

Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, Exercise sheet 12 (21.01.2026)

(12.1) Exercise: Limits.

a) Let \mathcal{I} be a partially ordered set, let \mathcal{C} be a category, let $\{A_i\}$ be an inverse system in \mathcal{C} , with respect to $\psi_{ji}: A_j \rightarrow A_i$ for $j \geq i$, and let P together with morphisms $\{\pi_i\}$ be a limit. Show that P is unique up to isomorphism, and that $\{\pi_i\}$ induces an equivalence of contravariant functors $\text{Hom}_{\mathcal{C}}(?, P) \cong \text{Hom}_{\mathcal{C}}(?, \{A_i\})$.

b) Now let \mathcal{C} be the category of sets, and let $\mathcal{P} := \prod_{i \in \mathcal{I}} A_i$ be the Cartesian product, with projection maps π_i . Show that $P := \{x \in \mathcal{P}; \pi_i(x) = \psi_{ji}(\pi_j(x)) \text{ for } j \geq i\}$ is a limit in \mathcal{C} . What happens if \mathcal{I} carries the trivial partial order, or if $\mathcal{I} = \emptyset$?

c) Let additionally $\mathcal{I} := \{0, 1, 2\}$ with non-trivial relations $1 \geq 0$ and $2 \geq 0$. Show that the pullback $A_1 \times_{A_0} A_2 := \{[x_1, x_2] \in A_1 \times A_2; \psi_1(x_1) = \psi_2(x_2)\}$ is a limit, with respect to the maps $\psi_i: A_i \rightarrow A_0$. Moreover, show that if both ψ_i are surjective, then both $\pi_i: A_1 \times_{A_0} A_2 \rightarrow A_i$ are surjective as well.

d) Let $\mathcal{I} := \mathbb{N}$, partially ordered by divisibility, let $\psi_{mn}: \mathbb{Z}/\langle m \rangle \rightarrow \mathbb{Z}/\langle n \rangle$ be the natural maps, for $n \mid m$, and let $\widehat{\mathbb{Z}} := \varprojlim \{\mathbb{Z}/\langle n \rangle\}$ in the category of sets. Show that the profinite completion $\widehat{\mathbb{Z}}$ is a limit in the category of rings.

Let $p \in \mathbb{Z}$ be a prime, let $\mathcal{I}_p := \{p^n \in \mathbb{N}; n \in \mathbb{N}_0\}$, and let $\widehat{\mathbb{Z}}_p := \varprojlim \{\mathbb{Z}/\langle p^n \rangle\}$ in the category of sets. Show that the p -adic completion $\widehat{\mathbb{Z}}_p$ is a limit in the category of rings. Moreover, show that $\widehat{\mathbb{Z}}_p$ is a domain, having $\langle p \rangle$ as its unique maximal ideal. Finally, show that $\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \widehat{\mathbb{Z}}_p$ as rings.

(12.2) Exercise: Colimits.

a) Let \mathcal{I} be a partially ordered set, let \mathcal{C} be a category, let $\{A_i\}$ be a direct system in \mathcal{C} , with respect to $\psi_{ij}: A_i \rightarrow A_j$ for $i \leq j$, and let I together with morphisms $\{\iota_i\}$ be a direct limit. Show that I is unique up to isomorphism, and that $\{\iota_i\}$ induces an equivalence of (covariant) functors $\text{Hom}_{\mathcal{C}}(I, ?) \cong \text{Hom}_{\mathcal{C}}(\{A_i\}, ?)$.

b) Now let \mathcal{C} be the category of sets, and let $\mathcal{Q} := \coprod_{i \in \mathcal{I}} A_i$ be the disjoint union, with inclusion maps ι_i . Show that there is an equivalence relation \sim on \mathcal{Q} , such that $I := \mathcal{Q}/\sim$, together with the induced maps $\{\iota_i\}$, is a colimit in \mathcal{C} . What happens if \mathcal{I} carries the trivial partial order, or if $\mathcal{I} = \emptyset$?

c) Let additionally $\mathcal{I} := \{0, 1, 2\}$ with non-trivial relations $0 \leq 1$ and $0 \leq 2$. Show that the pushout $A_1 \sqcup_{A_0} A_2 := (A_1 \sqcup A_2)/\sim$, where \sim is the relation generated by reflexive closure from $\iota_1(\psi_1(x)) \sim \iota_2(\psi_2(x))$, for $x \in A_0$, is a colimit, with respect to the maps $\psi_i: A_0 \rightarrow A_i$. Moreover, show that if both ψ_i are injective, then both $\pi_i: A_1 \sqcup_{A_0} A_2 \rightarrow A_i$ are injective as well.

d) We consider the category of K -vector spaces, where K is a field, let V_i be K -vector spaces, for $i \in \{0, 1, 2\}$, and let $\psi_i: V_0 \rightarrow V_i$ be K -linear, for $i \in \{1, 2\}$. If $V_0 = \{0\}$, show that the direct sum $V_1 \oplus V_2$ of K -vector spaces is a fibre sum. Show that a fibre sum $V_1 \oplus_{V_0} V_2$ exists in general, and give an explicit description.

We consider the category of K -algebras, let A_i be K -algebras, and let $\psi_i: K \rightarrow A_i$ be the structure homomorphisms, for $i \in \{1, 2\}$. Show that the tensor product $A_1 \otimes_K A_2$ of K -algebra is a fibre sum. What is its structure homomorphism?