$, is called a *short exact sequence.*

We recall some properties of D-modules that will be important in the course of the paper.

Definition 3.4. [24] Let D be an integral domain which is a (left) Ore ring, and let M be a finitely generated (left) D-module.

1. The D-module M is *free* if it is isomorphic to D^r for a certain $r \in \mathbb{Z}_{\geq 0}$.
2. M is a *projective* D-module if there exist a free D-module F and a D-module N such that $F \cong M \oplus N$.
3. Define by $\hom_D(M, D)$ the right D-module consisting of all D-morphisms $M \to D$. The set $\hom_D(\hom_D(M, D), D)$ is again a left D-module and there is a canonical map
$$\varepsilon_M : M \longrightarrow \hom_D(\hom_D(M, D), D), \quad \varepsilon_M(m)(f) = f(m),$$
$m \in M, f \in \hom_D(M, D)$. If ε_M is an isomorphism, then M is called *reflexive.*
4. The set $t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : \; P\, m = 0\}$ is a submodule of M which is called the *torsion submodule* of M. The non-zero elements of $t(M)$ are all the *torsion elements* of M. We have the exact sequence $0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0$.
5. The D-module M is called *torsion-free* if its torsion submodule is trivial, i.e. $t(M) = 0$.

Proposition 3.5. *[24] Let D be an integral domain and M be a left D-module. Then, we have the following implications among these concepts:*

free ⇒ projective ⇒ reflexive ⇒ torsion-free.

Theorem 3.6.

- *[10, 24] If D is a* Dedekind domain *(e.g. $D = A_1$), then a torsion-free generated D-module is projective. Moreover, if D is a* principal ideal domain *(e.g. the commutative polynomial ring $D = k[x]$ with coefficients in a field k), then a torsion-free generated D-module is free.*
- *[10, 24] Every projective module over a commutative polynomial ring with coefficients in a field is free.*

In the following sections, we shall develop effective algorithms based on Gröbner bases in order to check whether or not a left D-module M associated with a linear control system (e.g. the differential time-delay system (4)) is torsion-free, reflexive, ..., or free. In the next section, we shall give some system interpretations to these properties of modules.

4 System interpretations of module properties

In the sequel, we shall not precise the Ore algebra D that we use (e.g. if we study time-varying ordinary differential systems (resp. differential time-delay systems, ...), then $D = A_1$ (resp. $D = D_h$, ...)). Let us give some system interpretations of the properties of modules.

Definition 4.1.

- [18, 27] An *observable* of a linear control system $R\,y = 0$ is a *scalar* D-linear combination of the components of y (i.e. of the system variables including inputs, states, outputs ...). An observable $\phi(y)$ is called *autonomous* if it satisfies some linear equations by itself, namely $P_1\,\phi(y) = 0, \ldots, P_r\,\phi(y) = 0$, where $P_i \in D$. An observable is said to be *free* if it is not autonomous.
- [6, 18, 27] A linear control system is said to be *controllable* if every observable is free.
- [18, 19, 20] A linear control system $R\,y = 0$ is *parametrizable* if there exist a matrix R_{-1} with entries in D and arbitrary functions z such that the compatibility conditions of the inhomogeneous system $y = R_{-1}\,z$ is exactly generated by $R\,y = 0$ or, equivalently, if there exists $R_{-1} \in D^{p\times m}$ such that $M = D^p/D^q\,R \cong D^p\,R_{-1}$. Then, R_{-1} is called a *parametrization* of the system $R\,y = 0$ and z is the *potential* of the system.
- [6, 13, 23] A linear control system is *π-free* if it is parametrizable and there exist a matrix S_{-1} with entries in D and $0 \neq \pi \in D$ such that $S_{-1}\,R_{-1} = \pi\,I$, where I is the identity matrix.
- [6, 13] A linear control system is *flat* (or *free*) if it is parametrizable and every component z_i of the potential z is an observable of the system or, equivalently, if there exists a parametrization $R_{-1} \in D^{p\times m}$ which admits a left-inverse $S_{-1} \in D^{m\times p}$, i.e. $S_{-1}\,R_{-1} = I_m$. Then, z is called a *flat output* of the system.

Proposition 4.2. *Let D be an Ore algebra, $R \in D^{q\times p}$ and $M = D^p/D^q\,R$ be the left D-module associated to the system $R\,y = 0$ (see Definition 3.1).*

1. *[18, 19, 27] An observable of the system is an element of the left D-module M.*
2. *[6, 18, 19, 27] The autonomous elements of the system are in one-to-one correspondence with the torsion elements of M.*
3. *[6, 13, 18, 19] The system is controllable if and only if M is a torsion-free left D-module.*
4. *[18, 19, 20] The system is parametrizable iff M is a torsion-free left D-module.*
5. *[6, 13] If D is a commutative polynomial ring, then the system is π-free iff there exists a polynomial $0 \neq \pi \in D$ such that $D_\pi \otimes_D M = \left\{\frac{m}{a} \mid m \in M,\ a = \pi^n,\ n \in \mathbb{Z}_{\geq 0}\right\}$ is a free $D_\pi = \left\{\frac{b}{a} \mid b \in D,\ a = \pi^n,\ n \in \mathbb{Z}_{\geq 0}\right\}$-module.*
6. *[6, 13] The system is flat if and only if M is a free left D-module. Then, a basis of M is a flat output of the system.*

Let us recall the following well-known concepts of coprimeness developed in the literature of multidimensional systems. These concepts allow us to classify the systems.

Definition 4.3. [6, 11, 18, 20, 28] Let $R \in D^{q\times p}$ be a *full row rank matrix* with entries in a commutative polynomial ring $D = \mathbb{R}[x_1, \ldots, x_n]$ (namely, its q rows are D-linearly independent). Then,

- R is *minor left-prime* if the greatest common factor of all the q by q minors of R is 1.
- R is *weakly zero left-prime* if all the q by q minors of R only vanish simultaneously on a finite number of points of $\mathbb{C}^n$.
- R is *zero left-prime* if all the q by q minors of R does not vanish simultaneously in $\mathbb{C}^n$.

Theorem 4.4. *[6, 11, 18, 20] Let $R \in D^{q\times p}$ be a* full row rank matrix *with entries in a commutative polynomial ring $D = \mathbb{R}[x_1, \ldots, x_n]$. Then, we have:*

1. *R is minor left-prime iff the D-module $M = D^p/D^q\,R$ is torsion-free.*
2. *If $n = 3$, then R is weakly zero left-prime iff the D-module $M = D^p/D^q\,R$ is reflexive.*
3. *R is zero left-prime iff the D-module $M = D^p/D^q\,R$ is free.*

Hence, the concepts of torsion-freeness, reflexiveness, projectiveness and freeness generalize to non-commutative polynomial rings the well-known concepts of primenesses developed for multidimensional systems [20]. The next sections are dedicated to give effective algorithms which check the modules properties and compute the parametrizations and flat outputs.

5 Syzygy modules

Let M be a finitely generated left module over a left noetherian ring D. We can reformulate the fact that M is finitely generated by saying that there exists a surjective D-morphism $\varphi : D^p \to M$ which maps the ith vector of the canonical basis $\{e_i\}_{1\leq i\leq p}$ of D^p to some m_i. We have the exact sequence:

$$\begin{array}{ccccc} D^p & \xrightarrow{\varphi} & M & \longrightarrow & 0, \\ e_i & \longmapsto & m_i. & & \end{array}$$

This map may fail to be injective since there may be linear relations among the $\{m_i\}_{1\leq i\leq p}$:

$$\ker\varphi = \{P = (P_1 : \ldots : P_p) \in D^p \mid \varphi(P) = \sum_{i=1}^{p} P_i\,\varphi(e_i) = \sum_{i=1}^{p} P_i\, m_i = 0\}. \tag{7}$$

Definition 5.1. [21, 24] The D-linear relations among the $m_1, \ldots, m_p$ form the left D-module $S(M)$ defined by (7) and is called a *syzygy module* of M (this module is uniquely defined up to *projective equivalence*).

Since D is a left noetherian ring, $S(M)$ is a finitely generated left D-module. Therefore, we can again find a suitable free D-module D^q and a map ψ sending the canonical basis vectors of D^q to the generators of $S(M)$ and we have the following exact sequence:

$$D^q \xrightarrow{\psi} D^p \xrightarrow{\varphi} M \longrightarrow 0.$$

This exact sequence is called a *finite presentation* of the left D-module M and M is said to be *finitely presented.* Let us notice that, in the canonical bases of D^q and D^p, ψ is defined by the matrix whose ith row corresponds to the ith generators of $S(M)$. Finally, iterating the preceding construction, we get the definition of a *free resolution* of the D-module M.

Definition 5.2.

1. [24] An exact sequence of the form

$$\ldots\ldots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0 \tag{8}$$

is called a *free resolution* of M if the D-modules F_i are free left D-modules. The D-module $S_i(M) = \ker d_i$ is called the *ith syzygy module of* M.

2. [24] If the D-modules F_i in (8) are projective, then (8) is a *projective resolution* of M.

3. [24] Let $0 \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$ be a projective resolution of M. We define the *length* of this resolution to be n.

4. [24] The minimal length of a left projective resolution of M is called the *projective dimension* $\mathrm{pd}_D(M)$ of the left D-module M. The projective dimension may be infinite.

5. [24] We set $\mathrm{lgld}\, D = \sup\{\mathrm{pd}_D(M) \mid M \text{ a left } D\text{-module}\} \in \mathbb{N} \cup \{\infty\}$, which is called the *left global dimension* of D.

We describe the computational tools for the construction of free resolutions. The techniques to compute syzygy modules use Gröbner bases and elimination technique (see section 6.1 of [2]). Let D be an Ore algebra which satisfies (2) and L a finitely generated left D-module which is a submodule of a free D-module D^p, $p \in \mathbb{N}$. Thus, a set of generators of L consists of row vectors in D^p.

Algorithm 5.3.
Input: Set of generators $\{R_1, \ldots, R_q\} \subset D^p$ of the D-module L, where $R_i = (R_{i1} : \ldots : R_{ip})$, $i = 1, \ldots, q$.
Output: $S \in D^{r\times q}$ such that $D^r\, S$ is a set of generators of the syzygy module of L.

Syzygies $(R_1, \ldots, R_q)$
$P \leftarrow \{\sum_{j=1}^{p} R_{ij}\,\lambda_j - \mu_i \mid i = 1, \ldots, q\}$.
Compute the Gröbner basis G of P in the left free D-module generated by $\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q$, namely $\bigoplus_{i=1}^{p} D\,\lambda_i \oplus \bigoplus_{i=1}^{q} D\,\mu_i$, w.r.t. an elimination order (eliminating the λ_i's).
$S = (S_{ij}) \in D^{r\times q} \leftarrow G \cap \oplus_{i=1}^{q} D\,\mu_i = \{\sum_{j=1}^{q} S_{ij}\,\mu_j \mid i = 1, \ldots, r\}$.

Remark 5.4. Let us suppose that we have $R \in D^{q\times p}$ and the left D-module $M = D^p/D^q\,R$. Then, we can apply the preceding algorithm to the set formed by

$$R_i = (R_{i1} : \ldots : R_{ip}) \in L = D^q\,R \subseteq D^p, \quad i = 1, \ldots, q,$$

in order to obtain the matrix $S = (S_{ij}) \in D^{r\times q}$ such that

$$S_2(M) = \ker .R = \{(P_1 : \ldots : P_q) \in D^q \mid \sum_{i=1}^{q} P_i\,R_i = 0\} = D^r\,S$$

and we obtain the exact sequence $D^r \xrightarrow{.S} D^q \xrightarrow{.R} D^p \xrightarrow{\pi} M \longrightarrow 0$. Iterating the process, we obtain a free resolution of the left D-module M.

For further developments and optimization of the technique, see [8].

