Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 13** (18.11.2025)

(13.1) Homogenisation of ideals. a) We keep the above notation. Given $f \in A^\sharp$, the specialisation epimorphism of K-algebras $A^\sharp \to A$ defined by $X_0 \mapsto 1$ yields the **dehomogenisation** $f' := f(1, X_1, \ldots, X_n) \in A$. Conversely, for $0 \neq f \in A$ we let $f^\sharp := X_0^{\deg(f)} \cdot f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}) \in A^\sharp$ be its **homogenisation** with respect to X_0; for completeness we let $0^\sharp := 0$. Then we have $X_0 \nmid f^\sharp$ for $0 \neq f \in A$, and multiplicativity $(fg)^\sharp = f^\sharp g^\sharp$ holdds for $f, g \in A$.

For $0 \neq f \in A$ we have $(f^\sharp)' = (X_0^{\deg(f)} \cdot f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}))' = f(X_1, \ldots, X_n) = f$. Conversely, for $0 \neq f \in A^\sharp$ homogeneous, letting $\nu(f) = \nu_{X_0}(f) := \deg(f) - \deg(f') \in \mathbb{N}_0$, that is $X_0^{\nu(f)}$ is the highest power of X_0 dividing f, we have $(f')^\sharp = f(1, X_1, \ldots, X_n)^\sharp = X_0^{\deg(f')} \cdot f(1, \frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}) = X_0^{-\nu(f)} \cdot f(X_0, X_1, \ldots, X_n)$.

b) In terms of ideals, for $I \trianglelefteq A^\sharp$ homogeneous let $I' := \{f' \in A; f \in I\} \trianglelefteq A$ be its **dehomogenisation**, and conversely for $I \trianglelefteq A$ let $I^\sharp := \langle f^\sharp \in A^\sharp; f \in I \rangle \trianglelefteq A^\sharp$ be its **homogenisation**.

For $I \trianglelefteq A$ we have $(I^\sharp)' = (\langle f^\sharp \in A^\sharp; f \in I \rangle_{A^\sharp})' = \langle (f^\sharp)' \in A; f \in I \rangle_A = I$. Conversely, for $I \trianglelefteq A^\sharp$ homogeneous we have $(I')^\sharp = \{f' \in A; f \in I\}^\sharp = \langle (f')^\sharp \in A^\sharp; f \in I \rangle_{A^\sharp} = \langle X_0^{-\nu(f)} \cdot f \in A^\sharp; f \in I \text{ homogeneous} \rangle_{A^\sharp}$; in particular we have $I \subseteq (I')^\sharp$, such that for any $f \in (I')^\sharp$ homogeneous we have $X_0^{\nu(f)} \cdot f \in I$.

Proposition. i) An ideal $I \trianglelefteq A$ is radical if and only if $I^\sharp \trianglelefteq A^\sharp$ is so.
ii) A homogeneous ideal $I \trianglelefteq A^\sharp$ is radical if and only if $I' \trianglelefteq A$ is so.

Proof. i) Let $I^\sharp \trianglelefteq A^\sharp$ be radical, and let $f \in \sqrt{I} \trianglelefteq A$. Then we have $f^k \in I$ for some $k \in \mathbb{N}$. Thus we get $(f^\sharp)^k = (f^k)^\sharp \in I^\sharp = \sqrt{I^\sharp}$. This implies $f^\sharp \in I^\sharp$, so that $f = (f^\sharp)' \in (I^\sharp)' = I$.

Conversely, let $I \trianglelefteq A$ be radical, and let $f \in \sqrt{I^\sharp} \trianglelefteq A^\sharp$ homogeneous. Then we have $f^k \in I^\sharp$ for some $k \in \mathbb{N}$. Thus we get $(f')^k = (f^k)' \in (I^\sharp)' = I = \sqrt{I}$. This implies $f' \in I$, so that $(f')^\sharp = X_0^{-\nu(f)} \cdot f \in I^\sharp$, thus $f \in I^\sharp$.

ii) Let $I \trianglelefteq A^\sharp$ be radical, and let $f \in \sqrt{I'} \trianglelefteq A$. Then we have $f^k \in I'$ for some $k \in \mathbb{N}$. Thus we get $(f^\sharp)^k = (f^k)^\sharp \in (I')^\sharp$. Since $\nu(f^\sharp) = 0$ we conclude that actually $(f^\sharp)^k \in I = \sqrt{I}$. This implies $f^\sharp \in I$, so that $f = (f^\sharp)' \in I'$.

