PARAMETRIZING ALL SOLUTIONS OF UNCONTROLLABLE MULTIDIMENSIONAL LINEAR SYSTEMS

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Abstract: Using an algebraic analysis approach, we derive a necessary and sufficient condition so that we can parametrize all solutions of a multidimensional linear system by glueing the controllable sub-behaviour with the autonomous elements. Effective algorithms checking this condition are obtained. This result generalizes a result of 1-D linear systems for a class of multidimensional linear systems.

Keywords: Multidimensional linear systems, controllability, parametrizability, autonomous elements, linear systems over Ore algebras, Gröbner bases.

1. INTRODUCTION

Let us first show how to parametrize all solutions of a time-invariant linear control system defined by ordinary differential equations.

We consider the commutative polynomial ring $D = \mathbb{R}\left[\frac{d}{dt}\right]$ of differential operators in $\frac{d}{dt}$ with coefficients in the field $\mathbb{R}$. An element of D has the form $\sum_{i=0}^{n} a_i \frac{d^i}{dt^i}$ where $a_i \in \mathbb{R}$. Let us consider a *full row rank matrix* $R \in D^{q\times p}$, i.e., the rows of R are D-linearly independent. Then, computing a Smith form for R, we obtain

$$R = U\,(\mathrm{diag}(d_1, \dots, d_q) \quad 0)\,V,$$

where the matrices $U \in D^{q\times q}$ and $V \in D^{p\times p}$ are unimodular, i.e., $\det U$ and $\det V$ are non-zero constants, diag denotes the diagonal matrix and $0 \neq d_i \in D$. Hence, R can be written as:

$$\begin{cases} R = R''\,R', \\ R'' = U\,\mathrm{diag}(d_1, \dots, d_q) \in D^{q\times q}, \\ R' = (I_q \quad 0)\,V \in D^{q\times p}. \end{cases}$$

If we denote by $r = p - q$ and $V = (V_1^T \quad V_2^T)^T$, $V_1 \in D^{q\times p}$, $V_2 \in D^{r\times p}$, then, from the latter of the previous equations, we obtain $R' = V_1$. Using the fact that V is unimodular, then the entries of V^{-1} belong to D. If we denote by $V^{-1} = (S \quad Q)$, where $S \in D^{p\times q}$ and $Q \in D^{p\times r}$, then we obtain:

$$\begin{pmatrix} R' \\ V_2 \end{pmatrix} (S \quad Q) = I_p, \tag{1}$$

$$(S \quad Q) \begin{pmatrix} R' \\ V_2 \end{pmatrix} = I_p. \tag{2}$$

Now, solving the system $R\,\eta = 0$ is equivalent to solve the following system:

$$\begin{cases} R''\,\tau = 0, \\ \tau = R'\,\eta. \end{cases}$$

The first system $R''\,\tau = 0$ is equivalent to

$$d_1\,\tau_1 = 0, \dots, d_q\,\tau_q = 0, \tag{3}$$

where $\tau = (\tau_1 \dots \tau_q)^T$. We denote by $\overline{\tau}$ a fundamental solution of $R''\,\tau = 0$ in a signal space $\mathcal{F}$ which has a D-module structure (e.g., $C^{\infty}, \mathcal{D}'$).

Then, we need to solve the inhomogeneous system $R' \eta = \overline{\tau}$. But, using (1), we obtain $R' S = I_q$, and thus, a particular solution for $R' \eta = \overline{\tau}$ is given by $\overline{\eta} = S \overline{\tau} \in \mathcal{F}^q$. Moreover, (2) is equivalent to $S R' + Q V_2 = I_p$, and thus, if $R' \eta = 0$, then we have $\eta = S (R' \eta) + Q (V_2 \eta) = Q (V_2 \eta)$, showing that a general solution of the homogeneous system $R' \eta = 0$ is given by $\eta = Q \xi$ for a certain $\xi \in \mathcal{F}^r$.

Therefore, $\mathrm{Sol}_{\mathcal{F}}(R) = \{\eta \in \mathcal{F}^p \mid R \eta = 0\}$ has the following explicit parametrization:

$$\eta = S \overline{\tau} + Q \xi = (S \quad Q) \begin{pmatrix} \overline{\tau} \\ \xi \end{pmatrix}, \; \forall \xi \in \mathcal{F}^r. \quad (4)$$

We note that

$$\begin{aligned} \mathrm{Sol}_{\mathcal{F}}(R'') &= \mathrm{Sol}_{\mathcal{F}}(\mathrm{diag}(d_1, \ldots, d_q)) \\ &= \{\tau \in \mathcal{F}^q \mid d_1 \tau_1 = 0, \ldots, d_q \tau_q = 0\} \end{aligned}$$

is a finite-dimensional $\mathbb{R}$-vector space. Let us call l its dimension and let us denote by $\{\theta_i\}_{1 \leq i \leq l}$ one of its bases. Then, the general solution $\overline{\tau}$ can be written as $\overline{\tau} = \sum_{i=1}^{l} c_i \theta_i$, where $c_i \in \mathbb{R}$. Therefore, we obtain the parametrization $\mathrm{Sol}_{\mathcal{F}}(R) = \Phi(\mathbb{R}^l \times \mathcal{F}^r)$ where Φ is defined by:

$$\begin{aligned} \Phi : \qquad \mathbb{R}^l \times \mathcal{F}^r &\longrightarrow \mathcal{F}^p, \\ (c_1 \ldots c_l, \xi)^T &\longmapsto \sum_{i=1}^{l} c_i (S \theta_i) + Q \xi. \end{aligned} \quad (5)$$

Finally, if the set of initial conditions of the system $R \eta = 0$ is known, then the corresponding constants c_i can be explicitly computed.