Example 5.5. Let D_h be the differential time-delay Ore algebra introduced in Example 3.2, $R \in D_h^{3\times 4}$ defined by (5) and the D_h-module $L = D_h^4\,R^T$ generated by the rows of the matrix:

$$R^T = \begin{pmatrix} \partial + a & 0 & 0 \\ -k\,a\,\delta_h & \partial & \omega^2 \\ 0 & -1 & \partial + 2\,\zeta\,\omega \\ 0 & 0 & -\omega^2 \end{pmatrix} \in D_h^{4\times 3}.$$

The Gröbner basis of

$$P = \{(\partial + a)\,\lambda_1 - \mu_1, -k\,a\,\delta_h\,\lambda_1 + \partial\,\lambda_2 + \omega^2\,\lambda_3 - \mu_2, -\lambda_2 + (\partial + 2\,\zeta\,\omega)\,\lambda_3 - \mu_3, -\omega^2\,\lambda_3 - \mu_4\}$$

w.r.t. the elimination ordering induced by the degree reverse lexicographical orderings on $\lambda_1 > \lambda_2$ and $\mu_1 > \mu_2 > \delta_h > \partial$ resp. is:

$$\begin{aligned} G = \{\,&\omega^2\,\lambda_2 + \partial\,\mu_4 + \omega^2\,\mu_3 + 2\,\zeta\,\omega\,\mu_4,\ \omega^2\,k\,a\,\delta_h\,\lambda_1 + \omega^2\,\mu_2 + \omega^2\,\partial\,\mu_3 + (\partial^2 + 2\,\zeta\,\omega\,\partial + \omega^2)\,\mu_4, \\ &\omega^2\,k\,a\,\delta_h\,\mu_1 + (\omega^2\,\partial + \omega^2\,a)\,\mu_2 + (\omega^2\,\partial^2 + \omega^2\,a\,\partial)\,\mu_3 + (\partial^3 + 2\,\zeta\,\omega\,\partial^2 + a\,\partial^2 + \omega^2\,\partial \\ &+2\,a\,\zeta\,\omega\,\partial + a\,\omega^2)\,\mu_4,\ (\partial + a)\,\lambda_1 - \mu_1\}. \end{aligned}$$

Intersecting G with $\oplus_{i=1}^{3} D\,\mu_i$ we get

$$\begin{aligned} S = \{\ &\omega^2\,k\,a\,\delta_h\,\mu_1 + (\omega^2\,\partial - \omega^2\,a)\,\mu_2 + (\omega^2\,\partial^2 + \omega^2\,a\,\partial)\,\mu_3 + (\partial^3 + 2\,\zeta\,\omega\,\partial^2 + a\,\partial^2 + \omega^2\,\partial \\ &+2\,a\,\zeta\,\omega\,\partial + a\,\omega^2)\,\mu_4\}. \end{aligned}$$

If we denote by R_{-1}^T the following row vector with entries in D_h

$$R_{-1}^T = (\omega^2\,k\,a\,\delta_h :\ \omega^2\,\partial + \omega^2\,a :\ \omega^2\,\partial^2 + \omega^2\,a\,\partial :\ \partial^3 + 2\,\zeta\,\omega\,\partial^2 + a\,\partial^2 + \omega^2\,\partial + 2\,a\,\zeta\,\omega\,\partial + a\,\omega^2),$$

then we obtain the following free resolution of the D_h-module $N = D_h^3/D_h^4\,R^T$:

$$0 \longrightarrow D_h \xrightarrow{.R_{-1}^T} D_h^4 \xrightarrow{.R^T} D_h^3 \xrightarrow{\pi} N \longrightarrow 0. \tag{9}$$

Proposition 5.6. *[10] Let A be an integral domain with a finite left global dimension* lgld A *and σ an automorphism. Then, the left global dimension of $A[\partial; \sigma, \delta]$ satisfies:*

$$\mathrm{lgld}\,A \leq \mathrm{lgld}\,A[\partial; \sigma, \delta] \leq \mathrm{lgld}\,A + 1.$$

Moreover, if $\mathbb{Q} \subseteq k$ is a field, then we have $\mathrm{lgld}\,k[x_1, \ldots, x_n] = n$ *and* $\mathrm{lgld}\,A_n = n$.

6 Involutions

Definition 6.1. Let k be a field and D be a (non-commutative) k-algebra. An *involution* θ of D is a k-linear map $\theta : D \to D$ satisfying

$$\begin{aligned} \theta(a_1 \cdot a_2) &= \theta(a_2) \cdot \theta(a_1), \qquad a_1, a_2 \in D, \\ \theta \circ \theta &= \mathrm{id}_D, \end{aligned} \tag{10}$$

i.e. θ is an anti-automorphism of order two of the k-algebra D.

Proposition 6.2. *By means of an involution θ of D, left D-modules can be turned into right D-modules and vice versa: let D be a k-algebra and M be a right D-module and θ an involution of D, then we can define the left D-module $\widetilde{M}$, which is equal to M as a set and which is endowed with the same addition as M, but with the following right action of D:*

$$a\, m = m\, \theta(a), \quad m \in \widetilde{M}, \quad a \in D.$$

Property (10) *of θ ensures that $\widetilde{M}$ is a well-defined left D-module.*

Example 6.3.

1. Let $D = k[x_1, \ldots, x_n]$ be a commutative polynomial ring. Then, a trivial involution of D can be defined by $x_i \mapsto x_i, \quad 1 \leq i \leq n$.
2. Let $A_n = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ be the Weyl algebra (see Example 2.3). An involution θ of A_n can be defined by $x_i \mapsto x_i, \quad \partial_i \mapsto -\partial_i, \quad 1 \leq i \leq n$.
3. Let $S_h = k[t][\delta_h; \sigma_h, \delta]$ be the shift Ore algebra (see Example 2.4). An involution θ of S_h can be defined by $t \mapsto -t, \quad \delta_h \mapsto \delta_h$.
4. Let $H_h = k[t][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2][\tau_h; \sigma_3, \delta_3]$ be the Ore algebra of differential time-delay and advance operators (see Example 2.5 for more details). An involution θ of H_h can be defined by $t \mapsto t, \quad \partial \mapsto -\partial, \quad \delta_h \mapsto \tau_h, \quad \tau_h \mapsto \delta_h$.

Definition 6.4. Let D be an Ore algebra with an involution θ, $R \in D^{q \times p}$ and $M = D^p / D^q\, R$ is a left module. Then, the *transposed module* of M is the left D-module defined by:

$$N = D^q / D^p\, \theta(R). \tag{11}$$

Hence, the left D-module $N = D^q / D^p\, \theta(R)$ corresponds to the linear system $\theta(R)\, z = 0$, where $z = (z_1 : \ldots : z_q)^T$.

Example 6.5.

1. If D is a commutative ring (e.g. $D_h = \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ defined in Example 3.2), then the involution θ is just the transposition of matrices, namely $\theta(R) = R^T$, and the transposed D-module is defined by $N = D^q / D^p\, R^T$.
2. Let us consider the Ore algebra $H_h = k[t][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2][\tau_h; \sigma_3, \delta_3]$ defined in Example 2.5 and $R = [t\, \partial :\, -t^2\, \delta_h] \in H_h^{1 \times 2}$. Then, using 4 of Example 6.3, we obtain:

$$\theta(R) = \begin{pmatrix} \partial\, t \\ -\tau_h\, t^2 \end{pmatrix} = \begin{pmatrix} t\, \partial + 1 \\ -(t+h)^2\, \tau_h \end{pmatrix} \in H_h^{2 \times 1}.$$

7 Computation of extension modules

Definition 7.1. [24] Let M be a finitely generated left D-module, N a left D-module and a free resolution $\ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$ of M. Then, the defects of exactness of the complex

$$\ldots \longleftarrow \hom_D(F_2, N) \xleftarrow{d_2^*} \hom_D(F_1, N) \xleftarrow{d_1^*} \hom_D(F_0, N) \longleftarrow 0,$$

where d_i^* is defined by $d_i^*(f) = f \circ d_i$ for $f \in \hom_D(F_{i-1}, N)$, $i \geq 1$, are defined by:

$$\begin{cases} \operatorname{ext}_D^0(M, N) &= \ker d_1^* = \hom_D(M, N), \\ \operatorname{ext}_D^i(M, N) &= \ker d_{i+1}^* / \operatorname{im} d_i^*, \quad i \geq 1. \end{cases}$$

In the following, we shall only consider the case $N = D$, and thus, $\operatorname{ext}_D^i(M, D)$, $i \in \mathbb{Z}_{\geq 0}$.

Proposition 7.2. *[24] The right D-module $\operatorname{ext}_D^i(M, D)$ only depends on M, i.e. one can choose any free resolution of M to compute $\operatorname{ext}_D^i(M, D)$, $i \in \mathbb{Z}_{\geq 0}$.*

The next algorithm gives a description of a left D-module $\widetilde{\operatorname{ext}_D^1(M, D)}$, which corresponds to the right D-module $\operatorname{ext}_D^1(M, D)$ (see Proposition 6.2).

Algorithm 7.3.
Input: Ore algebra D satisfying (2) with an involution θ and $R = (R_1^T : \ldots : R_q^T)^T \in D^{q \times p}$.
Output: A list $L = [L_1,\, L_2]$ of two matrices such that:
$L_1 \in D^{m \times q}$ is such that $\widetilde{\operatorname{ext}_D^1(M, D)} = D^m\, L_1 / D^p\, \theta(R)$, where $M = D^p / D^q\, R$,
$L_2 \in D^{q \times r}$ is such that $L_1 = \textsc{Syzygies}\,(L_2)$.

PRE-EXT1 (R)
  $R_2 \leftarrow$ SYZYGIES (R),
  $L_2 \leftarrow \theta(R_2)$,
  $L_1 \leftarrow$ SYZYGIES (L_2),
  $L \leftarrow [L_1,\, L_2]$.

Example 7.4. Let us compute the $\text{ext}^i_{D_h}(N, D_h)$ of the D_h-module N defined in Example 5.5. In Example 5.5, we have already computed the free resolution (9) of $N = D_h^3/D_h^4\, R^T$. Thus, we have $\ker .R^T = D_h\, R_{-1}^T$, where R_{-1}^T is defined in Example 5.5. Then, using the fact that $D_h = \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ is a commutative polynomial ring, we obtain that $\theta(R_{-1}^T) = R_{-1}$ (see 1 of Example 6.3). Hence, we have the following complex

$$0 \longleftarrow D_h \xleftarrow{.R_{-1}} D_h^4 \xleftarrow{.R} D_h^3 \longleftarrow 0$$

and its defects of exactness are defined by:

$$\begin{aligned} \text{ext}^1_{D_h}(N, D_h) &= \ker .R_{-1}/D_h^3\, R, \quad (12)\\ \text{ext}^2_{D_h}(N, D_h) &= D_h/D_h^4\, R_{-1}. \quad (13) \end{aligned}$$

Following Algorithm 7.3, we need to compute the syzygy module of $D_h^4\, R_{-1}$. The Gröbner basis of

$$P = \{\, (\omega^2\, k\, a\, \delta_h)\, \lambda - \mu_1,\ (\omega^2\, \partial + \omega^2\, a)\, \lambda - \mu_2, (\omega^2\, \partial^2 + \omega^2\, a\, \partial)\, \lambda - \mu_3, \\ (\partial^3 + 2\, \zeta\, \omega\, \partial^2 + a\, \partial^2 + \omega^2\, \partial + 2\, a\, \zeta\, \omega\, \partial + a\, \omega^2)\, \lambda - \mu_4\}$$

w.r.t. the elimination ordering induced by the degree reverse lexicographical orderings on λ and $\mu_1 > \mu_2 > \mu_3 > \mu_4 > \delta_h > \partial$ resp. is:

$$G = \{\, \omega^2\, \mu_2 + (\partial + 2\, \zeta\, \omega)\, \mu_3 - \omega^2\, \mu_4, \partial\, \mu_2 - \mu_3,\ -(\partial + a)\, \mu_1 + k\, a\, \delta\, \mu_2, \\ (\partial^2 + a\, \partial)\, \mu_1 - k\, a\, \delta\, \mu_3,\ \omega^2\, (\partial + a)\, \lambda - \mu_2,\ \omega^2\, k\, a\, \delta\, \lambda - \mu_1\}.$$

Therefore, we obtain that the syzygy module of $D_h^4\, R_{-1}$ is defined by the matrix

$$L = \begin{pmatrix} 0 & \omega^2 & \partial + 2\, \zeta\, \omega & -\omega^2 \\ 0 & \partial & -1 & 0 \\ -\partial - a & k\, a\, \delta_h & 0 & 0 \\ \partial^2 + a\, \partial & 0 & -k\, a\, \delta_h & 0 \end{pmatrix} \in D_h^{4\times 4}, \quad (14)$$

and thus, we obtain $\text{ext}^1_{D_h}(N, D_h) = D_h^4\, L/D_h^3\, R$. Finally, using (13), $\text{ext}^2_{D_h}(N, D_h)$ corresponds to the system associated with $R_{-1}\, z = 0$, namely:

$$\left\{ \begin{array}{l} (\omega^2\, k\, a\, \delta_h)\, y = 0, \\ (\omega^2\, \partial + \omega^2\, a)\, y = 0, \\ (\omega^2\, \partial^2 + \omega^2\, a\, \partial)\, y = 0, \\ (\partial^3 + (2\, \zeta\, \omega + a)\, \partial^2 + (\omega^2 + 2\, a\, \zeta\, \omega) + a\, \omega^2)\, y = 0. \end{array} \right. \quad (15)$$

Note that $\text{ext}^2_{D_h}(N, D_h) \neq 0$ because, otherwise, in G, we should have had $\lambda - \sum_{i=1}^4 P_i\, \mu_i$, for some $P_i \in D_h$.