Conversely, let $I' \trianglelefteq A$ be radical, and let $f \in \sqrt{I} \trianglelefteq A^\sharp$ homogeneous. Then we have $f^k \in I$ for some $k \in \mathbb{N}$. Thus we get $(f')^k = (f^k)' \in I' = \sqrt{I'}$. This implies $f' \in I'$, so that $(f')^\sharp = X_0^{-\nu(f)} \cdot f \in (I')^\sharp$. Since $\nu(X_0^{-\nu(f)} \cdot f) = 0$ we conclude that actually $X_0^{-\nu(f)} \cdot f \in I$, thus $f \in I$. ♯

(13.2) Projective closure. Keeping the above notation, we consider $U_0 = \mathbf{P} \setminus \mathbf{H}_0$, which is open with respect to the Zariski topology on $\mathbf{P}$. Recall that U_0 can be identified with L^n via (de)homogenisation $L^n \to U_0 \colon v = [x_1, \ldots, x_n] \mapsto [1 \colon x_1 \colon \ldots \colon x_n] =: v^\sharp$ and $U_0 \to L^n \colon v = [x_0 \colon \ldots \colon x_n] \mapsto [\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] =: v'$.

a) We show that the topology on L^n induced by the Zariski topology on $\mathbf{P}$ and the Zariski topology on L^n coincide; in other words, the identification $L^n \to U_0$ is a homeomorphism, and thus $U_0 \subseteq \mathbf{P}$ is called an **affine open** subset:

For $I \trianglelefteq A^\sharp$ homogeneous we have $\mathbf{V}_L^\sharp(I) \cap L^n = \{v \in L^n; v^\sharp \in \mathbf{V}_L^\sharp(I)\} = \{v \in L^n; f'(v) = 0 \text{ for all } f \in I \text{ homogeneous}\} = \mathbf{V}_L(I')$; thus any closed subset of L^n with respect to the induced topology is Zariski closed. Conversely, for $I \trianglelefteq A$ we have $\mathbf{V}_L^\sharp(I^\sharp) \cap L^n = \mathbf{V}_L((I^\sharp)') = \mathbf{V}_L(I)$; thus any Zariski closed subset of L^n is closed with respect to the induced topology. $\sharp$

In particular, if L is infinite, from $\mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{P})) = \mathbf{P}$ we get $\mathbf{V}_L(\mathbf{I}_K^\sharp(\mathbf{P})') = L^n$, so that $\mathbf{I}_L^\sharp(\mathbf{P})' = \{0\}$, entailing $\mathbf{I}_L^\sharp(\mathbf{P}) = \{0\}$; thus we have $K[\mathbf{P}] \cong A^\sharp$.

b) We compare affine closed sets and projective closed sets: For $I \trianglelefteq A$ and $V := \mathbf{V}_L(I) \subseteq L^n$ affine closed, let $\overline{V} \subseteq \mathbf{P}$ be its **projective closure**, that is the smallest projective closed subset containing V.

Letting $\mathbf{W} := \mathbf{V}_L^\sharp(I^\sharp) \subseteq \mathbf{P}$, being projective closed such that $\mathbf{W} \cap L^n = \mathbf{V}_L((I^\sharp)') = \mathbf{V}_L(I) = V$, we get $\overline{V} \subseteq \mathbf{W}$, hence $V \subseteq \overline{V} \cap L^n \subseteq \mathbf{W} \cap L^n = V$, so that $\overline{V} \cap L^n = V$. (We will comment on the relationship $\overline{V} \subseteq \mathbf{W}$ below.) The elements of $\overline{V} \setminus V \subseteq \mathbf{P} \setminus L^n = \mathbf{H}_0$ are called the **points at infinity** of V; recall that $\overline{V}$ is irreducible if and only if V is irreducible.

Conversely, if $\mathbf{V} \subseteq \mathbf{P}$ is projective closed and irreducible, such that $\mathbf{V} \not\subseteq \mathbf{H}_0$, we have $\emptyset \neq \mathbf{V} \cap L^n \subseteq \mathbf{V}$ open, and hence dense, entailing $\overline{\mathbf{V} \cap L^n} = \mathbf{V}$. In conclusion, we have shown that mapping $V \mapsto \overline{V}$ yields a bijection

$$\{V \subseteq L^n \text{ closed, irreducible}\} \to \{\mathbf{V} \subseteq \mathbf{P} \text{ closed, irreducible}; \mathbf{V} \not\subseteq \mathbf{H}_0\},$$

whose inverse is given by $\mathbf{V} \mapsto \mathbf{V} \cap L^n$.

In general, if $V = \bigcup_{i=1}^r V_i \subseteq L^n$ are the irreducible components of V, then we have $\overline{V} = \bigcup_{i=1}^r \overline{V}_i \subseteq \mathbf{P}$, where the $\overline{V}_i$ are irreducible, and since $\overline{V}_i \cap L^n = V_i$ the decomposition of $\overline{V}$ is irredundant, so that the $\overline{V}_i$ are the irreducible components of $\overline{V}$. Hence the above bijections extend to bijections between the set of affine closed subsets of L^n, and the set of projective closed subsets of $\mathbf{P}$ having no irreducible component being contained in the hyperplane at infinity.