We point out that the existence of non-trivial d_i in the Smith form of R (i.e., existence of $d_i \in D \backslash \mathbb{R}$) is equivalent to the lack of controllability of the system $R \eta = 0$. The $\mathbb{R}$-vector space

$$\mathrm{Sol}_{\mathcal{F}}(R') = \{\eta \in \mathcal{F}^p \mid R' \eta = 0\} \quad (6)$$

is called the *controllable sub-behaviour* of the *behaviour* $\mathrm{Sol}_{\mathcal{F}}(R) = \{\eta \in \mathcal{F}^p \mid R \eta = 0\}$, whereas

$$\mathrm{Sol}_{\mathcal{F}}(R'') = \{\tau \in \mathcal{F}^q \mid R'' \tau = 0, \quad \tau = R' \eta\}$$

is called the *autonomous behaviour*. For time-invariant ordinary differential equations, it is well-known that $\mathrm{Sol}_{\mathcal{F}}(R'')$ can be interpreted as a sub-behaviour of $\mathrm{Sol}_{\mathcal{F}}(R)$ and we have:

$$\mathrm{Sol}_{\mathcal{F}}(R) = \mathrm{Sol}_{\mathcal{F}}(R') \oplus \mathrm{Sol}_{\mathcal{F}}(R''). \quad (7)$$

The controllable sub-behaviour $\mathrm{Sol}_{\mathcal{F}}(R')$ can be parametrized. See (Polderman and Willems, 1998; Pommaret and Quadrat, 1998) for more details.

The main interest of (4) is to parametrize the behaviour $\mathrm{Sol}_{\mathcal{F}}(R)$ and not simply the controllable sub-behaviour $\mathrm{Sol}_{\mathcal{F}}(R')$. Parametrizations (4) and (5) show how to glue elements of $\mathrm{Sol}_{\mathcal{F}}(R'')$ with those of $\mathrm{Sol}_{\mathcal{F}}(R')$ in order to obtain all trajectories of the system $\mathrm{Sol}_{\mathcal{F}}(R)$.

The purpose of this paper is to show when and how it is possible to extend the previous construction to multidimensional linear systems defined over *Ore algebras* (e.g., differential time-delay systems, partial differential equations, discrete systems) with constant or variable coefficients (Chyzak *et al.*, 2005).

2. MODULE-THEORETIC APPROACH

For multidimensional linear systems defined over (non-commutative) multivariate polynomial rings, no Smith form exists. Therefore, we cannot copy the results obtained in the introduction in order to parametrize all solutions of such systems.

In order to cope with this problem, we introduce concepts of module theory. In what follows, D denotes a (non-commutative) Ore algebra which is a left and right *noetherian domain* (Chyzak *et al.*, 2005). Then, D satisfies the *left* and *right Ore properties*, namely

$$\begin{aligned} &\forall d_1, d_2 \in D, \; \exists (u_1, u_2), (v_1, v_2) \in D^2 \backslash (0, 0) : \\ &\qquad u_1 d_1 = u_2 d_2, \quad d_1 v_1 = d_2 v_2, \end{aligned}$$

and

$$K = \{d^{-1} n = \tilde{n} \tilde{d}^{-1} \mid 0 \neq d, n, 0 \neq \tilde{d}, \tilde{n} \in D\}$$

is the (left and right) *quotient division ring* of D.

Let $R \in D^{q \times p}$ and $M = D^{1 \times p} / (D^{1 \times q} R)$ be the *finitely presented* left D-module defined as the cokernel of the D-morphism:

$$.R : D^{1 \times q} \longrightarrow D^{1 \times p}, \quad \lambda \longmapsto \lambda R. \quad (8)$$

The left D-module $M = D^{1 \times p} / (D^{1 \times q} R)$ is associated with the system $R \eta = 0$ in the sense that

$$\hom_D(M, \mathcal{F}) = \{\eta \in \mathcal{F}^p \mid R \eta = 0\}, \quad (9)$$

where $\hom_D(M, \mathcal{F})$ denotes the abelian group formed by the D-morphisms from M to the left D-module $\mathcal{F}$. See (Pommaret and Quadrat, 2003) for more details. Moreover, M is defined by the D-linear combinations of the equations $R y = 0$, where the components of $y = (y_1 \ldots y_p)^T$ are the generators of M, i.e., y_i is the class in M of the i^{th} vector e_i of the standard basis of $D^{1 \times p}$. See (Chyzak *et al.*, 2005) for more details.

