Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 4** (05.11.2025)

(4.1) Exercise: Maximal ideals of coordinate algebras.

a) Let $K \subseteq L$ be a field extension, let $\mathbf{V} \subseteq L^n$ be closed, let $\mathrm{Hom}(K[\mathbf{V}], L)$ be the set of all K-algebra homomorphisms from $K[\mathbf{V}]$ to L, and for $v \in \mathbf{V}$ let $\epsilon_v \colon K[\mathbf{V}] \to L \colon f \mapsto f(v)$ be the associated **evaluation map**. Show that the map $\mathbf{V} \to \mathrm{Hom}(K[\mathbf{V}], L) \colon v \mapsto \epsilon_v$ is a bijection.

b) Now let $K = L$ be algebraically closed. Show that for $v \in \mathbf{V}$ the inclusion map $\iota_v \colon \{v\} \to \mathbf{V}$ is an embedding such that $(\iota_v)^* = \epsilon_v$. Conclude that $\mathrm{Hom}(K[\mathbf{V}], K)$ is in bijective correspondence with the maximal ideals of $K[\mathbf{V}]$.

(4.2) Exercise: Coordinate algebras.

Let K be an algebraically closed field, let $A := K[X, Y]$, and let $\mathbf{V} := \mathbf{V}(Y - X^2)$ and $\mathbf{W} := \mathbf{V}(XY - 1)$. Which of the coordinate algebras $K[\mathbf{V}]$ and $K[\mathbf{W}]$ are isomorphic to a polynomial algebra?

(4.3) Exercise: Parametrisation of curves.

a) Let $K \subseteq \mathbb{C}$, and let $\mathbf{V} := \{[t^2 - 1, t(t^2 - 1)] \in K^2; t \in K\}$. Show that $\mathbf{V} \subseteq K^2$ is closed, irreducible and defined over $\mathbb{Q}$, and determine $\mathbf{I}(\mathbf{V})$. Is the given parametrisation an isomorphism? Try to depict $\mathbf{V}$ for $K = \mathbb{R}^2$.

b) Let $\mathbf{W} := \{[t^3, t^4, t^5] \in K^3; t \in K\}$. Show that $\mathbf{W} \subseteq K^3$ is closed, irreducible and defined over $\mathbb{Q}$, and determine $\mathbf{I}(\mathbf{W})$. Is the given parametrisation an isomorphism? Can you show that $\mathbf{I}(\mathbf{W})$ cannot be generated by two elements?

(4.4) Exercise: Parametrisation of surfaces.

a) Let $K \subseteq \mathbb{C}$, let $W := \{[uv, v, u^2] \in K^3; u, v \in K\}$, and let $\mathbf{W} := \overline{W} \subseteq K^3$ be the **Whitney umbrella** surface over K. Show that $\mathbf{W}$ is irreducible and defined over $\mathbb{Q}$, and determine $\mathbf{I}(\mathbf{W})$. For $K = \mathbb{C}$ show that the given parametrisation is bijective. Is it an isomorphism? For $K = \mathbb{R}$ determine $\mathbf{W} \setminus W$.

b) Let $Z := \{[uv, uv^2, u^2] \in K^3; u, v \in K\}$, and let $\mathbf{Z} := \overline{Z} \subseteq K^3$. Show that $\mathbf{Z}$ is irreducible and defined over $\mathbb{Q}$, and determine $\mathbf{I}(\mathbf{W})$. Show that the given parametrisation is injective. For $K \in \{\mathbb{R}, \mathbb{C}\}$ determine $\mathbf{W} \setminus W$.

(4.5) Exercise: Computing the image of a morphism.

Let $K \subseteq L$ be a field extension, let $A := K[X_1, \ldots, X_n]$ and $B := K[Y_1, \ldots, Y_m]$, for $n, m \in \mathbb{N}_0$, and $C := A \otimes_K B \cong K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$.

a) Show that $\pi \colon L^{n+m} \to L^m \colon [x_1, \ldots, x_n, y_1, \ldots, y_m] \mapsto [y_1, \ldots, y_m]$ is a morphism. Determine the associated comorphism π^*.

b) Now let $\varphi \colon L^n \to L^m \colon v \mapsto [f_1(v), \ldots, f_m(v)]$ be regular, where $f_j \in A$, and let $J := \langle Y_1 - f_1, \ldots, Y_m - f_m \rangle \trianglelefteq C$. Show that we have $\pi(\mathbf{V}_L(J)) = \varphi(L^n)$ and $\overline{\varphi(L^n)} = \mathbf{V}_L(J \cap B)$.

c) Letting $K = \mathbb{Q}$ and $L = \mathbb{C}$, apply this to the twisted cubic $\mathbf{C} := \{[t, t^2, t^3] \in \mathbb{C}^3; t \in \mathbb{C}\}$, and compute $\mathbf{I}_{\mathbb{Q}}(\mathbf{C})$ from the given parametrisation.