Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 3** (29.10.2025)

(3.1) Exercise: Topological spaces.
a) Recall that a maximal (closed) irreducible subset of a topological space $\mathcal{V}$ is called an irreducible component. Show that any irreducible subset of $\mathcal{V}$ is contained in an irreducible component, and ideduce that $\mathcal{V}$ is the irredundant union of its irreducible components.
b) A topological space is called **quasi-compact**, or has the **Heine–Borel property**, if any open cover has a finite subcover. Show that any Noetherian topological space is quasi-compact.
c) A topological space $\mathcal{V}$ is called **Hausdorff**, if for any $x \neq y \in \mathcal{V}$ there are open neighbourhoods $\mathcal{U}_x \subseteq \mathcal{V}$ and $\mathcal{U}_y \subseteq \mathcal{V}$ of x and y, respectively, such that $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$. Show that $\mathcal{V}$ is Hausdorff Noetherian if and only if it is finite and **discrete**, that is all subsets of $\mathcal{V}$ are open.

(3.2) Exercise: A generalised Nullstellensatz.
Let K be a field which is *not* algebraically closed, let A be a finitely generated polynomial K-algebra, and let $I \trianglelefteq A$.
a) Show that there is $g \in A$ such that $\mathbf{V}_K(I) = \mathbf{V}_K(g)$.
b) Show that $\mathbf{V}_K(I) \neq \emptyset$ if and only if $\mathbf{V}_K(f) \neq \emptyset$ for all $f \in I$. (This holds for algebraically closed fields as well, by Hilbert's Nullstellensatz.)

Hint. Show first that there is $h \in A$ such that $\mathbf{V}_K(h) = \{0\}$.

(3.3) Exercise: Radical membership test.
Let T be an indeterminate, let R be a ring, let $I \trianglelefteq R$, and let $f \in R$. Show that $f \in \sqrt{I}$ if and only if $\langle I, fT - 1\rangle = R[T]$.

(3.4) Exercise: Hypersurfaces.
Let $K \subseteq L$ be a field extension such that L is algebraically closed, let A be a finitely generated polynomial K-algebra, and let $f = \prod_{i=1}^{r} f_i^{a_i} \in A$, where $r \in \mathbb{N}$ and $a_i \in \mathbb{N}$, and the $f_i \in A$ are pairwise non-associated and irreducible. Determine the irreducible components of the hypersurface $\mathbf{V}_L(f)$.

(3.5) Exercise: Linear subspaces.
Let $K = L$ be an infinite field, and let $A := K[X_1, \ldots, X_n]$ for some $n \in \mathbb{N}_0$.
a) Let $V \leq K^n$ be a K-subspace. Show that $V = \mathbf{V}_K(f_1, \ldots, f_m)$ for some $m \leq n$, where $f_j = \sum_{i=1}^{n} a_{ji} X_i$ for some $a_{ji} \in K$. How is m related to $\dim_K(V)$?
b) Let m be chosen minimal. Show that $\mathbf{I}_K(V) = \langle f_1, \ldots, f_m\rangle \lhd A$, that V is irreducible, and that $K[V]$ is a polynomial algebra in $n - m$ indeterminates.

(3.6) Exercise: Irreducible components.
Let K be an algebraically closed field, let $A := K[X, Y, Z]$, and let $\mathbf{V} := \mathbf{V}_K(X^2 - YZ, XZ - X)$. Determine the irreducible components of $\mathbf{V}$. Moreover, compute the coordinate algebra of $\mathbf{V}$ and of its irreducible components.