Definition 1. The left D-module M is said to be:

- *Free* if there exists $r \in \mathbb{Z}_+$ such that M is isomorphic to $D^{1 \times r}$ (denoted by $M \cong D^{1 \times r}$).
- *Projective* if there exist a left D-module N and $r \in \mathbb{Z}_+$ such that $M \oplus N \cong D^{1 \times r}$.
- *Torsion-free* if the *torsion-submodule*

  $$t(M) = \{m \in M \mid \exists \, 0 \neq d \in D : \; d m = 0\}$$

  of M is trivial, i.e., $t(M) = 0$. Elements of $t(M)$ are called *torsion elements* of M.
- *Torsion* if $t(M) = M$.

We give characterizations of the previous properties. We refer to (Pommaret and Quadrat, 1998; Pommaret and Quadrat, 2003) for the proofs.

Theorem 1. Let us consider $R \in D^{q\times p}$ and the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$. Then, we have:

(1) M is a free left D-module iff there exist $Q \in D^{p\times m}$ and $T \in D^{m\times p}$ such that:

$$\begin{cases} \ker(.Q) \triangleq \{\lambda \in D^{1\times p} \mid \lambda\, Q = 0\} = D^{1\times q}\, R, \\ T\, Q \qquad = I_m. \end{cases}$$

(2) M is a projective left D-module iff there exists $S \in D^{p\times q}$ such that $R\, S\, R = R$.

(3) $t(M) = ((K^{1\times q}\, R) \cap D^{1\times p})/(D^{1\times q}\, R)$, and thus, M is a torsion-free left D-module iff:

$$(K^{1\times q}\, R) \cap D^{1\times p} = D^{1\times q}\, R.$$

(4) $M/t(M) = D^{1\times p}/((K^{1\times q}\, R) \cap D^{1\times p})$.

(5) M is a torsion left D-module iff:

$$(K^{1\times q}\, R) \cap D^{1\times p} = D^{1\times p}.$$

We recall some results (Rotman, 1979).

Theorem 2. (1) The following implications hold:

$$\text{free} \;\Rightarrow\; \text{projective} \;\Rightarrow\; \text{torsion-free}.$$

(2) If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field k, then every projective D-module is free.

(3) If D is a *(left) principal ideal domain*, i.e., every (left) ideal of D can be generated by means of one element, (e.g., $D = \mathbb{R}\left[\frac{d}{dt}\right]$, $\mathbb{R}(t)\left[\frac{d}{dt}\right]$), then every torsion-free (left) D-module M is free.

There exists an algorithm which computes a matrix $R' \in D^{q'\times p}$ such that:

$$(K^{1\times q}\, R) \cap D^{1\times p} = D^{1\times q'}\, R'.$$

See (Chyzak *et al.*, 2005; Pommaret and Quadrat, 1998) for more details. This algorithm is implemented in the Maple package OreModules (Chyzak *et al.*, 2003) (see procedure Exti) for rings of differential operators with polynomial or rational coefficients (i.e., $D = A[\partial_1, \ldots, \partial_n]$, where $A = \mathbb{R}$, $\mathbb{R}[x_1, \ldots, x_n]$, $\mathbb{R}(x_1, \ldots, x_n)$), for time-invariant time-delay systems (i.e., $D = \mathbb{R}[\frac{d}{dt}, \delta_{h_1}, \ldots, \delta_{h_r}]$ with incommensurable delays $\delta_{h_i} f(t) = f(t - h_i)$) and multidimensional discrete systems (i.e., $D = A[\sigma_1, \ldots, \sigma_r]$, where $\sigma_i z(n_1, \ldots, n_r) = z(n_1, \ldots, n_{i+1}, \ldots n_r)$ and $A = \mathbb{R}$, $\mathbb{R}[n_1, \ldots, n_r]$, $\mathbb{R}(n_1, \ldots, n_r)$). Using OreModules, we can effectively test whether or not a finitely presented left D-module has non-trivial torsion elements, is torsion-free or projective.

Definition 2. A sequence of D-modules P_i and D-morphisms $d_i : P_i \rightarrow P_{i-1}$ satisfying $d_i \circ d_{i+1} = 0$, i.e., $\operatorname{im} d_{i+1} \subseteq \ker d_i$, is said to be *exact at P_r* if $\ker d_r = \operatorname{im} d_{r+1}$ and *exact* if it is exact at all P_r.

Example 1. The following sequence of morphisms

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \qquad (10)$$

is exact iff f is injective, $\operatorname{im} f = \ker g$ and g is surjective. (10) is called a *short exact sequence.*

Using the embedding i of $t(M)$ into M and the canonical projection ρ of M onto $M/t(M)$, we obtain the following short exact sequence:

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0. \qquad (11)$$

If $M = D^{1\times p}/(D^{1\times q}\, R)$, then, from the D-morphism (8), we have the exact sequence

$$0 \longrightarrow \ker(.R) \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \qquad (12)$$

where $\pi(\lambda)$ is the class of $\lambda \in D^{1\times p}$ in M.

