Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 1** (15.10.2025)

(1.1) Exercise: Noetherian rings and modules.
Let R be a (commutative, associative, unital) ring, and let M be a finitely generated R-module. Recall that the set of R-submodules of M is partially ordered by set-theoretic inclusion.

a) Let $M \neq \{0\}$. Show that M has a maximal (proper) R-submodule.

b) Let $\mathcal{S} \subseteq M$ be a (possibly infinite) generating set of M as an R-module. Show that there is a finite subset $\mathcal{S}' \subseteq \mathcal{S}$ generating M.

c) Show that the following properties are equivalent:
i) M is **Noetherian**, that is any R-submodule of M is finitely generated.
ii) Any strictly ascending chain of R-submodule of M terminates.
iii) Any non-empty set of R-submodules of M contains a maximal element.

d) Let $N \leq M$ be an R-submodule. Show that M is Noetherian if and only if both N and M/N are Noetherian.

e) Let R be Noetherian. Show that M is Noetherian as well. Moreover, show that any surjective endomorphism of R is injective.

(1.2) Exercise: Non-Noetherian rings.
Let K be a field, and let $\mathcal{X} := \{X_1, X_2, \ldots\}$ be an infinite set of indeterminates. Show that $K[\mathcal{X}]$ is a non-Noetherian ring.

(1.3) Exercise: Polynomial functions.
Let $\mathcal{X} := \{X_1, \ldots, X_n\}$ be indeterminates, for some $n \in \mathbb{N}_0$, let L be a field, and let $A := L[\mathcal{X}]$. For $v = [x_1, \ldots, x_n] \in L^n$ let $\epsilon_v^*\colon A \to L\colon f \mapsto f(v)$ be the associated **evaluation function**, and for $f \in A$ let $f^\bullet\colon L^n \to L\colon v \mapsto \epsilon_v^*(f) = f(v)$ be the **polynomial function** afforded by f.

a) Show that the set $\mathcal{A}$ of all polynomial functions $L^n \to L$ carries the structure of an L-algebra, with respect to which the map $f \mapsto f^\bullet\colon A \to \mathcal{A}$ is a homomorphism of L-algebras.

b) Show that the map $f \mapsto f^\bullet\colon A \to \mathcal{A}$ is injective if and only if L is infinite.

(1.4) Exercise: Finite fields.
Let $\mathbb{F}$ be a finite field, and let $n \in \mathbb{N}_0$. Show that any function $\mathbb{F}^n \to \mathbb{F}$ is polynomial. Conclude that any subset of $\mathbb{F}^n$ is $\mathbb{F}$-algebraic.

(1.5) Exercise: Algebraically closed fields.
Let $\mathbf{K}$ be an algebraically closed field, let $n \geq 2$, and let $A := \mathbf{K}[X_1, \ldots, X_n]$.
a) Show that $\mathbf{K}$ is infinite.
b) Let $f \in A$ be non-constant. Show that the curve $\mathbf{V}_{\mathbf{K}}(f) \subseteq \mathbf{K}^n$ is infinite.

(1.6) Exercise: Algebraic sets in the affine plane.
Let $\mathbf{K}$ be an algebraically closed field, let $A := \mathbf{K}[X, Y]$, and let $f, g \in A$ be coprime. Show that $\mathbf{V}_{\mathbf{K}}(f, g)$ is finite. Use this, and Exercise (1.5), to give a description of all ($\mathbf{K}$-)algebraic sets in the affine plane $\mathbf{A}^2(\mathbf{K})$.