Algebraic Geometry (WS 2025)

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(15.1) Example: The twisted cubic. Let $K = L = \mathbb{C}$, and let $n = 3$, hence we have $A = \mathbb{C}[X, Y, Z]$ and $A^\sharp = \mathbb{C}[W, X, Y, Z]$. We consider the projective closed set $\mathbf{W} := \mathbf{V}^\sharp(I) \subseteq \mathbf{P} = \mathbf{P}^3(\mathbb{C})$, where $I := \langle XZ - Y^2, YW - X^2 \rangle \trianglelefteq A^\sharp$.

Note that the graded $\mathbb{C}$-algebra automorphism of $A^\sharp$ given by $X \leftrightarrow Y$ and $W \leftrightarrow Z$ leaves I invariant, so that it induces a graded $\mathbb{C}$-algebra automorphism of $A^\sharp/I$ and thus of $\mathbb{C}[\mathbf{W}] = A^\sharp/\mathbf{I}^\sharp(\mathbf{W}) = A^\sharp/\sqrt{I}$. (But we do not know yet what an 'automorphism' of a projective variety should be.)

i) We observe that $\mathbf{W} \cap \mathbf{H}_0 = \mathbf{V}^\sharp(XZ - Y^2, YW - X^2, W) = \mathbf{V}^\sharp(W, X, Y) = \{[0\colon 0\colon 0\colon 1]\}$. Hence $\mathbf{W}$ either has no irreducible component contained in $\mathbf{H}_0$, or $\{[0\colon 0\colon 0\colon 1]\}$ is the only one. We will exclude the latter case by showing (on the fly) that $\{[0\colon 0\colon 0\colon 1]\}$ is contained in a larger closed irreducible subset.

We proceed to examine the affine closed set $\mathbf{W} \cap \mathbb{C}^3 = \mathbf{V}(I')$, where we have $I' = \langle XZ - Y^2, Y - X^2 \rangle = \langle X(Z - X^3), Y - X^2 \rangle \trianglelefteq A$. Then both $P := \langle Z - X^3, Y - X^2 \rangle \trianglelefteq A$ and $Q := \langle X, Y - X^2 \rangle = \langle X, Y \rangle \trianglelefteq A$ divide I'.

ii) We show that P is prime: To this end, we consider the homomorphism of $\mathbb{C}$-algebras $\varphi^*\colon A/P \to \mathbb{C}[T]\colon X \mapsto T, Y \mapsto T^2, Z \mapsto T^3$, which is well-defined by the definition of P. Moreover, we have a homomorphism of $\mathbb{C}$-algebras $\psi^*\colon \mathbb{C}[T] \to A/P\colon T \mapsto X$. Then we have $\psi^*(\varphi^*(X)) = X$, and $\psi^*(\varphi^*(Y)) = X^2 = Y$, and $\psi^*(\varphi^*(Z)) = X^3 = Z$, thus $\varphi^*\psi^* = \mathrm{id}_{A/P}$. Similarly, we get $\varphi^*(\psi^*(T)) = T$, thus $\psi^*\varphi^* = \mathrm{id}_{\mathbb{C}[T]}$. This shows that $A/P \cong \mathbb{C}[T]$ is a domain.

Let $V := \mathbf{V}(P) \subseteq \mathbb{C}^3$, which is irreducible. Then φ^* is the comorphism associated with the isomorphism $\varphi\colon \mathbb{C} \to V\colon t \mapsto [t, t^2, t^3]$.

Similarly, $A/Q \cong \mathbb{C}[Z]$ is a domain, hence Q is prime. Let $U := \mathbf{V}(Q) = \{0\} \times \{0\} \times \mathbb{C} \subseteq \mathbb{C}^3$, that is the '$z$-axis'; we have the isomorphism $\mathbb{C} \to U\colon z \mapsto [0, 0, z]$.

