Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 13** (04.02.2026)

(13.1) Exercise: Morphisms of projective varieties.
Let L be an algebraically closed field such that $\mathrm{char}(L) \neq 2$, and let $\mathbf{P}$ be the projective plane having homogeneous coordinate algebra $L[X, Y, Z]$. We consider the curve $\mathbf{V} := \mathbf{V}(f) \subseteq \mathbf{P}$ of degree 3 given by $f := ZY^2 - X(X^2 - Z^2)$.

a) Let $V_Z := \mathbf{V} \cap D_Z$, $V_Y := \mathbf{V} \cap D_Y$, $V_X := \mathbf{V} \cap D_X$, and let $p_0 := [0\colon 0\colon 1]$, $p_\infty := [0\colon 1\colon 0]$, $p_1 := [1\colon 0\colon 1]$, and $p_{-1} := [(-1)\colon 0\colon 1]$. Show that $V_Z = \mathbf{V} \setminus \{p_\infty\}$, $V_Y = \mathbf{V} \setminus \{p_0, p_1, p_{-1}\}$, and $V_X = \mathbf{V} \setminus \{p_0, p_\infty\}$ are affine open.

b) Given any point $p_0 \neq p := [x\colon y\colon 1] \in V_Z$, show that the line in D_Z through p and p_0 intersects V_Z in $p' := [\frac{-1}{x}\colon \frac{-y}{x^2}\colon 1] = [x\colon y\colon (-x^2)]$.

We aim at showing that the map φ given by $p \mapsto p'$, and interchanging p_0 and p_∞, defines an involutory automorphism of $\mathbf{V}$. Convince yourself that to do so it suffices to show that φ is a morphism. To this end, let $V' := \mathbf{V} \setminus \{p_\infty, p_1, p_{-1}\}$.

c) Show that $\mathbf{V} = V' \cup V_X \cup V_Y = V_Z \cup V_Y$, and that $\varphi(V_X \cup V_Y) \subseteq V_Z$ and $\varphi(V') \subseteq V_Y$. Moreover, show that $\varphi|_{V_X}\colon V_X \to V_Z$, $\varphi|_{V_Y}\colon V_Y \to V_Z$, and $\varphi|_{V'}\colon V' \to V_Y$ are morphisms. Conclude that φ is a morphism.

Hint for c). Show that $\frac{1}{X^2-1}$ is regular on V'.

(13.2) Exercise: Hausdorff spaces.
Let V be a topological space. Show that the following are equivalent:
i) V is a Hausdorff space.
ii) The set $\{[v, v] \in V \times V; v \in V\} \subseteq V \times V$ is closed in the product topology.
iii) For any topological space U and any continuous map $\varphi\colon U \to V$, the graph $\{[u, \varphi(u)] \in U \times V; u \in U\} \subseteq U \times V$ is closed in the product topology.
iv) For any topological space U and continuous maps $\varphi, \psi\colon U \to V$, the difference kernel $\{u \in U; \varphi(u) = \psi(u)\} \subseteq U$ is closed.

(13.3) Exercise: Graphs of morphisms.
Let $K \subseteq L$ be a field extension, where L is algebraically closed, let U and V be prevarieties, and let $W \subseteq U \times V$ be a subprevariety. Show that W is the graph of a morphism $\varphi\colon U \to V$, if and only if the restriction $\pi_U|_W\colon W \to U$ is an isomorphism. In this case, show that $\varphi = (p_U|_W)^{-1} \cdot \pi_V$.

(13.4) Exercise: Affine open coverings.
Let $K \subseteq L$ be a field extension, where L is algebraically closed, and let V be a prevariety. Assume that V has an affine open covering $\{V_i; i \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set, such that $V_i \cap V_j$ is affine open having coordinate algebra $K[V_i \cap V_j] = K[V_i] \cdot K[V_j]$, for all $i, j \in \mathcal{I}$. Show that V is a variety.