Definition 3. A left D-module $\mathcal{F}$ is said to be *injective* if, for every short exact sequence (10), we have the following short exact sequence

$$0 \longleftarrow \hom_D(M', \mathcal{F}) \xleftarrow{f^\star} \hom_D(M, \mathcal{F}) \xleftarrow{g^\star} \hom_D(M'', \mathcal{F}) \longleftarrow 0,$$

where $f^\star(\phi) \triangleq \phi \circ f$ for all $\phi \in \hom_D(M, \mathcal{F})$.

Theorem 3. (Malgrange, 1966) If Ω is a convex open subset of $\mathbb{R}^n$, then $C^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ are injective $D = \mathbb{R}[\partial_1, \ldots, \partial_n]$-modules ($\partial_i = \frac{\partial}{\partial x_i}$).

Definition 4. A short exact sequence (10) *splits* if one of the following equivalent conditions holds:

(1) There exists a D-morphism $h : M'' \longrightarrow M$ such that $g \circ h = id_{M''}$.

(2) There exists a D-morphism $k : M \longrightarrow M'$ such that $k \circ f = id_{M'}$.

(3) We have an isomorphism $M \cong M' \oplus M''$.

Proposition 1. (Rotman, 1979) If M'' is a projective D-module, then the short exact sequence (10) splits and we have $M \cong M' \oplus M''$.

3. A NECESSARY AND SUFFICIENT CONDITION

Let us investigate when the exact sequence (11) splits. A first case is when $M/t(M)$ is a projective left D-module. Indeed, by Proposition 1, the exact sequence (11) splits and we obtain:

$$M \cong t(M) \oplus M/t(M). \qquad (13)$$

In particular, if $D = \mathbb{R}\left[\frac{d}{dt}\right]$ or $\mathbb{R}(t)\left[\frac{d}{dt}\right]$, then, using 3 of Theorem 2, we obtain that $M/t(M)$ is free, and thus, projective by 1 of Theorem 2. The same result holds over the ring $\mathbb{R}[t][\frac{d}{dt}]$ as every torsion-free left D-module is projective (Chyzak *et al.*, 2005). We shall show in Section 5 how the direct sum (13) of left D-modules implies the direct sum (7) between the controllable and autonomous sub-behaviours.

Lemma 1. Let us consider $M = D^{1\times p}/(D^{1\times q}\,R)$ and $M/t(M) = D^{1\times p}/(D^{1\times q'}\,R')$. Then, there exists $R'' \in D^{q\times q'}$ such that $R = R''\,R'$ and we have the following commutative exact diagram

$$\begin{tikzcd}
D^{1\times q} \arrow[r, ".R"] \arrow[d, ".R''"'] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\rho"] & 0 \\
D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p} \arrow[r, "\pi'"] & M/t(M) \arrow[r] \arrow[d] & 0, \\
 & & 0 &
\end{tikzcd} \qquad (14)$$

where π' is defined by $\pi' = \pi \circ \rho$.

PROOF. By 4 of Theorem 1, we have

$$(K^{1\times q}\,R) \cap D^{1\times p} = D^{1\times q'}\,R'$$

and we check that $D^{1\times q}\,R \subseteq (K^{1\times q}\,R) \cap D^{1\times p}$, which proves that $D^{1\times q}\,R \subseteq D^{1\times q'}\,R'$. Then, every row R_i of R belongs to $D^{1\times q'}\,R'$, and thus, there exists $R''_i \in D^{1\times q'}$ such that $R_i = R''_i\,R'$. If we denote by $R'' = ((R''_1)^T \dots (R''_q)^T)^T$, then we obtain $R = R''\,R'$. The commutative diagram (14) directly follows from (11), (12) and $R = R''\,R'$.

The matrix $R'' \in D^{q\times q'}$ can be computed using the procedure FACTORIZE of OREMODULES.

Theorem 4. Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$ be a left D-module and $R' \in D^{q'\times p}$ a matrix such that $M/t(M) = D^{1\times p}/(D^{1\times q'}\,R')$. Then, the short exact sequence (11) splits, i.e., we have (13), iff there exist $S \in D^{p\times q'}$ and $V \in D^{q'\times q}$ such that:

$$R' - R'\,S\,R' = V\,R. \qquad (15)$$

PROOF. Let us suppose that there exist matrices $S \in D^{p\times q'}$ and $V \in D^{q'\times q}$ satisfying (15) and let us denote by $U = I_p - S\,R'$. Then, we have:

$$U = I_p - S\,R', \quad R'\,U = V\,R. \qquad (16)$$

From the last equality, we obtain the following commutative exact diagram:

$$\begin{tikzcd}
D^{1\times q'} \arrow[r, ".R'"] \arrow[d, ".V"'] & D^{1\times p} \arrow[r, "\pi'"] \arrow[d, ".U"'] & M/t(M) \arrow[r] & 0 \\
D^{1\times q} \arrow[r, ".R"] & D^{1\times p} \arrow[r, "\pi"] & M \arrow[r] & 0.
\end{tikzcd} \qquad (17)$$

