Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 5** (12.11.2025)

(5.1) Example: Automorphisms.
Let K be an algebraically closed field. An isomorphism of an affine K-variety $\mathbf{V} \subseteq K^n$ onto itself, for some $n \in \mathbb{N}_0$, is called an **automorphism** of $\mathbf{V}$. Let $\mathrm{Aut}(\mathbf{V})$ be the group of automorphisms of $\mathbf{V}$.

a) Determine $\mathrm{Aut}(K)$.

b) For $\varphi\colon K^n \to K^n\colon v \mapsto [f_1(v), \ldots, f_n(v)]$ regular, where $f_i \in K[\mathcal{X}]$, let

$$J(\varphi) := \det\big(\big[\frac{\partial f_i}{\partial X_j}\big]_{ij}\big) \in K[X_1, \ldots, X_n]$$

be the **Jacobian** of φ, where $\frac{\partial}{\partial X_j}$ denotes the partial derivative with respect to X_j. Show that $J\colon \mathrm{Aut}(K^n) \to K^*\colon \varphi \mapsto J(\varphi)$ is a group homomorphism.

(5.2) Example: Frobenius morphisms.
Let q be a prime power, let $\mathbf{F}$ be an algebraic closure of $\mathbb{F}_q$, let $\mathbf{V} \subseteq \mathbf{F}^n$ be $\mathbf{F}$-closed, for some $n \in \mathbb{N}_0$, and let $\Phi = \Phi_q\colon \mathbf{F}^n \to \mathbf{F}^n\colon [x_1, \ldots, x_n] \mapsto [x_1^q, \ldots, x_n^q]$ be the associated **(geometric) Frobenius morphism**.

a) Show that Φ is regular and bijective, but is not an isomorphism.

b) Show the equivalence of the following assertions:
i) We have $\Phi(\mathbf{V}) = \mathbf{V}$. **ii)** The set $\mathbf{V}$ is $\mathbb{F}_q$-closed.
iii) There are $f_1, \ldots, f_s \in \mathbb{F}_q[X_1, \ldots, X_n]$ such that $\mathbf{V} = \mathbf{V}_{\mathbf{F}}(f_1, \ldots, f_s) \subseteq \mathbf{F}^n$.
iv) There are $f_1, \ldots, f_r \in \mathbb{F}_q[X_1, \ldots, X_n]$ such that $\mathbf{I}_{\mathbf{F}}(\mathbf{V}) = \langle f_1, \ldots, f_r\rangle \trianglelefteq \mathbf{F}[\mathbf{V}]$.
v) There is a finitely generated $\mathbb{F}_q$-algebra A such that $\mathbf{F}[\mathbf{V}] \cong A \otimes_{\mathbb{F}_q} \mathbf{F}$.

c) Let $\mathbf{V}$ be $\mathbb{F}_q$-closed. Determine the $\mathbb{F}_q$-closed points of $\mathbf{V}$.

(5.3) Exercise: Graded rings.
Let R be a **(non-negatively) graded ring**, that is $R = \bigoplus_{d \geq 0} R_d$ as $\mathbb{Z}$-modules, such that $R_i R_j \subseteq R_{i,j}$ for $i, j \geq 0$. Then the following are equivalent:
i) R is Noetherian.
ii) R_0 is Noetherian and R is a finitely generated R_0-algebra.
iii) R_0 is Noetherian and $R_+ := \bigoplus_{d \geq 1} R_d \trianglelefteq R$ is a finitely generated ideal.
iv) The subring $R^{(n)} := \bigoplus_{n \mid d} R_d \subseteq R$ is Noetherian for all $n \in \mathbb{N}$.

(5.4) Exercise: Affine cones.
Let $K \subseteq L$ be a field extension, let $\mathbf{V} \subseteq L^n$ be an irreducible variety, for some $n \in \mathbb{N}_0$, and let $\widetilde{\mathbf{V}} := \bigcup_{v \in \mathbf{V}} \langle v\rangle_L$. Show that both $\widetilde{\mathbf{V}} \subseteq L^n$ and its closure $\overline{\widetilde{\mathbf{V}}} \subseteq L^n$ are irreducible cones, and determine the vanishing ideal $\mathbf{I}_K(\widetilde{\mathbf{V}}) = \mathbf{I}_K(\overline{\widetilde{\mathbf{V}}})$.

Hint. If $P \lhd A = \bigoplus_{d \geq 0} A_d$ is prime, then so is $\widetilde{P} := \bigoplus_{d \geq 0}(P \cap A_d) \subseteq P \lhd A$.