Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 8** (03.12.2025)

(8.1) Exercise: Localisation.
Let R be a ring, let $\mathcal{S} \subseteq R$ be multiplicatively closed, and let M be an R-module.

a) Show that a localisation $M_\mathcal{S}$ of M at $\mathcal{S}$ is unique up to isomorphism.

b) Let $\sim$ be the relation on $M \times \mathcal{S}$ given by letting $[m, f] \sim [m', f']$ if there is $g \in \mathcal{S}$ such that $(mf' - m'f)g = 0$. Show that $\sim$ is an equivalence relation.

c) Show that the set of equivalence classes $M/\mathcal{S} := (M \times \mathcal{S})/\sim$ becomes an R-module, such that there is a natural R-module homomorphism $\sigma\colon M \to M/\mathcal{S}$. Moreover, show that $M/\mathcal{S}$, together with σ, is a localisation of M at $\mathcal{S}$ indeed.

d) Show that the localisation $R_\mathcal{S}$ of R at $\mathcal{S}$ becomes a ring again, and derive the universal property of $R_\mathcal{S}$ in the category of rings from its universal property in the category of R-modules.

e) Show that $\frac{f}{1} \in R_\mathcal{S}$, where $f \in \mathcal{S}$, acts bijectively on any $R_\mathcal{S}$-module. Conversely, if any $f \in \mathcal{S}$ acts bijectively on M, show that M becomes an $R_\mathcal{S}$-module.

(8.2) Exercise: Localisation functors.
Let R be a ring, and let $\mathcal{S} \subseteq R$ be multiplicatively closed.

a) Let $\alpha\colon M \to N$ be a homomorphism of R-modules. Show that there is a unique homomorphism $\alpha_\mathcal{S}\colon M_\mathcal{S} \to N_\mathcal{S}$ of $R_\mathcal{S}$-modules, called the **localisation** of α at $\mathcal{S}$, such that $\alpha \cdot \sigma_N = \sigma_M \cdot \alpha_\mathcal{S}$, where $\sigma_?$ denotes the natural map.

b) Show that localisation at $\mathcal{S}$ induces covariant functors $?_\mathcal{S}\colon \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}R_\mathcal{S}$ and $?_\mathcal{S}\colon \mathsf{mod}\text{-}R \to \mathsf{mod}\text{-}R_\mathcal{S}$, where $\mathsf{Mod}\text{-}?$ and $\mathsf{mod}\text{-}?$ denotes the category of all modules and of finitely generated modules, respectively. Is the map $?_\mathcal{S}\colon \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{R_\mathcal{S}}(M_{R_\mathcal{S}}, N_{R_\mathcal{S}})$ injective? Is it surjective?

c) Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} P$ be an **exact sequence** of R-modules, that is we have $\mathrm{im}(\alpha) = \ker(\beta)$. Show that $M_\mathcal{S} \xrightarrow{\alpha_\mathcal{S}} N_\mathcal{S} \xrightarrow{\beta_\mathcal{S}} P_\mathcal{S}$ is an exact sequence of $R_\mathcal{S}$-modules; in other words, $?_\mathcal{S}$ is an **exact functor**. In particular, conclude that $\alpha_\mathcal{S}$ is injective if α is so, that $\alpha_\mathcal{S}$ is surjective if α is so, and that $M_\mathcal{S}/N_\mathcal{S} \cong (M/N)_\mathcal{S}$ as $R_\mathcal{S}$-modules.

d) If $M' \leq M$ and $M'' \leq M$ are R-submodules, show that $(M' \cap M'')_\mathcal{S} = (M')_\mathcal{S} \cap (M'')_\mathcal{S} \subseteq M_\mathcal{S}$. Similarly, if $\{M_i \leq M; i \in \mathcal{I}\}$ are R-submodules, where $\mathcal{I}$ is an index set, show that $(\sum_{i\in\mathcal{I}} M_i)_\mathcal{S} = \sum_{i\in\mathcal{I}}(M_i)_\mathcal{S} \leq M_\mathcal{S}$. In which sense can the various localised modules be considered as submodules of $M_\mathcal{S}$?

(8.3) Exercise: Varying the denominator set.
Let R be a ring, let $\mathcal{S}, \mathcal{T} \subseteq R$ be multiplicatively closed subsets, let $\mathcal{S}' := \sigma_\mathcal{T}(\mathcal{S}) \subseteq R_\mathcal{T}$ and $\mathcal{T}' := \sigma_\mathcal{S}(\mathcal{T}) \subseteq R_\mathcal{S}$, and let M be an R-module.

a) Show that there are a natural ring homomorphism $\rho\colon R_\mathcal{T} \to (R_\mathcal{S})_{\mathcal{T}'}$, and a natural homomorphism of $R_\mathcal{T}$-modules $\tau\colon M_\mathcal{T} \to (M_\mathcal{S})_{\mathcal{T}'}$. In which sense does $(M_\mathcal{S})_{\mathcal{T}'}$ become an $R_\mathcal{T}$-module?

b) Show that if $\mathcal{S}'$ consists of units in $R_\mathcal{T}$, then ρ and τ are an isomorphism of rings and $R_\mathcal{T}$-modules, respectively. What happens in the case $\mathcal{S} \subseteq \mathcal{T}$? What happens if $\mathcal{T}'$ consists of units in $R_\mathcal{S}$ as well?

c) Let $f, g \in R$, and let R_f denote the localisation of R at $\{f^k \in R; k \in \mathbb{N}_0\}$. Show that there are a natural ring isomorphism $(R_f)_g \cong R_{fg}$, and a natural isomorphism of R_{fg}-modules $(M_f)_g \cong M_{fg}$.