The quotient module $\widetilde{\text{ext}^1_D(M, D)} = D^m\, L_1/D^p\, \theta(R)$ can be computed using elimination techniques similar to Algorithm 5.3.

Algorithm 7.5.
Input: $R \in D^{q\times p}$ and $L_1 = (L_1^T : \ldots : L_m^T)^T \in D^{m\times q}$ computed by PRE-EXT1(R).
Output: A set S of generating equations satisfied by the residue class z_i of L_i^T in the left D-module $D^m\, L_1/D^p\, \theta(R)$.

QUOTIENT (L_1, R)
  Compute $\theta(R)$.
  for $i = 1, \ldots, m$ **do**
    $L \leftarrow \{\sum_{j=1}^q L_{ij}\, \lambda_j - \mu_i\} \cup \{\sum_{j=1}^q \theta(R)_{kj}\, \lambda_j \mid k = 1, \ldots, p\}$
    compute the Gröbner basis G_i of L in $\bigoplus_{j=1}^q D\, \lambda_j \oplus D\, \mu_i$
      w.r.t. an elimination order (eliminating the λ_j's).
  endfor
  $S \leftarrow \bigcup_{i=1}^m (G_i \cap D\, \mu_i)$.

Remark 7.6. In the result of the preceding algorithm, each $G_i \cap D[\mu_i]$ is a generating set of the relations fulfilled by z_i. Let us notice that every polynomial in $G_i \cap D[\mu_i]$ has the form $P\,\mu_i$, for a certain $P \in D$.

Example 7.7. In Example 7.4, we proved that $\mathrm{ext}^1_{D_h}(N, D_h) = D_h^4\, L/D_h^3\, R$, where R (resp. L) is defined by (5) (resp. (14)) and $N = D_h^3/D_h^4\, R^T$ and $D_h = \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$. In order to check whether or not $\mathrm{ext}^1_{D_h}(N, D_h)$ is equal to 0, we apply Quotient (L, R) described in Algorithm 7.5. If we denote

$$R = (R_1^T \,:\, R_2^T \,:\, R_3^T)^T, \quad L = (L_1^T \,:\, L_2^T \,:\, L_3^T \,:\, L_4^T)^T,$$

then we easily check that we have $G_i \cap D[\mu_i] = \{\mu_i\}$, for $i = 1 \ldots 4$, because we have $L_1 = R_3$, $L_2 = R_2$, $L_3 = -R_1$ and $L_4 = \partial\, R_1 + k\, a\, \delta_h\, R_2$. Hence, we have $D_h^4\, L = D_h^3\, R$, which shows that $\mathrm{ext}^1_{D_h}(N, D_h) = D_h^4\, L/D_h^3\, R = 0$.

The next algorithm gives a description of $\widetilde{\mathrm{ext}^1_D(M, D)} = D_h^m\, L_1/D_h^p\, \theta(R)$ in which the quotient is explicitly computed.

Algorithm 7.8.
Input: Ore algebra D satisfying (2), with an involution θ
and $R = (R_1^T : \ldots : R_q^T)^T \in D^{q\times p}$.
Output: A list $L = [L_0, L_1, L_2]$ such that:
$L_1 = (L_1^T : \ldots : L_m^T)^T \in D^{m\times q}$ is such that
$\widetilde{\mathrm{ext}^1_D(M, D)} = D^m\, L_1/D^p\, \theta(R)$, where $M = D^p/D^q\, R$,
$L_2 \in D^{q\times r}$ is such that $L_1 = \text{Syzygies}(L_2)$,
L_0 is the set of the relations satisfied by the residue class z_i
of L_i^T in the left D-module $D^m\, L_1/D^p\, \theta(R)$.

Ext1 (R)
$(L_1, L_2) \leftarrow$ Pre-Ext(R),
$L_0 \leftarrow$ Quotient(L_1, R),
$L \leftarrow [L_0, L_1, L_2]$.

Remark 7.9. First of all, if L_0 is an identity matrix, then we have $\mathrm{ext}^1_D(M, D) = 0$ (e.g. see Example 7.7). Secondly, let us notice that while using Ext1, we start to compute a free resolution of the left D-module M (see Algorithm 7.3). If we denote this resolution by

$$\ldots \xrightarrow{.R_3} D^{p_2} \xrightarrow{.R_2} D^{p_1} \xrightarrow{.R_1} D^{p_0} \xrightarrow{\pi} M \longrightarrow 0,$$

where $p_0 = p$, $p_1 = q$ and $R_1 = R$, then Ext1(R_i) computes $\widetilde{\mathrm{ext}^i_D(M, D)}$, $i \geq 1$.

The main motivation of introducing the concept of extension functor is explained in the next theorem which gives an effective way to check the module properties, and thus, the structural properties of the corresponding linear control system (see Section 4).

Theorem 7.10. *Let $M = D^p/D^q\, R$ be a left D-module and $N = D^q/D^p\, \theta(R)$ the transposed module of M. Then, we have the following equivalences:*

1. *[20, 22] $t(M) \cong \mathrm{ext}^1_D(N, D)$.*
2. *[20, 22] M is a torsion-free left D-module if and only if $\mathrm{ext}^1_D(N, D) = 0$.*
3. *[19, 20, 22] The system $R\, y = 0$ is parametrizable iff $t(M) \cong \mathrm{ext}^1_D(N, D) = 0$. Then, the matrix L_2 in Ext1(R) is a parametrization of the system $R\, y = 0$.*
4. *[20, 23] M is a reflexive left D-module if and only if $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, 2$.*
5. *[20, 23] M is a projective left D-module if and only if $\mathrm{ext}^i_D(N, D) = 0$ for $1 \leq i \leq \mathrm{lgld}\, D$.*
6. *[20, 23] If R has a full row rank, namely $S(D^q\, R) = 0$, then M is a projective left D-module iff $N = \widetilde{\mathrm{ext}^1_D(M, D)} = 0 \Leftrightarrow \exists\, S \in D^{p\times q} :\ R\, S = I_q$.*

Example 7.11. Let us check whether or not the differential time-delay system defined by (4) is controllable, and thus, parametrizable. By Proposition 4.2, we know that (4) is controllable iff the D_h-module $M = D_h^4/D_h^2\, R$ is torsion-free, where R is defined by (5). By 2 of Theorem 7.10, this is equivalent to check $\mathrm{ext}^1_{D_h}(N, D_h) = 0$, where $N = D^3/D^4\, R^T$ (see 1 of Example 6.5). Therefore, system (4) is controllable and, using 3 of Theorem 7.10, we deduce that a parametrization of (4) is given by the matrix R_{-1} defined in Example 5.5:

$$\left\{\begin{array}{l} (\omega^2\, k\, a\, \delta_h)\, z(t) = x_1(t), \\ (\omega^2\, \partial + \omega^2\, a)\, z(t) = x_2(t), \\ (\omega^2\, \partial^2 + \omega^2\, a\, \partial)\, z(t) = x_3(t), \\ (\partial^3 + (2\,\zeta\,\omega + a)\,\partial^2 + (\omega^2 + 2\,a\,\zeta\,\omega)\,\partial + a\,\omega^2)\, z(t) = u(t), \end{array}\right. \Rightarrow (4). \qquad (16)$$

Finally, the fact that $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$ implies that $M = D^4/D^3\, R$ is not a projective, and thus, a free D_h-module (see 2 of Theorem 3.6). The obstruction for M to be free is given by (15): the fact that system (15) is not equivalent to $y = 0$ means that it is not possible to express z in terms of a D_h-linear combination of x_1, x_2, x_3 and u. Therefore, (4) is not a flat differential time-delay system.

In order to check the flatness of a linear control system, we need to know whether or not the associated D-module is projective, and thus, by 5 of Theorem 7.10, we need to compute $\mathrm{ext}^i_D(N, D)$ for $i = 1 \ldots \mathrm{lgld}\, D$. However, if the system is defined by a full row rank matrix (which seems to be always the case in real systems), then 6 of Theorem 7.10 gives a more economic way to check projectiveness. We need to check if R admits a right-inverse S.

Algorithm 7.12.
Input: Ore algebra D satisfying (2) and a matrix $R \in D^{q\times p}$.
Output: A matrix $S \in D^{p\times q}$ satisfying $S R = I_p$ if it exists and [ ] otherwise.

LEFT-INVERSE (R)
 $P \leftarrow \{\sum_{j=1}^p R_{ij}\, \lambda_j - \mu_i \mid i = 1, \ldots, q\}$,
 Compute the Gröbner basis G of P in $\bigoplus_{i=1}^p D\,\lambda_i \oplus \bigoplus_{i=1}^q D\,\mu_i$
  w.r.t. an elimination order (eliminating the λ_i's).
 Define matrices L and M such that the rows in $L \cdot (\lambda_1 : \ldots : \lambda_p)^T$
  and $M \cdot (\mu_1 : \ldots : \mu_q)^T$ are equations in G.
 If L is invertible and $L^{-1}\, M \in D^{p\times q}$, then return $S = L^{-1}\, M$, else return [ ].

Now, we can compute a right-inverse $S \in D^{q\times p}$ of $R \in D^{q\times p}$ ($R\,S = I_q$), when such an inverse exists, by doing RIGHT-INVERSE(R) $= \theta$(LEFT-INVERSE($\theta(R)$)). Therefore, if $R \in D^{q\times p}$ has a full row rank, by 6 of Theorem 7.10, the left D-module $M = D^p/D^q\, R$ is projective iff RIGHT-INVERSE(R) $\neq$ [].

Example 7.13. Let us reconsider the differential time-delay system defined by (4). Applying Algorithm 7.12 to $\theta(R) = R^T$, where R is defined by (5), we are led to the Gröbner basis G defined in Example 5.5. We easily check that G does not contain some relations of the form $\lambda_i - \sum_{j=1}^4 S_{ij}\, \mu_j$, where $S_{ij} \in D_h$, for $i = 1, 2$. Therefore, $M = D_h^4/D_h^3\, R$ is not a projective D_h-module, and thus, (4) is not a flat system [13] (see also Example 7.11).

If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring, then we can study the obstructions for a system to be flat. In this case, they are given by polynomials containing a certain number of variables x_i which depends on the properties of the corresponding D-module M.

