Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 14** (19.12.2025)

(14.1) Projective closure. We keep the earlier notation, and let L be infinite.

From $\mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{P})) = \mathbf{P}$ we get $\mathbf{V}_L(\mathbf{I}_K^\sharp(\mathbf{P})') = L^n$, so that $\mathbf{I}_L^\sharp(\mathbf{P})' = \{0\}$, entailing $\mathbf{I}_L^\sharp(\mathbf{P}) = \{0\}$; thus we have $K[\mathbf{P}] \cong A^\sharp$.

Both L^n, having coordinate algebra A, and $\mathbf{P}$, having homogeneous coordinate algebra $A^\sharp$, are irreducible, so that $\mathbf{P} \cap L^n = L^n$ implies that the projective closure of L^n is $\overline{L^n} = \mathbf{P}$. (In contrast, if L is finite and $K = L$, then singleton sets are closed, so that $\mathbf{P}$ carries the discrete topology.)

(14.2) Vanishing ideal of projective closures. We proceed to describe the vanishing ideals of projective closures: To this end, let $\mathbf{V} \subseteq \mathbf{P}$ be projective closed having no irreducible component contained in $\mathbf{H}_0$, and let $V := \mathbf{V} \cap L^n \subseteq L^n$, being affine closed such that $\overline{V} = \mathbf{V}$.

i) If $I \trianglelefteq A^\sharp$ is homogeneous such that $\mathbf{V} = \mathbf{V}_L^\sharp(I)$, then we have $V = \mathbf{V} \cap L^n = \mathbf{V}_L(I')$, saying that a defining ideal of V is given as the dehomogenisation of any defining ideal of $\mathbf{V}$. In particular, if L is algebraically closed, the vanishing ideal of V is given as the dehomogenisation of the vanishing ideal of $\mathbf{V}$.

ii) Let conversely $I \trianglelefteq A$ such that $V = \mathbf{V}_L(I)$, and let $\mathbf{W} := \mathbf{V}_L^\sharp(I^\sharp) \subseteq \mathbf{P}$. Then we have already seen that $\mathbf{W} \cap L^n = \mathbf{V}_L((I^\sharp)') = \mathbf{V}_L(I) = V$, implying that $\mathbf{V} = \overline{V} \subseteq \mathbf{W}$. If L is algebraically closed we show that actually $\mathbf{V} = \mathbf{W}$, by showing that any projective closed set containing V already contains $\mathbf{W}$:

Let $J \trianglelefteq A^\sharp$ homogeneous such that $\mathbf{U} := \mathbf{V}_L^\sharp(J)$ contains V, that is we have $V \subseteq \mathbf{U} \cap L^n = \mathbf{V}_L(J')$. Thus we have $J' \subseteq \mathbf{I}_K(V) = \sqrt{I} \trianglelefteq A$. Hence for any $f \in J$ we have $(f^k)' = (f')^k \in I$, for some $k \in \mathbb{N}$. This implies $(X_0^{-\nu(f)} \cdot f)^k = ((f^k)')^\sharp \in I^\sharp$, thus $X_0^{-\nu(f)} \cdot f \in \sqrt{I^\sharp} = \mathbf{I}_K^\sharp(\mathbf{W})$. This entails $f \in \mathbf{I}_K^\sharp(\mathbf{W})$, hence $J \subseteq \mathbf{I}_K^\sharp(\mathbf{W})$, thus $\mathbf{W} = \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{W})) \subseteq \mathbf{V}_L^\sharp(J) = \mathbf{U}$. $\sharp$

Thus, if L is algebraically closed, a defining ideal of $\mathbf{V}$ is given as the homogenisation of any defining ideal of V; in particular, the vanishing ideal of $\mathbf{V}$ is given as the homogenisation of the vanishing ideal of V. The latter assertions do not necessarily hold if L is not algebraically closed:

Example. Let $K = L = \mathbb{R}$ and $n = 2$, hence $A^\sharp = \mathbb{R}[T, X, Y]$. Let $I = \langle X^2 + Y^4 \rangle \trianglelefteq A$. Then we have $V = \mathbf{V}(I) = \{[0, 0]\}$, so that $\overline{V} = V = \{[1\colon 0\colon 0]\}$. But we have $I^\sharp = \langle T^2X^2 + Y^4 \rangle$, so that $\mathbf{V}^\sharp(I^\sharp) = \{[1\colon 0\colon 0], [0\colon 1\colon 0]\}$.