Then, the D-morphism $h : M/t(M) \longrightarrow M$, defined by $h(m') = \pi(\lambda\,U)$, where $\lambda \in D^{1\times p}$ is any element satisfying $m' = \pi'(\lambda)$, is well-defined. Then, using $\pi' = \rho \circ \pi$, for $m' \in M/t(M)$, we have $(\rho \circ h)(m') = \rho(\pi(\lambda\,U)) = \pi'(\lambda\,U)$, and thus,

$$\begin{aligned}(\rho \circ h - id_{M/t(M)})(m') &= \pi'(\lambda\,U) - \pi'(\lambda) \\ &= \pi'(\lambda\,(U - I_p)) = -\pi'(\lambda\,(S\,R')) = 0,\end{aligned}$$

because $(\lambda\,S)\,R' \in D^{1\times q'}\,R'$ and π' is the canonical projection onto $M/t(M) = D^{1\times p}/(D^{1\times q'}\,R')$. Therefore, we have $\rho \circ h = id_{M/t(M)}$ showing that (11) splits by 1 of Definition 4.

Conversely, let us suppose that there exists a D-morphism h satisfying $\rho\circ h = id_{M/t(M)}$. We denote by $e_i \in D^{1\times p}$ the vector with 1 in the i^{th} position and 0 elsewhere. Then, we have $(h \circ \pi')(e_i) \in M$, and thus, there exists $U_i \in D^{1\times p}$ such that

$$(h \circ \pi')(e_i) = \pi(U_i), \quad i = 1, \dots, p,$$

as π is surjective. If we define $U = (U_1^T \dots U_p^T)^T \in D^{p\times p}$, then we have $\pi \circ (.U) = h \circ \pi'$ and:

$$\begin{aligned}&\pi \circ (.U \circ (.R')) = h \circ \pi' \circ (.R') = 0, \\ &\Rightarrow D^{1\times q'}\,(R'\,U') \subseteq \ker \pi = D^{1\times q}\,R.\end{aligned}$$

In particular, if R'_j denotes the j^{th} row of R', then we have $R'_j\,U \in D^{1\times q}\,R$, and thus, there exists $V_j \in D^{1\times q}$ such that $R'_j\,U = V_j\,R$. If we denote by $V = (V_1^T \dots V_{q'}^T)^T \in D^{q'\times q}$, then we obtain $R'\,U = V\,R$ and the commutative exact diagram (17). Composing (17) and (14), we obtain the following commutative diagram:

$$\begin{tikzcd}
D^{1\times q'} \arrow[r, ".R'"] \arrow[d, ".(V\,R'')"'] & D^{1\times p} \arrow[r, "\pi'"] \arrow[d, ".U"'] & M/t(M) \arrow[r] \arrow[d, "\rho\circ h"'] & 0 \\
D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p} \arrow[r, "\pi'"] & M/t(M) \arrow[r] & 0.
\end{tikzcd} \qquad (18)$$

We have $(\rho \circ h)(\pi'(e_i)) = \pi'(e_i\,U)$, and using the fact that $\rho \circ h = id_{M/t(M)}$, we obtain:

$$\begin{aligned}&\pi'(e_i) = \pi'(e_i\,U) \Rightarrow \pi'(e_i\,(I_p - U)) = 0, \\ &\Rightarrow \exists\; S_i \in D^{1\times q'} :\; e_i\,(I_p - U) = S_i\,R',\; 1 \le i \le p, \\ &\Rightarrow \exists\; S = (S_1^T \dots S_p^T)^T \in D^{p\times q'} :\; U - I_p = S\,R'.\end{aligned}$$

Therefore, we have just proved the existence of $U \in D^{p\times p}$, $V \in D^{q'\times q}$ and $S \in D^{p\times q'}$ satisfying (16), or equivalently, (15) by eliminating U.

We note that we have $h(M/t(M)) \oplus t(M) = M$, where h is defined in the beginning of the proof. Condition (15) corresponds to the existence of a *generalized inverse* S of R' modulo $D^{1\times q}\,R$.

Pommaret has just pointed out to us that a similar result had already appeared in (Zerz and Lomadze, 2001) with a different proof. We want to acknowledge this priority. However, the purposes of the last paper are different and we also study here the non-commutative case.