Proposition 7.14. *[23] Let $D = k[x_1, \ldots, x_n]$, $R \in D^{q\times p}$ be a full row rank matrix and the D-modules $M = D^p/D^q\, R$ and $N = D^q/D^p\, R^T$. Let $i(M)$ be the* torsion-free degree *of M:*

$$i(M) = \min_{i\geq 1}\{i - 1 \mid \mathrm{ext}^i_D(N, D) \neq 0\} \in \{0, \ldots, n-1, +\infty\}.$$

Then, there exist

$$\left\{\begin{array}{ll} \pi^\sigma_{n-i(M)} \in D^\sigma_{n-i(M)} = k[x_{\sigma(1)}, \ldots, x_{\sigma(n-i(M))}], & \text{if } 0 \leq i(M) \leq n-1, \\ \pi_{n-i(M)} \in k, & \text{if } i(M) = +\infty, \end{array}\right.$$

such that the $D_{\pi^\sigma_{n-i(M)}}$-module $D_{\pi^\sigma_{n-i(M)}} \otimes_D M = \{\frac{m}{a} \mid m \in M,\, a = (\pi^\sigma_{n-i(M)})^m,\, m \in \mathbb{Z}_{\geq 0}\}$ is free, where σ is an element of the group Σ_n of the permutations of n elements. Then, there exist $R_{-1} \in D^{p\times(p-q)}$, $S \in D^{p\times q}$, $S_{-1} \in D^{(p-q)\times p}$ and $\nu \in \mathbb{Z} \geq 0$ such that we have:

$$(S :\ R_{-1}) \left(\begin{array}{c} R \\ S_{-1} \end{array}\right) = (\pi^\sigma_{n-i(M)})^\nu\, I_p, \qquad \left(\begin{array}{c} R \\ S_{-1} \end{array}\right) (S :\ R_{-1}) = (\pi^\sigma_{n-i(M)})^\nu\, I_p.$$

In order to compute such polynomial $\pi^{\sigma}_{n-i(M)}$, we can follow π-POLYNOMIAL [23].

Algorithm 7.15.

Input: A commutative polynomial ring $D = k[x_1, \ldots, x_n]$, a left D-module $M = D^p/D^q\, R$ and a set $\{x_{\sigma(1)}, \ldots, x_{\sigma(m)}\}$ for a certain permutation $\sigma \in \Sigma_n$ and $1 \leq m \leq n$.

Output: An ideal J of $k[x_{\sigma(1)}, \ldots, x_{\sigma(m)}]$ such that any element $\pi \in J$ satisfies that $D_\pi \otimes_D M$ is a free D_π-module.

π-POLYNOMIAL $(R, \{x_{\sigma(1)}, \ldots, x_{\sigma(m)}\})$
  $R_1 \leftarrow R$,
  for $i = 1, \ldots, \mathrm{gld}(D) + 1$ **do**
    $L_i = [L_{i0},\, L_{i1},\, L_{i2}] \leftarrow$ EXT1(R_i),
    $P_{ij} \leftarrow \{d \in D \mid d \cdot \mu_j \in L_{i0}\}$, $j = 1, \ldots, \mathrm{rowdim}(L_{i1})$ (cf. Remark 7.6),
    $R_{i+1} \leftarrow$ SYZYGIES(R_i),
  endfor
  $I \leftarrow \bigcap_{i,j} \langle P_{ij} \rangle$,
  $J \leftarrow I \cap k[x_{\sigma(1)}, \ldots, x_{\sigma(m)}]$.

In the last step of π-POLYNOMIAL, the intersection of ideals of Ore algebras can be computed by means of elimination algorithms (see [2] and Example 2.14).

Example 7.16. In Examples 7.11 and 7.13, we proved that the linear differential time-delay defined by (4) is not flat, i.e. the associated D_h-module $M = D_h^4/D_h^3\, R$ is not free. Let us find a polynomial $\pi \in \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\delta_h; \sigma_2, \delta_2]$ such that the D_π-module $D_\pi \otimes_D M$ is free. In Examples 7.4, we saw that $\mathrm{ext}^1_{D_h}(N, D_h) = 0$ and $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$. Therefore, the torsion-free degree $i(M)$ of M is 1. Applying Algorithm 7.15 and using (15), we find that:

$$I = (\omega^2\, k\, a\, \delta_h,\ \omega^2\, \partial + \omega^2\, a,\ \omega^2\, \partial^2 + \omega^2\, a\, \partial,\ \partial^3 + (2\,\zeta\,\omega + a)\,\partial^2 + (\omega^2 + 2\,a\,\zeta\,\omega) + a\,\omega^2).$$

Thus, we have $I \cap \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\delta_h; \sigma_2, \delta_2] = (\delta_h)$, which shows that (7.11) is δ_h-free [13].

8 Conclusion

We hope that we have convinced the reader that the simultaneous use of module theory, homological algebra, effective algebra and computational methods allows us to study effectively the structural properties of linear non-commutative multidimensional systems (e.g. time-varying OD systems, time-varying differential time-delay systems, underdetermined systems of PDs or multidimensional discrete systems systems with variables coefficients). In particular, in this unified mathematical framework, we presented effective algorithms which check controllability/flatness/π-freeness ... and compute the parametrizations/autonomous elements/flat outputs/π-polynomials ... Certain of these problems were still open for some classes of linear multidimensional systems [27, 28] and, in particular, for linear differential time-delay systems [6, 13]. The Maple package *OreModules*, based on *Mgfun* [4], as well as Maple worksheets containing the explicit examples of the appendix are available at `http://wwwb.math.rwth-aachen.de/OreModules`.

A Appendix: Examples

In this appendix, we give some Maple worksheets which deal with controllability, parametrizability, flatness and π-freeness of some time-invariant/time-varying linear OD systems, differential time-delay systems and systems of PDE with constant or variable coefficients. These results have been obtained using the Maple package *OreModules*[1] which is crucially based on the library *Mgfun* [4] (e.g. Ore algebras and non-commutative Gröbner bases are developed in *Mgfun*). In these examples, the most time comsuming computation is that of the $\mathrm{ext}^i_D(N, D)$. We give some timings for these operations. All examples were run on a Pentium III, 1 GHz (2 processors) with 1 GB RAM using Maple 8 (*OreModules* is available for Maple V release 5, Maple 6, and Maple 8).

[1] *OreModules* and all examples of this appendix are available on the web page `http://wwwb.math.rwth-aachen.de/OreModules`.

A.1 Two pendula mounted on a car

The first example that we consider is the linearized time-invariant OD system formed by two pendula mounted on a car. See for more details Examples 5.2.1 and 5.2.12 in [16].

```
> with(Ore_algebra): with(OreModules):
```

After loading the required Maple packages, the first step is to define the commutative Ore algebra $D = \mathbb{R}(m_1, m_2, M, L_1, L_2, g)[\partial; \sigma, \delta]$, where $\sigma = \mathrm{id}_{\mathbb{R}(m_1,m_2,M,L_1,L_2,g)}$ and $\delta = \frac{d}{dt}$. In the sequel, this Ore algebra is denoted by *Alg*.

```
> Alg:=diff_algebra([D[t],t], polynom={t}, comm={m1, m2, L1, L2, M, g}):
```

In *Alg*, we need to declare the constants m_1, m_2, M, L_1, L_2 and g which occur in the system. Then, we define the matrix $R \in D^{3\times 4}$ which corresponds to the system.

```
> R:=evalm([[m1*L1*D[t]^2, m2*L2*D[t]^2, -1, (M+m1+m2)*D[t]^2],
> [m1*L1^2*D[t]^2-m1*L1*g, 0, 0, m1*L1*D[t]^2],
> [0, m2*L2^2*D[t]^2-m2*L2*g, 0, m2*L2*D[t]^2]]);
```

$$R := \begin{bmatrix} m1\, L1\, D_t^2 & m2\, L2\, D_t^2 & -1 & (M + m1 + m2)\, D_t^2 \\ m1\, L1^2\, D_t^2 - m1\, L1\, g & 0 & 0 & m1\, L1\, D_t^2 \\ 0 & m2\, L2^2\, D_t^2 - m2\, L2\, g & 0 & m2\, L2\, D_t^2 \end{bmatrix}$$

We compute $R_{\mathrm{adj}} = \theta(R) \in D^{4\times 3}$ using the involution θ defined in 2 of Example 6.3.

```
> R_adj:=adjoint(R, Alg);
```

$$R\_adj := \begin{bmatrix} m1\, L1\, D_t^2 & m1\, L1^2\, D_t^2 - m1\, L1\, g & 0 \\ m2\, L2\, D_t^2 & 0 & m2\, L2^2\, D_t^2 - m2\, L2\, g \\ -1 & 0 & 0 \\ (M + m1 + m2)\, D_t^2 & m1\, L1\, D_t^2 & m2\, L2\, D_t^2 \end{bmatrix}$$

We compute $\mathrm{ext}^1_D(N, D)$, where $N = D^4/D^3\, R_{\mathrm{adj}}$, using the procedure $\mathrm{EXT1}(R_{\mathrm{adj}})$.

```
> st:=time(): Ext1:=exti(R_adj, Alg, 1): time()-st;
```

$$.591$$

The computation of $\mathrm{ext}^1_D(N, D)$ only takes .591 s. Let us notice that all the computations are done generically, i.e. in the case where $L_1 \neq L_2$.

```
> Ext1[1];
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ext1[1] corresponds to the matrix L_0 returned by $\mathrm{EXT1}(R)$ and defined in Algorithm 7.8. In our case, L_0 is the identity matrix which shows that $\mathrm{ext}^1_D(N, D) = 0$ (see Remark 7.9). Therefore, the system is controllable, and thus, parametrizable (see Proposition 4.2 and Theorem 7.10). The parametrization of the system corresponds to the matrix L_2 =*Ext1[3]* of $\mathrm{EXT1}(R)$ (see Algorithm 7.8).

```
> Ext1[3];
```

$$\begin{bmatrix} -L2\, D_t^4 + D_t^2\, g \end{bmatrix}$$
$$\begin{bmatrix} D_t^2\, g - D_t^4\, L1 \end{bmatrix}$$
$$\Big[-L2\, D_t^4\, M\, g + L2\, D_t^6\, M\, L1 - L2\, m1\, g\, D_t^4 + M\, D_t^2\, g^2 + m1\, g^2\, D_t^2 - g\, M\, D_t^4\, L1$$
$$+ D_t^2\, m2\, g^2 - D_t^4\, g\, L1\, m2 \Big]$$
$$\begin{bmatrix} -L2\, g\, D_t^2 + L2\, D_t^4\, L1 + g^2 - g\, L1\, D_t^2 \end{bmatrix}$$

Therefore, we obtain the parametrization $(x_1 : x_2 : x_3 : u)^T = L_2\, z$ of the system $R\,(x_1 : x_2 : x_3 : u)^T = 0$. Using the fact that D is a principal ideal domain, then, by 1 of Theorem 3.6, we know that the D-module $M = D^4/D^3\, R$ is free, and thus, the system defined by R is flat.

```
> LeftInverse(Ext1[3], Alg);
```

$$\left[\begin{array}{cccc} \dfrac{L1^2}{g^2\,(-L2+L1)} & -\dfrac{L2^2}{g^2\,(-L2+L1)} & 0 & -\dfrac{-L1+L2}{g^2\,(-L2+L1)} \end{array}\right]$$

LeftInverse(Ext1[3], Alg) computes a left-inverse of the parametrization $L_2 = Ext1[3]$. We deduce that $z = (L_1^2\, x_1 - L_2^2\, x_2 + (L_1 - L_2)\, u)/(g^2\,(L_1 - L_2))$ is a flat output of the system.

Let us notice that the difference of the pendula lengths $L_1 - L_2$ appears in the denominator of the flat output of the system. Thus, we need to study the non-generic case where $L_1 = L_2$.

