Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 6** (19.11.2025)

(6.1) Exercise: Homogenisation.
Let K be a field, let $\mathcal{X} := \{X_1, \dots, X_n\}$ and $\mathcal{X}^\sharp := \{X_0\} \dot\cup \mathcal{X}$ be indeterminates, for some $n \in \mathbb{N}_0$, let $A := K[\mathcal{X}]$ and $A^\sharp := K[\mathcal{X}^\sharp]$.
a) Show that an ideal $I \trianglelefteq A$ is prime if and only if $I^\sharp \trianglelefteq A^\sharp$ is prime.
b) Let $I \trianglelefteq A^\sharp$ be homogeneous and prime. Show that $I' = A$ or $I' \trianglelefteq A$ is prime.
c) Now assume that $I = \langle f_1, \dots, f_r \rangle \trianglelefteq A^\sharp$, for some $r \in \mathbb{N}_0$, where $((f_i)')^\sharp = f_i$ for all i. Show that if $I' = A$ or $I' \trianglelefteq A$ is prime, then $I = A$ or $I \trianglelefteq A$ is prime.

(6.2) Exercise: Euler Identity.
Let K be a field, let $\mathcal{X} := \{X_1, \dots, X_n\}$, for some $n \in \mathbb{N}_0$, and let $A := K[\mathcal{X}]$.
a) Let $0 \neq f \in A$ be homogeneous. Show that f fulfills the **Euler Identity**

$$\sum_{i=1}^{n} X_i \cdot \frac{\partial f}{\partial X_i} = \deg(f) \cdot f.$$

b) Assume that $\mathrm{char}(K) = 0$, and let conversely $0 \neq f \in A$ fulfill the Euler Identity. Show that f is homogeneous.

(6.3) Exercise: Projective closure.
Let $L \subseteq \mathbb{C}$, let $\mathcal{Q} := \mathbf{V}_L(Y - X^2) \subseteq L^2$ be a **parabola**, and let $\mathbf{Q} := \overline{\mathcal{Q}} \subseteq \mathbf{P}^2(L)$ be its projective closure.

a) Show that $\mathbf{Q}$ is irreducible, and determine $\mathbf{I}^\sharp(\mathbf{Q})$ and the points at infinity of $\mathcal{Q}$. Determine the affine closed subsets $\mathbf{Q} \cap U_X$ and $\mathbf{Q} \cap U_Y$ in the '(y,z)-' and the '(x,z)-plane', respectively, where Z denotes the homogenizing coordinate.

b) Now let $L = \mathbb{R}$. Depict $\mathcal{Q}$, and the affine sets $\mathbf{Q} \cap U_X$ and $\mathbf{Q} \cap U_Y$. Show that $\mathbf{Q}$ is 'tangent' to the line at infinity $\mathbf{H}_Z = \mathbf{V}^\sharp(Z) \subseteq \mathbf{P}^2(\mathbb{R})$.

(6.4) Exercise: Rational parametrisations.
Let $L \subseteq \mathbb{C}$, let $\mathcal{H} := \mathbf{V}_L(X^2 - Y^2 - 1) \subseteq L^2$ be a **hyperbola**, and let $\mathbf{H} := \overline{\mathcal{H}} \subseteq \mathbf{P}^2(L)$ be its projective closure.

a) Show that $\mathbf{H}$ is irreducible, and determine $\mathbf{I}^\sharp(\mathbf{H})$ and the points at infinity of $\mathcal{H}$. Determine the affine closed subsets $\mathbf{H} \cap U_X$ and $\mathbf{H} \cap U_Y$ in the '(y,z)-' and the '(x,z)-plane', respectively, where Z denotes the homogenizing coordinate.

b) Show that $\tau \colon L \setminus \{\pm 1\} \to \mathcal{H} \setminus \{[-1,0]\} \colon t \mapsto \frac{1}{1-t^2} \cdot [1+t^2, 2t]$ is well-defined and bijective, and determine its inverse. Moreover, show that τ can be extended to a bijective ('polynomial') map $\tau^\sharp \colon \mathbf{P}^1(L) \to \mathbf{H}$, and determine its inverse.

c) Use this to describe the behaviour of τ around $t = \pm 1$. Similarly, using the involution on $\mathbf{P}^1(L)$ given by $[s \colon t] \mapsto [t \colon s]$, describe behaviour of $\tau^\sharp$ around $t = \infty$; recall that we write $\mathbf{P}^1(L) = L \dot\cup \{\infty\}$.

d) Now let $L = \mathbb{R}$. Depict $\mathcal{H}$, and the affine sets $\mathbf{H} \cap U_X$ and $\mathbf{H} \cap U_Y$. How can τ^{-1} be understood geometrically? How does $\tau(t)$ 'travel' along $\mathcal{H}$ when t varies? In particular, what happens for $t \to (\pm 1)^\pm$, $t \to 0^\pm$, and $t \to \pm\infty$ (in the metric topology)?