4. ALGORITHMS

We first consider the case where D is a commutative polynomial ring. Then, we use the fact that the product $U \cdot V \cdot W$ of matrices $U \in D^{a\times b}$, $V \in D^{b\times c}$, $W \in D^{c\times d}$ can be written as a row

$$(V_1 \dots V_b) \cdot (U^T \otimes W), \qquad (19)$$

where $V_1, \ldots, V_b$ are the rows of V and $\otimes$ denotes the tensor product of matrices. We continue to use single subscripts to denote the rows of a matrix. Then, it is easy to see that (15) can be written as:

$$\begin{aligned}(R'_1 \ldots R'_{q'}) = (S_1 \ldots S_p)\,(R'^T \otimes R') \\ +(V_1 \ldots V_{q'})\,(I_{q'} \otimes R).\end{aligned}$$

We obtain the inhomogeneous system $f\,T = g$:

$$f = (S_1 \ldots S_p \quad V_1 \ldots V_{q'}) \in D^{1 \times (p+q)q'}$$

$$T = \begin{pmatrix} R'^T \otimes R' \\ I_{q'} \otimes R \end{pmatrix} \in D^{(p+q)q' \times pq'}, \tag{20}$$

$$g = (R'_1 \ldots R'_{q'}) \in D^{1 \times pq'}. \tag{21}$$

Algorithm 1. Input: $R \in D^{q \times p}$, $R' \in D^{q' \times p}$.
Output: $U \in D^{p \times p}$ such that the D-module $h(M/t(M)) \triangleq \pi(D^{1 \times p}\,U)$ is a direct complement of $t(M)$ in M, or $\emptyset$ if no such complement exists.

```
ComplementConstCoeff $(R, R')$
  Define $T$ and $g$ as in (20), (21).
  $G \leftarrow$ Gröbner basis of the rows of $T$.
  $r \leftarrow$ Normal form of $g$ modulo $G$.
  if $r = 0$ then
    From the reduction of $g$ modulo $G$,
      find $f \in D^{1 \times (p+q)q'}$ s.t. $f\,T = g$.
    Construct the matrix $S$ from $f$:
    $S_{i,j} \leftarrow f_{1,(i-1)\,q'+j}$, $1 \leq i \leq p$, $1 \leq j \leq q'$.
    return $U = I_p - S\,R'$
  else
    There exists no solution of (15) ; return $\emptyset$
  endif
```

If now we consider the non-commutative Weyl algebra $D = K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n]$, then, the product $U \cdot V \cdot W$ of matrices U, V, W can no longer be written as in (19). If $S \in D^{p \times q'}$ were given, (15) could be viewed as the problem to factorize the matrix $R' - R'\,S\,R'$ as a product $V\,R$ with a suitable $V \in D^{q' \times q}$. Assuming $S_{ij} = \sum_{k,l \in \{0,\ldots,d\}^n} a_{k,l}^{(i,j)}\, x^k\, \partial^l$ with indeterminates $a_{k,l}^{(i,j)}$, we derive a system of equations in $a_{k,l}^{(i,j)}$ that characterizes the above factorizability.

Algorithm 2. Input: $d \in \mathbb{N}$, $R \in D^{q \times p}$, $R' \in D^{q' \times p}$.
Output: $U \in D^{p \times p}$ of order d such that the left D-module $h(M/t(M)) \triangleq \pi(D^{1 \times p}\,U)$ is a direct complement of $t(M)$ in M, or $\emptyset$ if no such complement exists.

```
Complement $(R, R', d)$
  Introduce the indeterminates $\lambda_j$, $j = 1, \ldots, p$,
    $\mu_i$, $i = 1, \ldots, q$, and $a_{k,l}^{(i,j)}$ over $D$.
  $P \leftarrow \{\sum_{j=1}^{p} R_{ij}\,\lambda_j - \mu_i \mid i = 1, \ldots, q\}$.
  Compute the Gröbner basis $G$ of $P$ in
    $\bigoplus_{i=1}^{p} D\,\lambda_i \oplus \bigoplus_{i=1}^{q} D\,\mu_i \oplus \bigoplus D\,a_{k,l}^{(i,j)}$ w.r.t.
    an order which eliminates the $\lambda_i$'s.
  Let $S_{ij} = \sum_{k,l \in \{0,\ldots,d\}^n} a_{k,l}^{(i,j)}\, x^k\, \partial^l$.
  $X \leftarrow R' - R'\,S\,R' \in D^{q' \times p}$;   $H \leftarrow \emptyset$.
  for $i = 1, \ldots, q'$ do
    $F_i \leftarrow$ normal form of $\sum_{j=1}^{p} X_{ij}\,\lambda_j$ modulo $G$.
    Augment $H$ with all non-zero coefficients of
      $x^k \partial^l \lambda_j$ in $F_i$ for all $1 \leq k, l \leq n$, $1 \leq j \leq p$
  endfor
  Solve the linear system given by $H$ for $a_{k,l}^{(i,j)}$.
  if the linear system has a solution $(\tilde{a}_{k,l}^{(i,j)})$ then
    Plug all $\tilde{a}_{k,l}^{(i,j)}$ into $S$.
    return $U = I_p - S\,R'$.
  else
    No complement of order $d$ ; return $\emptyset$
  endif
```

See OreModules for implementations.