```
> Rmod:=evalm([[m1*L1*D[t]^2, m2*L1*D[t]^2, -1, (M+m1+m2)*D[t]^2],
> [m1*L1^2*D[t]^2-m1*L1*g, 0, 0, m1*L1*D[t]^2],
> [0, m2*L1^2*D[t]^2-m2*L1*g, 0, m2*L1*D[t]^2]]);
```

$$Rmod := \left[\begin{array}{cccc} m1\,L1\,\mathrm{D}_t^{\,2} & \mathrm{D}_t^{\,2}\,L1\,m2 & -1 & (M+m1+m2)\,\mathrm{D}_t^{\,2} \\ m1\,L1^2\,\mathrm{D}_t^{\,2} - m1\,L1\,g & 0 & 0 & m1\,L1\,\mathrm{D}_t^{\,2} \\ 0 & m2\,L1^2\,\mathrm{D}_t^{\,2} - L1\,m2\,g & 0 & \mathrm{D}_t^{\,2}\,L1\,m2 \end{array}\right]$$

```
> st:=time(): Ext1mod:=exti(adjoint(Rmod, Alg), Alg, 1)[1..3]; time()-st;
```

$$Ext1mod := \left[\left[\begin{array}{ccc} L1\,\mathrm{D}_t^{\,2} - g & 0 & 0 \\ 0 & L1\,\mathrm{D}_t^{\,2} - g & 0 \\ 0 & 0 & L1\,\mathrm{D}_t^{\,2} - g \end{array}\right],\right.$$

$$\left[\begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & m1\,g + g\,m2 & -1 & M\,\mathrm{D}_t^{\,2} \\ 0 & M\,\mathrm{D}_t^{\,2}\,L1 - M\,g - m1\,g - g\,m2 & 1 & 0 \end{array}\right],$$

$$\left.\left[\begin{array}{c} -\mathrm{D}_t^{\,2} \\ -\mathrm{D}_t^{\,2} \\ -g\,\mathrm{D}_t^{\,2}\,M - \mathrm{D}_t^{\,2}\,g\,m1 - \mathrm{D}_t^{\,2}\,g\,m2 + L1\,\mathrm{D}_t^{\,4}\,M \\ L1\,\mathrm{D}_t^{\,2} - g \end{array}\right]\right]$$

$$.490$$

The first matrix L_0 of *Ext1mod* is not an identity matrix, and thus, we know that $\mathrm{ext}^1_D(N_{\mathrm{mod}}, D) \neq 0$, where $N_{\mathrm{mod}} = D^3/D^4\,\theta(R_{\mathrm{mod}})$ and R_{mod} corresponds to R where $L_2 = L_1$. Thus, the system is not controllable (see Proposition 4.2 and Theorem 7.10). The second matrix L_1 of *Ext1mod* gives a (non minimal) family of generators of the torsion elements of $M_{\mathrm{mod}} = D^4/D^3\,R_{\mathrm{mod}}$, i.e. of the non-controllable elements of the system. For instance, the first row of respectively L_1 and L_0 shows that $d = x_1 - x_2$ satisfies $(L_1\,\partial^2 - g)\,d = 0$ (non-controllable element). The third matrix L_2 gives a parametrization of the torsion-free submodule $M_{\mathrm{mod}}/t(M_{\mathrm{mod}})$ of M_{mod}, i.e. of the controllable part of the system. Finally, let us notice that it has only taken .490 s to obtain all these informations.

A.2 Linear DAE

Let us consider the following example of a time-varying linear OD system which corresponds to a linear system of differential algebraic equations (DAE) studied in [9].

Therefore, we introduce the Weyl algebra $D = A_1$ (see Example 2.3).

```
> Alg:=diff_algebra([D[t],t], polynom={t}):
```

The time-varying linear system is defined by means of the following matrix.

```
> R:=evalm([[-t*D[t]+1, t^2*D[t], -1, 0], [-D[t], t*D[t]+1, 0, -1]]);
```

$$R := \left[\begin{array}{cccc} -t\,\mathrm{D}_t + 1 & t^2\,\mathrm{D}_t & -1 & 0 \\ -\mathrm{D}_t & t\,\mathrm{D}_t + 1 & 0 & -1 \end{array}\right]$$

Then, the system is defined by $R\,(x_1 : x_2 : u_1 : u_2)^T = 0$ and we define the left D-module $M = D^4/D^2\,R$.

```
> R_adj:=adjoint(R, Alg);
```

$$R\_adj := \begin{bmatrix} t\,\mathrm{D}_t + 2 & \mathrm{D}_t \\ -t^2\,\mathrm{D}_t - 2\,t & -t\,\mathrm{D}_t \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then, using the procedure Ext1(R_{adj}), we compute $\widetilde{\mathrm{ext}_D^1(N, D)}$, where $N = D^2/D^4\,R_{\mathrm{adj}}$ is the left D-module associated with $R_{\mathrm{adj}} = \theta(R) \in D^{4\times 2}$.

```
> Ext1:=exti(R_adj, Alg, 1):  Ext1[1];
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that the left D-module $M = D^4/D^2\,R$ is a torsion-free D-module (see Remark 7.9), and thus, a projective D-module (see 1 of Theorem 3.6).

Then, a parametrization L_2 of the system is given by the following matrix of operators.

```
> Ext1[3];
```

$$\begin{bmatrix} t & 1 \\ 1 & 0 \\ 0 & -t\,\mathrm{D}_t + 1 \\ 0 & -\mathrm{D}_t \end{bmatrix}$$

The system is flat, i.e. the left D-module $M = D^4/D^2\,R$ is free, iff the parametrization L_2 admits a left-inverse. Let us compute whether or not L_2 admits a left-inverse.

```
> LeftInverse(Ext1[3], Alg);
```

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \end{bmatrix}$$

We see that the system is flat.

A.3 Wind tunnel model

In this example, we reconsider the linear differential time-delay system (4) defined in Example 3.2. We first need to define the commutative polynomial ring $D_h = \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ of differential time-delay operators.

```
> Alg:=diff_algebra([D[t],t],[delta,s], polynom={t,s}, comm={a,omega,zeta,k}):
```

The linear differential time-delay system is defined by matrix (5).

```
> R:=evalm([[D[t]+a, -k*a*delta, 0, 0], [0, D[t], -1, 0],
> [0, omega^2, D[t]+2*zeta*omega, -omega^2]]);
```

$$R := \begin{bmatrix} \mathrm{D}_t + a & -k\,a\,\delta & 0 & 0 \\ 0 & \mathrm{D}_t & -1 & 0 \\ 0 & \omega^2 & \mathrm{D}_t + 2\,\zeta\,\omega & -\omega^2 \end{bmatrix}$$

```
> R_adj:=adjoint(R, Alg);
```

$$R\_adj := \begin{bmatrix} \mathrm{D}_t + a & 0 & 0 \\ -k\,a\,\delta & \mathrm{D}_t & \omega^2 \\ 0 & -1 & \mathrm{D}_t + 2\,\zeta\,\omega \\ 0 & 0 & -\omega^2 \end{bmatrix}$$

The Gröbner basis G defined in Example 5.5 can be computed as follows.

```
> integrability(R_adj, Alg);
```

$$[\omega^2\,k\,a\,\delta\,\mu_1 + \omega^2\,\mathrm{D}_t\,\mu_2 + \omega^2\,a\,\mu_2 + \omega^2\,{\mathrm{D}_t}^2\,\mu_3 + \omega^2\,a\,\mathrm{D}_t\,\mu_3 + {\mathrm{D}_t}^3\,\mu_4 + 2\,{\mathrm{D}_t}^2\,\zeta\,\omega\,\mu_4 + a\,{\mathrm{D}_t}^2\,\mu_4 + \mathrm{D}_t\,\omega^2\,\mu_4 + 2\,a\,\mathrm{D}_t\,\zeta\,\omega\,\mu_4 + a\,\omega^2\,\mu_4,\ \lambda_3\,\omega^2 + \mu_4,\ \omega^2\,\lambda_2 + \mathrm{D}_t\,\mu_4 + \omega^2\,\mu_3 + 2\,\zeta\,\omega\,\mu_4,\ \lambda_1\,\mathrm{D}_t + \lambda_1\,a - \mu_1,\ \omega^2\,\lambda_1\,k\,a\,\delta + {\mathrm{D}_t}^2\,\mu_4 + \omega^2\,\mathrm{D}_t\,\mu_3 + 2\,\mathrm{D}_t\,\zeta\,\omega\,\mu_4 + \omega^2\,\mu_2 + \omega^2\,\mu_4]$$

The syzygy module of $D_h^4\,R^T$ can be computed using $\textsc{Syzygy}(R_{\mathrm{adj}})$ (see Example 5.5).

```
> S:=syzygy_module(R_adj, Alg);
```

$$S := \Big[\omega^2\,k\,a\,\delta\,,\ \mathrm{D}_t\,\omega^2 + a\,\omega^2\,,\ \omega^2\,{\mathrm{D}_t}^2 + \omega^2\,a\,\mathrm{D}_t\,,\ \mathrm{D}_t\,\omega^2 + a\,\omega^2 + {\mathrm{D}_t}^3 + 2\,{\mathrm{D}_t}^2\,\zeta\,\omega + a\,{\mathrm{D}_t}^2 + 2\,a\,\mathrm{D}_t\,\zeta\,\omega\Big]$$

The syzygy module $D_h^4\,R_{-1}$ can be computed using $\textsc{Syzygy}(R_{-1})$ (see Example 7.4).

```
> L:=syzygy_module(adjoint(S, Alg), Alg);
```

$$L := \begin{bmatrix} 0 & \omega^2 & \mathrm{D}_t + 2\,\zeta\,\omega & -\omega^2 \\ 0 & \mathrm{D}_t & -1 & 0 \\ -\mathrm{D}_t - a & k\,a\,\delta & 0 & 0 \\ {\mathrm{D}_t}^2 + \mathrm{D}_t\,a & 0 & -k\,a\,\delta & 0 \end{bmatrix}$$

Finally, we compute $\textsc{Quotient}(L, R)$ (see Example 7.7) as follows.

```
> quotient(L, R, Alg);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We deduce that $\mathrm{ext}^1_{D_h}(N, D_h) = 0$, where $N = D^3/D^4\,R^T$ (see Remark 7.9), and thus, system (4) is controllable and parametrizable (see Proposition 4.2 and Theorem 7.10).

We can directly compute $\mathrm{ext}^1_{D_h}(N, D_h)$ using $\textsc{Ext1}(R_{\mathrm{adj}})$ defined in Algorithm 7.8.

```
> Ext1:=exti(adjoint(R, Alg), Alg, 1);
```

$$\mathit{Ext1} := \left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 0 & \omega^2 & \mathrm{D}_t + 2\,\zeta\,\omega & -\omega^2 \\ 0 & \mathrm{D}_t & -1 & 0 \\ -\mathrm{D}_t - a & k\,a\,\delta & 0 & 0 \\ {\mathrm{D}_t}^2 + \mathrm{D}_t\,a & 0 & -k\,a\,\delta & 0 \end{bmatrix},\ \begin{bmatrix} \omega^2\,k\,a\,\delta \\ \mathrm{D}_t\,\omega^2 + a\,\omega^2 \\ \omega^2\,{\mathrm{D}_t}^2 + \omega^2\,a\,\mathrm{D}_t \\ {\mathrm{D}_t}^3 + 2\,{\mathrm{D}_t}^2\,\zeta\,\omega + a\,{\mathrm{D}_t}^2 + \mathrm{D}_t\,\omega^2 + 2\,a\,\mathrm{D}_t\,\zeta\,\omega + a\,\omega^2 \end{bmatrix}\right]$$

The last matrix L_2 of *Ext1* gives the parametrization (16) of System (4).

We compute $\mathrm{ext}^2_{D_h}(N, D_h)$ in order to check whether or not System (16) is flat.

