Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 7** (26.11.2025)

(7.1) Exercise: Veronese embeddings.
Let $K \subseteq L$ be a field extension, where L is infinite, and let $\mathbf{P}^n$ be the projective space over L, for some $n \in \mathbb{N}_0$. Let $\{M_0, \ldots, M_N\}$ be the set of monomials of degree $d \in \mathbb{N}_0$ in the indeterminates $\mathcal{X} := \{X_0, \ldots, X_n\}$. Letting $N := \binom{n+d}{n} - 1$, we consider the d-th **Veronese embedding**

$$\varphi\colon \mathbf{P}^n \to \mathbf{P}^N\colon [x_0\colon \ldots\colon x_n] \mapsto [\ldots\colon M_i(x_0, \ldots, x_n)\colon \ldots].$$

a) Show that φ is well-defined and injective.
b) Show that the image $\varphi(\mathbf{P}^n) \subseteq \mathbf{P}^N$ is closed and irreducible.
c) Show that $\varphi(\mathbf{P}^n)$ is homeomorphic via φ to $\mathbf{P}^n$.
d) Show that the twisted cubic in $\mathbf{P}^3$ coincides with the image of the 3-uple embedding $\mathbf{P}^1 \to \mathbf{P}^3$, for a suitable choice of coordinates.

Hint. Consider the kernel of the homomorphism of K-algebras

$$K[Y_0, \ldots, Y_N] \to K[X_0, \ldots, X_n]\colon Y_i \mapsto M_i.$$

(7.2) Exercise: Segre embedding.
Let $K \subseteq L$ be a field extension, where L is infinite, and let $\mathbf{P}^n$ and $\mathbf{P}^m$ be projective spaces over L, for some $n, m \in \mathbb{N}_0$. Letting $N := (n+1)(m+1) - 1$, we consider the map

$$\varphi\colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^N\colon [[x_0\colon \ldots\colon x_n], [y_0\colon \ldots\colon y_m]] \mapsto [\ldots\colon x_i y_j\colon \ldots].$$

a) Show that φ is well-defined and injective; it is called the **Segre embedding**.
b) Show that the image $\varphi(\mathbf{P}^n \times \mathbf{P}^m) \subseteq \mathbf{P}^N$ is closed and irreducible.

Hint. Consider the kernel of the homomorphism of K-algebras

$$K[Z_{ij}; i \in \{0, \ldots, n\}, j \in \{0, \ldots, m\}] \to K[X_0, \ldots, X_n, Y_0, \ldots, Y_m]\colon Z_{ij} \mapsto X_i Y_j.$$

(7.3) Exercise: The quadric surface.
Let $K \subseteq L$ be a field extension, where L is infinite, and let $A^\sharp = K[W, X, Y, Z]$. We consider the **quadric surface** $\mathbf{V} := \mathbf{V}_L^\sharp(XY - ZW) \subseteq \mathbf{P}^3$.

a) Determine $\mathbf{I}_K^\sharp(\mathbf{V}) \trianglelefteq A^\sharp$, and show that $\mathbf{V}$ coincides with the image of the Segre embedding $\varphi\colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$, for a suitable choice of coordinates.

b) Show that $\mathbf{V}$ conains two sets of lines $\{\mathbf{L}_i; i \in \mathcal{I}\}$ and $\{\mathbf{M}_j; j \in \mathcal{J}\}$, each parametrized by $\mathbf{P}^1$, where $\mathcal{I}$ and $\mathcal{J}$ are index sets, such that $\mathbf{L}_i \cap \mathbf{L}_k = \emptyset$ whenever $i \neq k$, and $\mathbf{M}_j \cap \mathbf{L}_l = \emptyset$ whenever $j \neq l$, while $\mathbf{L}_i \cap \mathbf{M}_j$ is a singleton set, for all i and j. Here, a **line** in $\mathbf{P}^3$ is an intersection of two distinct hyperplanes.

c) Show that $\mathbf{V}$ contains further space curves, distinct from the above lines. Deduce that the Zariski topology on $\mathbf{V}$ is not homeomorphic via φ to the product topology of the Zariski topologies on $\mathbf{P}^1 \times \mathbf{P}^1$.