5. PARAMETRIZING ALL SOLUTIONS

Now, we only investigate the case where condition (15) of Theorem 4 is fulfilled. Then, we have (13). By applying the functor $\hom_D(\cdot, \mathcal{F})$ to (14) and (17), we obtain the commutative exact diagrams

$$\begin{tikzcd}
 & & 0 & \\
 & 0 & \hom_D(t(M), \mathcal{F}) \arrow[u] & \\
\mathcal{F}^q & \mathcal{F}^p \arrow[l, "R."'] \arrow[u] & \mathrm{Sol}_{\mathcal{F}}(R) \arrow[l] \arrow[u, "i^\star"] & 0 \arrow[l] \\
\mathcal{F}^{q'} \arrow[u, "R''."] & \mathcal{F}^p \arrow[l, "R'."'] \arrow[u, equal] & \mathrm{Sol}_{\mathcal{F}}(R') \arrow[l] \arrow[u, "\rho^\star"] & 0, \arrow[l] \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 & 0 & \\
\mathcal{F}^{q'} & \mathcal{F}^p \arrow[l, "R'."'] \arrow[u] & \mathrm{Sol}_{\mathcal{F}}(R') \arrow[l] \arrow[u] & 0 \arrow[l] \\
\mathcal{F}^q \arrow[u, "V."] & \mathcal{F}^p \arrow[l, "R."'] \arrow[u, "U."] & \mathrm{Sol}_{\mathcal{F}}(R) \arrow[l] \arrow[u, "h^\star"] & 0, \arrow[l] \\
 & 0 \arrow[u] & \hom_D(t(M), \mathcal{F}) \arrow[u, "k^\star"] & \\
 & & 0 \arrow[u] &
\end{tikzcd}$$

where $k : M \longrightarrow t(M)$ denotes the D-morphism satisfying $k \circ i = id_{t(M)}$. Then, we have:

$$\begin{aligned}\mathrm{Sol}_{\mathcal{F}}(R) = \rho^\star(\mathrm{Sol}_{\mathcal{F}}(R')) \oplus \\ k^\star(\hom_D((D^{1 \times q'}\,R')/(D^{1 \times q}\,R), \mathcal{F})).\end{aligned} \tag{22}$$

$\mathrm{Sol}_{\mathcal{F}}(R')$ is called the *controllable sub-behaviour* of $\mathrm{Sol}_{\mathcal{F}}(R)$, whereas $\hom_D(t(M), \mathcal{F})$ cannot generally be interpreted as a sub-behaviour of $\mathrm{Sol}_{\mathcal{F}}(R)$. However, in the previous case, $k^\star(\hom_D(t(M), \mathcal{F}))$ is a sub-behaviour of $\mathrm{Sol}_{\mathcal{F}}(R)$ which we call the *non-controllable sub-behaviour*.

From (22), it follows that computing $\mathrm{Sol}_{\mathcal{F}}(R)$ can be decomposed into two problems:

(1) Computing $\mathrm{Sol}_{\mathcal{F}}(R')$.
(2) Computing $\hom_D((D^{1\times q'}\, R')/(D^{1\times q}\, R), \mathcal{F})$.

Theorem 5. Let $M' = D^{1\times p}/(D^{1\times q'}\, R')$ be a torsion-free left D-module. Then, there exists a matrix $Q \in D^{p\times m}$ such that we have the following exact sequence $D^{1\times q'} \xrightarrow{.R'} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$.

We refer to (Chyzak *et al.*, 2005) for a constructive proof and an implementation in OreModules.

Corollary 1. Let $\mathcal{F}$ be an injective left D-module. With the hypothesis and the notations of Theorem 5, we obtain the following exact sequence

$$\mathcal{F}^{q'} \xleftarrow{R'.} \mathcal{F}^{p} \xleftarrow{Q.} \mathcal{F}^{m}, \tag{23}$$

i.e., we have $\mathrm{Sol}_{\mathcal{F}}(R') = Q\, \mathcal{F}^m$. This result holds for the $D = \mathbb{R}[\partial_1, \ldots, \partial_n]$-modules $\mathcal{F} = C^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ and an open convex subset Ω of $\mathbb{R}^n$.

If we denote by θ_i the class of the i^{th} row of R' in $t(M) = (D^{1\times q'}\, R')/(D^{1\times q}\, R)$, then $\{\theta_i\}_{1\leq i\leq q'}$ is a family of generators of the torsion submodule $t(M)$ of M (Chyzak *et al.*, 2005). Then, for every torsion element $\theta_i \neq 0$, there exists a family $\mathrm{ann}_D(\theta_i)$ of non-zero elements of D satisfying: $\forall\ d \in \mathrm{ann}_D(\theta_i),\ d\, \theta_i = 0$. We refer to (Chyzak *et al.*, 2005; Pommaret and Quadrat, 1998) for a description of the algorithm computing $\mathrm{ann}_D(\theta_i)$.