```
> Ext2:=exti(adjoint(R, Alg), Alg, 2);
```

$$\mathit{Ext2} := \left[\begin{bmatrix} \mathrm{D}_t + a \\ \delta \end{bmatrix},\ \begin{bmatrix} 1 \end{bmatrix},\ \mathrm{SURJ}(1)\right]$$

The first matrix is not an identity matrix, and thus, we know that $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$ (see Remark 7.9). Hence, $M = D^4/D^3\,R$ is a torsion-free but not a free D_h-module, and thus, (4) is not a flat system. Finally, let us notice that (15) is equivalent to the reduced system:

$$(\partial + a)\,y = 0, \quad \delta_h\,y = 0.$$

Then, the formal obstructions of flatness are defined by π-POLYNOMIAL$(R_{\text{adj}}, \{\partial, \delta_h\})$.

```
> pi_polynomial(R_adj, Alg);
```

$$[\mathrm{D}_t + a, \delta]$$

The π-polynomial, such that $(D_h)_\pi \otimes_D M$ is a free $(D_h)_\pi$-module, is defined by the generator of the principal ideal π-POLYNOMIAL$(R_{\text{adj}}, \{\delta_h\})$ of $\mathbb{R}(a, k, \zeta, \omega)[\delta_h; \sigma_2, \delta_2]$.

```
> pi_polynomial(R_adj, Alg, [delta]);
```

$$[\delta]$$

Therefore, we find that system (4) is δ_h-flat (see Example 7.16).

```
> pi_polynomial(R_adj, Alg, [D[t]]);
```

$$[\mathrm{D}_t + a]$$

The fact that (4) is not a flat system is coherent with the fact that its parametrization (16) does not admit a left-inverse (a linear system is flat iff it is parametrizable and its parametrization admits a left-inverse).

```
> LeftInverse(Ext1[3], Alg);
```

$$[]$$

Finally, the fact that (4) is not a flat system is also coherent with the fact that the full row-rank matrix R, defined by (5), does not admit a right-inverse (R admits a right-inverse iff the D_h-module M is projective, and thus, free by 2 of Theorem 3.6).

```
> RightInverse(R, Alg);
```

$$[]$$

A.4 A two reflector antenna

Let us consider the example of a two reflector antenna [13]. Any linear differential constant time-delay systems can be studied similarly. We refer to [13, 14, 15] for more examples.

```
> Alg:=diff_algebra([Dt,t],[delta,s],polynom={t,s},comm={K1,K2,Te,Kp,Kc}):
```

The system of two reflector antenna is defined by the following matrix with entries in the commutative polynomial ring $D_h = \mathbb{R}(K1, K2, Te, Kp, Kc)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ of differential time-delay operators.

```
> R :=evalm([[Dt, -K1, 0, 0, 0, 0, 0, 0, 0],
> [0, Dt+K2/Te, 0, 0, 0, 0, -Kp/Te*delta, -Kc/Te*delta, -Kc/Te*delta],
> [0, 0, Dt, -K1, 0, 0, 0, 0, 0],
> [0, 0, 0, Dt+K2/Te, 0, 0, -Kc/Te*delta, -Kp/Te*delta, -Kc/Te*delta],
> [0, 0, 0, 0, Dt, -K1, 0, 0, 0],
> [0, 0, 0, 0, 0, Dt+K2/Te, -Kc/Te*delta, -Kc/Te*delta, -Kp/Te*delta]]);
```

$$R := \begin{bmatrix} Dt & -K1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Dt + \dfrac{K2}{Te} & 0 & 0 & 0 & 0 & -\dfrac{Kp\,\delta}{Te} & -\dfrac{Kc\,\delta}{Te} & -\dfrac{Kc\,\delta}{Te} \\ 0 & 0 & Dt & -K1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Dt + \dfrac{K2}{Te} & 0 & 0 & -\dfrac{Kc\,\delta}{Te} & -\dfrac{Kp\,\delta}{Te} & -\dfrac{Kc\,\delta}{Te} \\ 0 & 0 & 0 & 0 & Dt & -K1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Dt + \dfrac{K2}{Te} & -\dfrac{Kc\,\delta}{Te} & -\dfrac{Kc\,\delta}{Te} & -\dfrac{Kp\,\delta}{Te} \end{bmatrix}$$

```
> R_adj:=adjoint(R, Alg):
```

Let us check whether or not the system, defined by the matrix R, is controllable and, if it is the case, let us find one of its parametrizations. In order to do that, we need to compute $\mathrm{ext}^1_{D_h}(N, D_h)$, where $N = D^6/D^9\, R$. The result is (partially) displayed below and corresponds to $[L_0, L_1, L_2]$ in the output of EXT1.

```
> Ext1:=exti(R_adj, Alg, 1): Ext1[1];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We obtain that $t(M) = \mathrm{ext}^1_{D_h}(N, D_h) = 0$ (see Remark 7.9), where $M = D^9/D^6\, R$, and thus, the system of two reflector antenna is controllable (see Proposition 4.2 and Theorem 7.10).

Then, the matrix L_2, defined by

```
> Ext1[3];
```

$$\begin{bmatrix} Kc\,\delta\,K1 & Kc\,\delta\,K1 & \delta\,Kp\,K1 \\ \delta\,Kc\,Dt & \delta\,Kc\,Dt & \delta\,Kp\,Dt \\ Kc\,\delta\,K1 & \delta\,Kp\,K1 & Kc\,\delta\,K1 \\ \delta\,Kc\,Dt & \delta\,Kp\,Dt & \delta\,Kc\,Dt \\ \delta\,Kp\,K1 & Kc\,\delta\,K1 & Kc\,\delta\,K1 \\ \delta\,Kp\,Dt & \delta\,Kc\,Dt & \delta\,Kc\,Dt \\ 0 & 0 & Dt^2\,Te + Dt\,K2 \\ 0 & Dt^2\,Te + Dt\,K2 & 0 \\ Dt^2\,Te + Dt\,K2 & 0 & 0 \end{bmatrix}$$

is a parametrization of the two reflector antenna. The two reflector antenna is not a flat system because we have $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$ as it is shown below.

```
> Ext2:=exti(R_adj, Alg, 2): Ext2[1];
```

$$\begin{bmatrix} \delta & 0 & 0 \\ Dt^2\,Te + Dt\,K2 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & Dt^2\,Te + Dt\,K2 & 0 \\ 0 & 0 & \delta \\ 0 & 0 & Dt^2\,Te + Dt\,K2 \end{bmatrix}$$

Since the torsion-free degree $i(M)$ of M is equal to 1 (see Proposition 7.14), then we can find a polynomial π containing only one of the indeterminates Dt and δ such that $(D_h)_\pi \otimes_{D_h} M$ is a free $(D_h)_\pi$-module. We study polynomials in δ_h.

```
> pi_polynomial(R_adj, Alg, [delta]);
```

$$[\delta]$$

So, we conclude that the two antenna reflector is δ-free [13].

A.5 An electric transmission line

To finish with linear differential time-delay system, we shall study the example of an electric transmission line [25]. We shall exhibit an explict parametrization of this system. It seems that no parametrization for such a system was previously known [13]. This can be easily explained by the fact that it was very difficult to guess such a parametrization (indeed, the characterization in terms of the extension functor of the existence of a parametrization has only been recently developed in [20]). Moreover, it becomes very difficult to avoid the use of a symbolic computation program in order to handle all the computations.

We first introduce $D_h = \mathbb{R}(a_0, a_1, a_2, a_3, a_4, a_5, b_0)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ the Ore algebra of differential time-delay operators.

```
> Alg:=diff_algebra([Dt,t],[delta,s], polynom={t,s}, comm={a[0],a[1],
> a[2],a[3],a[4],a[5],b[0]}):
```

The electric transmission line is defined by means of the following matrix.

```
> R:=evalm([[Dt+a[0], -(a[4]*Dt+a[0])*delta, -a[0], 0, -b[0]*Dt],
> [-delta*(a[5]*Dt+a[1]), Dt+a[1], 0, a[1], 0],
> [a[2], -a[2]*a[4]*delta, Dt, 0, -a[2]*b[0]],
> [a[3]*a[5]*delta, -a[3], 0, Dt, 0]]);
```

$$R := \begin{bmatrix} Dt + a_0 & -(a_4\,Dt + a_0)\,\delta & -a_0 & 0 & -b_0\,Dt \\ -\delta\,(a_5\,Dt + a_1) & Dt + a_1 & 0 & a_1 & 0 \\ a_2 & -a_2\,a_4\,\delta & Dt & 0 & -a_2\,b_0 \\ a_3\,a_5\,\delta & -a_3 & 0 & Dt & 0 \end{bmatrix}$$

```
> R_adj:=adjoint(R, Alg):
> Ext1:=exti(R_adj, Alg, 1): Ext1[1];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, by Proposition 4.2 and Theorem 7.10, we find that the electric transmission line is controllable and parametrizable and a parametrization is given by the following matrix.

```
> Ext1[3];
```

$$\left[-b_0\,Dt^4 - Dt^2\,a_1\,b_0\,a_3 - b_0\,a_1\,a_0\,a_2\,Dt - a_1\,b_0\,Dt^3 - Dt^2\,a_0\,a_2\,b_0 - a_1\,b_0\,a_3\,a_0\,a_2\right]$$

$$\left[-a_3\,a_1\,\delta\,a_5\,Dt^2\,b_0 - a_5\,b_0\,\delta\,Dt^4 - a_1\,Dt\,\delta\,a_0\,a_2\,b_0 - a_1\,\delta\,b_0\,Dt^3 - a_5\,Dt^2\,b_0\,\delta\,a_0\,a_2 \right.$$
$$\left. - a_3\,a_0\,a_1\,a_2\,\delta\,b_0\,a_5\right]$$

$$\left[a_5\,a_0\,a_2\,a_1\,b_0\,a_3\,\delta^2 - Dt^2\,a_0\,a_2\,b_0 + a_1\,a_0\,a_2\,b_0\,Dt\,\delta^2 - b_0\,a_1\,a_0\,a_2\,Dt + a_5\,Dt^2\,\delta^2\,a_0\,a_2\,b_0 \right.$$
$$\left. - a_1\,b_0\,a_3\,a_0\,a_2\right]$$

$$\left[-a_3\,b_0\,\delta\,a_1\,Dt^2 + a_3\,a_0\,a_1\,a_2\,\delta\,b_0\,a_5 - a_3\,a_1\,\delta\,a_0\,a_2\,b_0 + a_3\,a_1\,\delta\,a_5\,Dt^2\,b_0\right]$$

$$\left[-Dt^4 - a_1\,Dt^2\,a_0 - a_1\,Dt^3 - Dt^3\,a_0 + a_3\,a_1\,a_5\,\delta^2\,Dt\,a_0 + a_3\,a_1\,a_5\,a_0\,a_2\,a_4\,\delta^2 + a_1\,Dt^2\,\delta^2\,a_0 \right.$$
$$+ a_5\,Dt^4\,\delta^2\,a_4 + a_5\,Dt^3\,\delta^2\,a_0 - a_1\,Dt\,a_0\,a_2 - Dt^2\,a_0\,a_2 - a_3\,a_1\,a_0\,a_2 - a_3\,a_1\,Dt\,a_0 + a_1\,Dt\,\delta^2\,a_0\,a_2\,a_4$$
$$\left. + a_1\,Dt^3\,\delta^2\,a_4 + a_5\,Dt^2\,\delta^2\,a_0\,a_2\,a_4 + a_3\,a_1\,a_5\,\delta^2\,Dt^2\,a_4 - a_3\,a_1\,Dt^2\right]$$

