Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 2** (22.10.2025)

(2.1) Exercise: Complements of algebraic sets.
Let $K \subseteq L$ be a field extension such that L is infinite, let $n \geq 1$, and let $\mathbf{V} \subset L^n$ be a proper K-algebraic subset. Show that $L^n \setminus \mathbf{V}$ is infinite.

(2.2) Exercise: Hilbert's Nullstellensatz.
Let K be a field, let $A := K[\mathcal{X}]$ be a finitely generated polynomial K-algebra, and let $P \lhd A$ be a maximal ideal.
a) Show that $K \subseteq (A/P)$ is a finite field extension.
b) Let $K \subseteq L$ be a field extension. Show that $\mathbf{V}_L(P)$ is finite.

(2.3) Exercise: Hypersurfaces.
Let $K \subseteq L$ be a field extension such that L is algebraically closed, let A be the polynomial K-algebra in $n \in \mathbb{N}$ indeterminates, and let $f \in A$ be non-constant. Writing $f = \prod_{i=1}^r f_i^{a_i} \in A$, where $r \in \mathbb{N}$ and $a_i \in \mathbb{N}$, and the $f_i \in A$ are pairwise non-associates and irreducible, show that $\mathbf{I}_K(\mathbf{V}_L(f)) = \prod_{i=1}^r \langle f_i \rangle \lhd A$.

(2.4) Exercise: Cartesian products.
Let $K \subseteq L$ be a field extension, let $A := K[X_1, \ldots, X_n]$ and $B := K[Y_1, \ldots, Y_m]$ be polynomial K-algebras, for $n, m \in \mathbb{N}_0$, let $C := K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$, and let $\mathbf{V} \subseteq L^n$ and $\mathbf{W} \subseteq L^m$ be K-algebraic. Show that $\mathbf{V} \times \mathbf{W} \subseteq L^{n+m}$ is K-algebraic again, having vanishing ideal

$$\mathbf{I}_K(\mathbf{V} \times \mathbf{W}) = \sqrt{\mathbf{I}_K(\mathbf{V}) \cdot C + \mathbf{I}_K(\mathbf{W}) \cdot C} \trianglelefteq C.$$

(2.5) Exercise: Scalar extensions.
Let $K \subseteq L$ be a field extension, let $A_K := K[\mathcal{X}]$ be the polynomial K-algebra in $n \in \mathbb{N}_0$ indeterminates, and let $A_L := L[\mathcal{X}]$ be its **scalar extension** to L.

a) Let $V \subseteq L^n$ be L-algebraic. Show that $U := V \cap K^n$ is K-algebraic, such that $\mathbf{I}_K(U) = \mathbf{I}_L(V) \cap A_K \trianglelefteq A_K$.

b) Now let $U \subseteq K^n$ be K-algebraic, and let $V \subseteq L^n$ be the smallest L-algebraic subset containing U. Show that $V \cap K^n = U$, and that $\mathbf{I}_L(V) = \mathbf{I}_K(U) \cdot A_L \trianglelefteq A_L$.

c) Let $K \subseteq M \subseteq L$ be an intermediate field, and let $W \subseteq L^n$ be K-algebraic. Show that W is M-algebraic, such that $\mathbf{I}_M(W) = \sqrt{\mathbf{I}_K(W) \cdot A_M} \trianglelefteq A_M$.