If $\eta \in \mathcal{F}^p$ is a solution of $R\, \eta = 0$, then, we have the following autonomous elements:

$$\tau_i \triangleq R'_i\, \eta \in \hom_{\mathcal{F}}(t(M), \mathcal{F}),\ i = 1, \ldots, q'. \tag{24}$$

Lemma 2. Let us consider the left D-modules

$M = D^{1\times p}/(D^{1\times q}\, R), M/t(M) = D^{1\times p}/(D^{1\times q'}\, R')$

and $R'' \in D^{q\times q'}$ the matrix defined by $R = R''\, R'$ (see Lemma 1). Then, we have:

(1) There exist $L \in D^{r\times q'}$ and $L' \in D^{r\times q}$ s.t.:

$$F \triangleq \ker\left(.\begin{pmatrix} R' \\ R \end{pmatrix}\right) = D^{1\times r}\,(L \quad L').$$

(2) If $\ker(.R') = D^{1\times r'}\, T$, then we have:

$$F = D^{1\times r'}\,(T \quad 0) + D^{1\times q}\,(R'' \quad -I_q).$$

If $\mathcal{F}$ is an injective left D-module, then τ_i defined in (24) satisfy the following equivalent systems:

$$L\, \tau = 0 \Leftrightarrow \begin{cases} R''\, \tau = 0, \\ T\, \tau = 0. \end{cases} \tag{25}$$

PROOF. 1 is satisfied as D is a noetherian ring.

2. Let us consider $\lambda = (\lambda_1 \quad \lambda_2) \in F$. Then, we have $\lambda_1\, R' + \lambda_2\, R = 0$ and, using $R = R''\, R'$, we obtain $(\lambda_1 + \lambda_2\, R'')\, R' = 0$, and thus, we have $\lambda_1 + \lambda_2\, R'' \in \ker(.R') = D^{1\times r'}\, T$. Then, there exists $\mu \in D^{1\times r'}$ satisfying $\lambda_1 = \mu\, T - \lambda_2\, R''$ implying $(\lambda_1 \quad \lambda_2) = \mu\,(T \quad 0) - \lambda_2\,(R'' \quad -I_q)$

$$\Rightarrow \lambda \in (D^{1\times r'}\,(T \quad 0) + D^{1\times q}\,(R'' \quad -I_q)).$$

The converse inclusion trivially holds proving 2.

Now, applying $(L \quad L')$ to the left of the system

$$R'\, \eta = \tau, \quad R\, \eta = 0, \tag{26}$$

we obtain $L\, \tau = 0$. Applying the matrix $\begin{pmatrix} T & 0 \\ R'' & -I_q \end{pmatrix}$ to the left of (26), we obtain $T\, \tau = 0$ and $R''\, \tau = 0$. The equivalences follow from the injectivity of $\mathcal{F}$.

Using $I_p = (I_p - S\, R') + S\, R'$, i.e., $I_p = U + S\, R'$, for all $\eta \in \mathrm{Sol}_{\mathcal{F}}(R)$, we finally obtain:

$$\begin{cases} \eta = U\, \eta + S\,(R'\, \eta) = U\, \eta + S\, \tau, \\ U\eta \in \mathrm{Sol}_{\mathcal{F}}(R'),\ \tau = (\tau_1 \ldots \tau_{q'})^T = R'\, \eta, \\ \tau_i \in \hom_{\mathcal{F}}(t(M), \mathcal{F}), \quad 1 \leq i \leq q'. \end{cases}$$

Theorem 6. Let $\mathcal{F}$ be an injective left D-module, $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ and $R' \in D^{q'\times p}$ a matrix satisfying $M/t(M) = D^{1\times p}/(D^{1\times q'}\, R')$. If there exist $S \in D^{p\times q'}$ and $V \in D^{q'\times q}$ satisfying (15), then every element $\eta \in \mathrm{Sol}_{\mathcal{F}}(R)$ has the form

$$\eta = S\, \overline{\tau} + Q\, \xi, \quad \forall\, \xi \in \mathcal{F}^m,$$

where $Q \in D^{p\times m}$ is a matrix as in Theorem 5 and $\overline{\tau}$ is a fundamental solution of (25) in $\mathcal{F}^{q'}$.

This result holds for the $D = \mathbb{R}[\partial_1, \ldots, \partial_n]$-modules $\mathcal{F} = C^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ and an open convex subset Ω of $\mathbb{R}^n$.

REFERENCES

Chyzak, F., A. Quadrat and D. Robertz (2003). `http://wwwb.math.rwth-aachen.de/OreModules`.

Chyzak, F., A. Quadrat and D. Robertz (2005). Effective algorithms for parametrizing linear control systems over Ore algebras. To appear in *Appl. Algebra Engrg. Comm. Comput.*

Malgrange, B. (1966). *Ideals of Differential Functions.* Oxford University Press.

Polderman, J.W. and J.C. Willems (1998). *Introduction to Mathematical Systems Theory. A Behavioral Approach, TAM 26.* Springer.

Pommaret, J.-F. and A. Quadrat (1998). Generalized Bezout identities. *Appl. Algebra Engrg. Comm. Comput.* **9**, 91–116.

Pommaret, J.-F. and A. Quadrat (2003). A functorial approach to the behaviour of multidimensional control systems. *Int. J. Appl. Math. Comput. Sci.* **13**, 7–13.

Rotman, J. J. (1979). *An Introduction to Homological Algebra.* Academic Press.

Zerz, E. and V. Lomadze (2001). A constructive solution to interconnection and decomposition problems with multidimensional behaviors. *SIAM J. Control Optim.* **40**, 1072–1086.