```
> Ext2:=exti(R_adj, Alg, 2): Ext2[1];
```

$$\left[-a_2\,a_0\,\delta\,Dt\,a_1 + \delta\,Dt\,a_3\,{a_1}^2 - \delta\,a_3\,{a_1}^2\,a_5\,Dt + a_5\,\delta\,a_1\,Dt\,a_0\,a_2 + \delta^3\,{a_1}^2\,a_2\,a_0 + {a_3}^2\,{a_1}^2\,{a_5}^2\,\delta^3 \right.$$
$$\left. - 2\,a_3\,a_1\,{a_5}^2\,\delta^3\,a_2\,a_0 + {a_5}^2\,{a_0}^2\,{a_2}^2\,\delta^3 - \delta\,{a_3}^2\,{a_1}^2\,a_5 + 2\,a_3\,a_0\,a_1\,a_2\,\delta\,a_5 - a_5\,{a_0}^2\,{a_2}^2\,\delta - {a_1}^2\,\delta\,a_2\,a_0\right]$$

$$\left[\delta^2\,Dt\,a_1 - Dt^2 + a_5\,a_1\,a_3\,\delta^2 - Dt\,a_1 - a_3\,a_1 - a_5\,\delta^2\,a_2\,a_0\right]$$

$$\left[\delta\,a_2\,a_0 + Dt^2\,\delta\right]$$

$$\left[a_1\,Dt^3 + Dt^2\,{a_1}^2 - a_3\,a_1\,a_5\,Dt^2 + a_5\,Dt^2\,a_0\,a_2 + \delta^2\,{a_1}^2\,a_2\,a_0 + {a_3}^2\,{a_1}^2\,{a_5}^2\,\delta^2 - 2\,a_3\,a_1\,{a_5}^2\,\delta^2\,a_2\,a_0 \right.$$
$$\left. + {a_5}^2\,{a_0}^2\,{a_2}^2\,\delta^2 + Dt\,a_3\,{a_1}^2 - a_3\,{a_1}^2\,a_5\,Dt + a_5\,a_1\,Dt\,a_0\,a_2 - {a_3}^2\,{a_1}^2\,a_5 + a_3\,a_0\,a_1\,a_2\,a_5\right]$$

Since we have $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$, where $N = D_h^{1\times 4}/D_h^{1\times 5}\,R^T$, the transmission line is not a flat system. Thus, we have $i(M) = 1$, and we can find a polynomial π that contains only Dt or δ such that $(D_h)_\pi \otimes_D M$ is a free $(D_h)_\pi$-module. The third argument for π-POLYNOMIAL selects the variable for the π-polynomial:

```
> pi:=pi_polynomial(R_adj, Alg, [delta]): factor(pi);
```

$$[\delta({a_5}^2 {a_2}^2 {a_0}^2 \delta^4 - 2\,\delta^2 a_5 {a_2}^2 {a_0}^2 + {a_2}^2 {a_0}^2 - 2\,{a_5}^2 a_1 a_3 \delta^4 a_2 a_0 + 4\,\delta^2 a_0 a_1 a_2 a_5 a_3 + \delta^4 {a_1}^2 a_2 a_0 + {a_1}^2 a_2 a_0 - 2\,a_0 a_1 a_2 a_3 - 2\,\delta^2 {a_1}^2 a_2 a_0 + {a_5}^2 {a_1}^2 {a_3}^2 \delta^4 - 2\,\delta^2 a_5 {a_1}^2 {a_3}^2 + {a_1}^2 {a_3}^2)]$$

We could have equally well chosen the variable Dt for the π-polynomial:

```
> pi:=pi_polynomial(R_adj, Alg, [Dt]): factor(pi);
```

$$[(Dt^2 + Dt\,a_1 + a_1 a_3)\,(Dt^2 + a_0 a_2)]$$

We conclude that the system is π-free.

A.6 Einstein equations

Let us show that the results exposed in this paper can also be interesting in the study of underdetermined systems of PDEs coming from mathematical physics. We shall study the parametrizability of the Einstein equations using the linearized Ricci equations in the vacuum [17] (see also [28]).

Let us introduce the Weyl algebra $D = A_4$ (x_1, x_2 and x_3 stand for the three space components and $x_4 = c\,t$ for the time t component up to the speed of the light factor).

```
> Alg:=diff_algebra([D[1], x[1]], [D[2], x[2]], [D[3], x[3]], [D[4], x[4]],
> polynom={x[1], x[2], x[3], x[4]}):
```

The linearized Ricci equations in the vacuum are defined by the following 10×10 matrix of partial differential operators.

```
> R:=evalm(
> [[D[2]^2+D[3]^2-D[4]^2, D[1]^2, D[1]^2, -D[1]^2, -2*D[1]*D[2], 0, 0,
> -2*D[1]*D[3], 0, 2*D[1]*D[4]],
> [D[2]^2, D[1]^2+D[3]^2-D[4]^2, D[2]^2, -D[2]^2, -2*D[1]*D[2],
> -2*D[2]*D[3], 0, 0, 2*D[2]*D[4], 0],
> [D[3]^2, D[3]^2, D[1]^2+D[2]^2-D[4]^2, -D[3]^2, 0, -2*D[2]*D[3],
> 2*D[3]*D[4], -2*D[1]*D[3], 0, 0],
> [D[4]^2, D[4]^2, D[4]^2, D[1]^2+D[2]^2+D[3]^2, 0, 0, -2*D[3]*D[4], 0,
> -2*D[2]*D[4], -2*D[1]*D[4]],
> [0, 0, D[1]*D[2], -D[1]*D[2], D[3]^2-D[4]^2, -D[1]*D[3], 0,
> -D[2]*D[3], D[1]*D[4], D[2]*D[4]],
> [D[2]*D[3], 0, 0, -D[2]*D[3],-D[1]*D[3], D[1]^2-D[4]^2, D[2]*D[4],
> -D[1]*D[2], D[3]*D[4], 0],
> [D[3]*D[4], D[3]*D[4], 0, 0, 0, -D[2]*D[4], D[1]^2+D[2]^2,
> -D[1]*D[4], -D[2]*D[3], -D[1]*D[3]],
> [0, D[1]*D[3], 0, -D[1]*D[3], -D[2]*D[3], -D[1]*D[2], D[1]*D[4],
> D[2]^2-D[4]^2, 0, D[3]*D[4]],
> [D[2]*D[4], 0, D[2]*D[4], 0, -D[1]*D[4], -D[3]*D[4], -D[2]*D[3], 0,
> D[1]^2+D[3]^2, -D[1]*D[2]],
> [0, D[1]*D[4], D[1]*D[4], 0, -D[2]*D[4], 0, -D[1]*D[3], -D[3]*D[4],
> -D[1]*D[2], D[2]^2+D[3]^2]]);
```

$$R := \left[\begin{array}{c} {D_2}^2 + {D_3}^2 - {D_4}^2\,,\ {D_1}^2\,,\ {D_1}^2\,,\ -{D_1}^2\,,\ -2\,D_1 D_2\,,\ 0\,,\ 0\,,\ -2\,D_1 D_3\,,\ 0\,,\ 2\,D_1 D_4 \\ {D_2}^2\,,\ {D_1}^2 + {D_3}^2 - {D_4}^2\,,\ {D_2}^2\,,\ -{D_2}^2\,,\ -2\,D_1 D_2\,,\ -2\,D_2 D_3\,,\ 0\,,\ 0\,,\ 2\,D_2 D_4\,,\ 0 \\ {D_3}^2\,,\ {D_3}^2\,,\ {D_1}^2 + {D_2}^2 - {D_4}^2\,,\ -{D_3}^2\,,\ 0\,,\ -2\,D_2 D_3\,,\ 2\,D_3 D_4\,,\ -2\,D_1 D_3\,,\ 0\,,\ 0 \\ {D_4}^2\,,\ {D_4}^2\,,\ {D_4}^2\,,\ {D_1}^2 + {D_2}^2 + {D_3}^2\,,\ 0\,,\ 0\,,\ -2\,D_3 D_4\,,\ 0\,,\ -2\,D_2 D_4\,,\ -2\,D_1 D_4 \\ 0\,,\ 0\,,\ D_1 D_2\,,\ -D_1 D_2\,,\ {D_3}^2 - {D_4}^2\,,\ -D_1 D_3\,,\ 0\,,\ -D_2 D_3\,,\ D_1 D_4\,,\ D_2 D_4 \\ D_2 D_3\,,\ 0\,,\ 0\,,\ -D_2 D_3\,,\ -D_1 D_3\,,\ {D_1}^2 - {D_4}^2\,,\ D_2 D_4\,,\ -D_1 D_2\,,\ D_3 D_4\,,\ 0 \\ D_3 D_4\,,\ D_3 D_4\,,\ 0\,,\ 0\,,\ 0\,,\ -D_2 D_4\,,\ {D_1}^2 + {D_2}^2\,,\ -D_1 D_4\,,\ -D_2 D_3\,,\ -D_1 D_3 \\ 0\,,\ D_1 D_3\,,\ 0\,,\ -D_1 D_3\,,\ -D_2 D_3\,,\ -D_1 D_2\,,\ D_1 D_4\,,\ {D_2}^2 - {D_4}^2\,,\ 0\,,\ D_3 D_4 \\ D_2 D_4\,,\ 0\,,\ D_2 D_4\,,\ 0\,,\ -D_1 D_4\,,\ -D_3 D_4\,,\ -D_2 D_3\,,\ 0\,,\ {D_1}^2 + {D_3}^2\,,\ -D_1 D_2 \\ 0\,,\ D_1 D_4\,,\ D_1 D_4\,,\ 0\,,\ -D_2 D_4\,,\ 0\,,\ -D_1 D_3\,,\ -D_3 D_4\,,\ -D_1 D_2\,,\ {D_2}^2 + {D_3}^2 \end{array}\right]$$

```
> R_adj:=adjoint(R, Alg);
```

$$
R\_adj := \left[\begin{array}{c}
D_2{}^2 + D_3{}^2 - D_4{}^2 , D_2{}^2 , D_3{}^2 , D_4{}^2 , 0 , D_2 D_3 , D_3 D_4 , 0 , D_2 D_4 , 0 \\
D_1{}^2 , D_1{}^2 + D_3{}^2 - D_4{}^2 , D_3{}^2 , D_4{}^2 , 0 , 0 , D_3 D_4 , D_1 D_3 , 0 , D_1 D_4 \\
D_1{}^2 , D_2{}^2 , D_1{}^2 + D_2{}^2 - D_4{}^2 , D_4{}^2 , D_1 D_2 , 0 , 0 , 0 , D_2 D_4 , D_1 D_4 \\
-D_1{}^2 , -D_2{}^2 , -D_3{}^2 , D_1{}^2 + D_2{}^2 + D_3{}^2 , -D_1 D_2 , -D_2 D_3 , 0 , -D_1 D_3 , 0 , 0 \\
-2 D_1 D_2 , -2 D_1 D_2 , 0 , 0 , D_3{}^2 - D_4{}^2 , -D_1 D_3 , 0 , -D_2 D_3 , -D_1 D_4 , -D_2 D_4 \\
0 , -2 D_2 D_3 , -2 D_2 D_3 , 0 , -D_1 D_3 , D_1{}^2 - D_4{}^2 , -D_2 D_4 , -D_1 D_2 , -D_3 D_4 , 0 \\
0 , 0 , 2 D_3 D_4 , -2 D_3 D_4 , 0 , D_2 D_4 , D_1{}^2 + D_2{}^2 , D_1 D_4 , -D_2 D_3 , -D_1 D_3 \\
-2 D_1 D_3 , 0 , -2 D_1 D_3 , 0 , -D_2 D_3 , -D_1 D_2 , -D_1 D_4 , D_2{}^2 - D_4{}^2 , 0 , -D_3 D_4 \\
0 , 2 D_2 D_4 , 0 , -2 D_2 D_4 , D_1 D_4 , D_3 D_4 , -D_2 D_3 , 0 , D_1{}^2 + D_3{}^2 , -D_1 D_2 \\
2 D_1 D_4 , 0 , 0 , -2 D_1 D_4 , D_2 D_4 , 0 , -D_1 D_3 , D_3 D_4 , -D_1 D_2 , D_2{}^2 + D_3{}^2
\end{array}\right]
$$

Let us study whether or not the linearized Ricci equations are parametrizable. Let us notice that this problem is related to a question asked by J. Wheeler on the existence of potentials for the Einstein equations [17].

```
> st:=time(): Ext1:=exti(R_adj, Alg, 1): time() - st;
```

$$30.860$$

```
> Ext1[1];
```

$$
\left[\begin{array}{c}
\%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1 , 0 \\
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , \%1
\end{array}\right]
$$

$$\%1 := D_1{}^2 + D_2{}^2 + D_3{}^2 - D_4{}^2$$

Therefore, we see that the linearized Ricci equations are not parametrizable because the system, defined by $R \in D^{10\times 10}$, admits a family of 20 torsion elements which generate the torsion submodule of the D-module $M = D^{10}/D^{10}\, R$. Let us notice that every torsion element of the system satisfies the Dalambertian equation, namely $(\triangle - c^2 \frac{\partial^2}{\partial t^2})\, y = 0$ (travelling wave in space-time).

The list of 20 torsion elements of the system is given by the following 20×10 matrix.

```
> Ext1[2];
```

$$\begin{bmatrix}
0, 0, D_4^2, D_3^2, 0, 0, -2 D_3 D_4, 0, 0, 0 \\
0, 0, -D_2 D_4, 0, 0, D_3 D_4, D_2 D_3, 0, -D_3^2, 0 \\
0, 0, 0, D_2 D_3, 0, D_4^2, -D_2 D_4, 0, -D_3 D_4, 0 \\
0, 0, 0, 0, 0, -D_1 D_4, 0, D_2 D_4, D_1 D_3, -D_2 D_3 \\
0, 0, -D_1 D_4, 0, 0, 0, D_1 D_3, D_3 D_4, 0, -D_3^2 \\
0, 0, 0, D_1 D_3, 0, 0, -D_1 D_4, D_4^2, 0, -D_3 D_4 \\
0, D_3 D_4, 0, 0, 0, -D_2 D_4, D_2^2, 0, -D_2 D_3, 0 \\
0, D_4^2, 0, D_2^2, 0, 0, 0, 0, -2 D_2 D_4, 0 \\
0, D_3^2, D_2^2, 0, 0, -2 D_2 D_3, 0, 0, 0, 0 \\
0, -D_1 D_4, 0, 0, D_2 D_4, 0, 0, 0, D_1 D_2, -D_2^2 \\
0, 0, 0, 0, D_3 D_4, -D_1 D_4, D_1 D_2, 0, 0, -D_2 D_3 \\
0, -D_1 D_3, 0, 0, D_2 D_3, D_1 D_2, 0, -D_2^2, 0, 0 \\
0, 0, 0, D_1 D_2, D_4^2, 0, 0, 0, -D_1 D_4, -D_2 D_4 \\
0, 0, D_1 D_2, 0, D_3^2, -D_1 D_3, 0, -D_2 D_3, 0, 0 \\
D_2 D_4, 0, 0, 0, -D_1 D_4, 0, 0, 0, D_1^2, -D_1 D_2 \\
D_3 D_4, 0, 0, 0, 0, 0, D_1^2, -D_1 D_4, 0, -D_1 D_3 \\
D_2 D_3, 0, 0, 0, -D_1 D_3, D_1^2, 0, -D_1 D_2, 0, 0 \\
D_4^2, 0, 0, D_1^2, 0, 0, 0, 0, 0, -2 D_1 D_4 \\
D_3^2, 0, D_1^2, 0, 0, 0, 0, -2 D_1 D_3, 0, 0 \\
D_2^2, D_1^2, 0, 0, -2 D_1 D_2, 0, 0, 0, 0, 0
\end{bmatrix}$$

The parametrization of the D-module $M/t(M) = D^{10}/D^{20}\,\mathrm{Ext1}[2]$ is defined by:

```
> Ext1[3];
```

$$\begin{bmatrix}
0 & 0 & 0 & -2 D_1 \\
0 & 0 & -2 D_2 & 0 \\
0 & -2 D_3 & 0 & 0 \\
-2 D_4 & 0 & 0 & 0 \\
0 & 0 & -D_1 & -D_2 \\
0 & -D_2 & -D_3 & 0 \\
-D_3 & -D_4 & 0 & 0 \\
0 & -D_1 & 0 & -D_3 \\
-D_2 & 0 & -D_4 & 0 \\
-D_1 & 0 & 0 & -D_4
\end{bmatrix}$$

Therefore, the underdetermined linear system of PDEs $\mathrm{Ext1}[2]\,y = 0$, which is associated with the D-module $M/t(M) = D^{10}/D^{20}\,\mathrm{Ext1}[2]$, is parametrized by Ext1[3], i.e. we have $y = \mathrm{Ext1}[3]\,z$, where $z = (z_1 : z_2 : z_3 : z_4)^T$ are 4 potentials.

```
> st:=time():  exti(R_adj, Alg, 2); time() - st;
```

$$\left[\left[\begin{array}{cccc} D_4 & 0 & 0 & 0 \\ {D_3}^2 & 0 & 0 & 0 \\ D_2\,D_3 & 0 & 0 & 0 \\ D_1\,D_3 & 0 & 0 & 0 \\ {D_2}^2 & 0 & 0 & 0 \\ D_1\,D_2 & 0 & 0 & 0 \\ {D_1}^2 & 0 & 0 & 0 \\ 0 & D_3 & 0 & 0 \\ 0 & {D_4}^2 & 0 & 0 \\ 0 & D_2\,D_4 & 0 & 0 \\ 0 & D_1\,D_4 & 0 & 0 \\ 0 & {D_2}^2 & 0 & 0 \\ 0 & D_1\,D_2 & 0 & 0 \\ 0 & {D_1}^2 & 0 & 0 \\ 0 & 0 & D_2 & 0 \\ 0 & 0 & {D_4}^2 & 0 \\ 0 & 0 & D_3\,D_4 & 0 \\ 0 & 0 & D_1\,D_4 & 0 \\ 0 & 0 & {D_3}^2 & 0 \\ 0 & 0 & D_1\,D_3 & 0 \\ 0 & 0 & {D_1}^2 & 0 \\ 0 & 0 & 0 & D_1 \\ 0 & 0 & 0 & {D_4}^2 \\ 0 & 0 & 0 & D_3\,D_4 \\ 0 & 0 & 0 & D_2\,D_4 \\ 0 & 0 & 0 & {D_3}^2 \\ 0 & 0 & 0 & D_2\,D_3 \\ 0 & 0 & 0 & {D_2}^2 \end{array}\right], \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \mathrm{SURJ}(4)\right]$$

$$2.981$$

```
> st:=time(): exti(R_adj, Alg, 3); time() - st;
```

$$[\mathit{undefined}, \mathit{ZERO}, \mathit{ZERO}]$$

$$1.500$$

We compute a free resolution of the linearized Ricci equations.

```
> st:=time(): resol(R, Alg, 4); time()-st;
```

$$\mathrm{table}([1 = R,\, 2 = \left[\begin{array}{cccccccccc} -D_4 & -D_4 & -D_4 & -D_4 & 0 & 0 & 2\,D_3 & 0 & 2\,D_2 & 2\,D_1 \\ -D_3 & -D_3 & D_3 & D_3 & 0 & 2\,D_2 & -2\,D_4 & 2\,D_1 & 0 & 0 \\ -D_2 & D_2 & -D_2 & D_2 & 2\,D_1 & 2\,D_3 & 0 & 0 & -2\,D_4 & 0 \\ D_1 & -D_1 & -D_1 & D_1 & 2\,D_2 & 0 & 0 & 2\,D_3 & 0 & -2\,D_4 \end{array}\right],$$

$3 = \mathrm{INJ}(4)$,
$4 = \mathit{ZERO}$
])

$$2.290$$

We compute a free resolution of the formal adjoint of the linearized Ricci equations.

```
> st:=time(): resol(R_adj, Alg, 4); time()-st;
```

$$\mathrm{table}([1 = \mathit{R\_adj},\, 2 = \left[\begin{array}{cccccccccc} 0 & 0 & 0 & 2\,D_4 & 0 & 0 & D_3 & 0 & D_2 & D_1 \\ 0 & 0 & 2\,D_3 & 0 & 0 & D_2 & D_4 & D_1 & 0 & 0 \\ 0 & 2\,D_2 & 0 & 0 & D_1 & D_3 & 0 & 0 & D_4 & 0 \\ 2\,D_1 & 0 & 0 & 0 & D_2 & 0 & 0 & D_3 & 0 & D_4 \end{array}\right],$$

$3 = \mathrm{INJ}(4)$,
$4 = \mathit{ZERO}$
])

$$1.460$$

A.7 Lie-Poisson structures

Finally, let us notice that the results developed in this paper can be applied to underdetermined linear systems of PDEs with variable coefficients that appear in some problems of mathematical physics. For instance, let us give an example coming from the study of *Lie-Poisson structures* [3, 26].

Let us consider the Ore algebra $B_3 = \mathbb{R}(x_1, x_2, x_3)[\partial_1; \sigma_1, \delta_1][\partial_2; \sigma_2, \delta_2][\partial_3; \sigma_3, \delta_3]$, where σ_i and δ_i are defined as in the last part of Example 2.3. Since the domain of coefficients of B_3 is the *quotient field* of $\mathbb{R}[x_1, x_2, x_3]$, we are going to use the Weyl algebra $D = A_3$ (see Example 2.3) and allow our algorithms to divide by non-zero polynomials in x_1, x_2, x_3. (This is taken into account by `exti_quot` below.)

```
> Alg:=diff_algebra([D[1],x[1]],[D[2], x[2]],[D[3], x[3]],
> polynom={x[1],x[2],x[3]}):
```

The following example appears in the study of the E_2 algebra [3]. The authors of [3] investigated the possibility to parametrize all the solutions of the system of PDEs defined by the following matrix.

```
> R:=evalm([[x[1]*D[3], x[2]*D[3], 0],
> [-x[1]*D[2]+x[2]*D[1], -1, x[2]*D[3]],
> [-1, -x[2]*D[1]+x[1]*D[2], x[1]*D[3]]]);
```

$$R := \begin{bmatrix} x_1 \, D_3 & x_2 \, D_3 & 0 \\ -x_1 \, D_2 + x_2 \, D_1 & -1 & x_2 \, D_3 \\ -1 & -x_2 \, D_1 + x_1 \, D_2 & x_1 \, D_3 \end{bmatrix}$$

This problem can be solved by computing $\widetilde{\mathrm{ext}_D^1(N, D)}$, where $N = D^3/D^3\,\theta(R)$ is a left D-module and θ is the involution defined in 2 of Example 6.3.

```
> Ext1:=exti_quot(adjoint(R, Alg), Alg, 1);
```

$$Ext1 := \left[\begin{bmatrix} D_3 & 0 \\ -x_1 \, D_2 + x_2 \, D_1 & 0 \\ 0 & D_3 \\ 0 & -x_1 \, D_2 + x_2 \, D_1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & -{x_1}^2 \, D_2 + x_1 \, D_1 \, x_2 - x_2 & -{x_1}^2 \, D_3 \end{bmatrix}, \begin{bmatrix} x_2 \, D_3 \\ -x_1 \, D_3 \\ -x_2 \, D_1 + x_1 \, D_2 \end{bmatrix} \right]$$

We find that the system is not parametrizable because there exist two torsion elements $\Phi_1 = x_1\,F + x_2\,G$ and $\Phi_2 = (-x_1^2\,\partial_2 + x_1\,x_2\,\partial_1 - x_2)\,G - x_1^3\,\partial_3\,H$ which both satisfy the system:

$$\begin{cases} \partial_3\,\Phi_i = 0, \\ (-x_1\,\partial_2 + x_2\,\partial_1)\,\Phi_i = 0, \end{cases} \quad i = 1, 2.$$

However, the system of PDEs $\mathrm{Ext1}[2]\,y = 0$ is parametrized by $y = \mathrm{Ext1}[3]\,z$. Up to the mistake underlined in [26] concerning the existence of the torsion elements, we recover the parametrization exhibited in [3]. We refer the interested reader to [26] for more details.

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