We show that $P \cap Q = I'$: We have $I' \subseteq P \cap Q$. From $P = \mathbf{I}(V)$ and $Q = \mathbf{I}(U)$ we get $\mathbf{I}(V \cup U) = P \cap Q = PQ$. Since $Y - X^2 \in I'$ we have $PQ = \langle X(Z - X^3) \rangle = \{0\} \trianglelefteq A/I'$. Thus we infer that $PQ \subseteq I'$, so that $I' = PQ = P \cap Q = \mathbf{I}(V \cup U)$.

Hence we have the decomposition $\mathbf{W} \cap \mathbb{C}^3 = \mathbf{V}(I') = \mathbf{V}(\mathbf{I}(V \cup U)) = V \cup U$ into irreducible components, where $V \cap U = \mathbf{V}(P + Q) = \mathbf{V}(\langle X, Y, Z \rangle) = \{[0, 0, 0]\}$.

Letting $\mathbf{V} := \overline{V} = \mathbf{V}^\sharp(P^\sharp)$ and $\mathbf{U} := \overline{U} = \mathbf{V}^\sharp(Q^\sharp)$ we have the decomposition $\mathbf{W} = \mathbf{V} \cup \mathbf{U} \cup \{[0\colon 0\colon 0\colon 1]\}$ into irreducible closed subsets, where it remains to be decided whether the last piece is redundant. Being homogenisations of prime ideals, both $P^\sharp$ and $Q^\sharp$ are prime. In order to determine $P^\sharp$ and $Q^\sharp$ explicitly, we apply homogenisation, recalling that we have to apply homogenisation not only to a generating set of the ideal in question, but to all its elements:

iii) We determine $Q^\sharp \trianglelefteq A^\sharp$: From $Q = \langle X, Y \rangle = \{Xf + Yg \in A; f, g \in A\} \trianglelefteq A$

we get $Q^\sharp = \langle X, Y\rangle \trianglelefteq A^\sharp$. Thus we have

$$\mathbf{U} = (\mathbf{U} \cap \mathbb{C}^3) \,\dot\cup\, (\mathbf{U} \cap \mathbf{H}_0) = \{[1\colon 0\colon 0\colon z] \in \mathbf{P}; z \in \mathbb{C}\} \,\dot\cup\, \{[0\colon 0\colon 0\colon 1]\},$$

having homogeneous coordinate algebra $\mathbb{C}[\mathbf{U}] = A^\sharp/Q^\sharp \cong \mathbb{C}[W, Z] = \mathbb{C}[\mathbf{P}^1]$. Indeed we have the homeomorphism $\mathbf{U} \to \mathbf{P}^1\colon [w\colon 0\colon 0\colon z] \mapsto [w\colon z]$. (We are tempted to call it an 'isomorphism', but so far we do not even have a definition of a 'morphism' between projective varieties.)

iv) We proceed to determine $P^\sharp \trianglelefteq A^\sharp$: We have $\langle YW - X^2, ZW^2 - X^3\rangle \subseteq P^\sharp$. But we have $Z = X^3 = XY \in A/P$ and $Y^2 = X^4 = XZ \in A/P$, which implies that $Z - XY \in P$ and $Y^2 - XZ \in P$. Hence we have $ZW - XY \in P^\sharp$ and $Y^2 - XZ \in P^\sharp$ as well. Letting

$$J := \langle Y^2 - XZ, ZW - XY, YW - X^2\rangle \subseteq P^\sharp \trianglelefteq A^\sharp,$$

we observe that $ZW^2 - X^3 = XYW - XYW = 0 \in A^\sharp/J$, making this generator redundant. We guess that we actually have $J = P^\sharp$, and set out to show this:

To this end, we consider the epimorphism of $\mathbb{C}$-algebras

$$\alpha\colon A^\sharp \to \mathbb{C}[S, T]_{3\mathbb{N}_0} := \bigoplus_{d\in\mathbb{N}_0} \mathbb{C}[S, T]_{3d}\colon W \mapsto S^3, X \mapsto S^2T, Y \mapsto ST^2, Z \mapsto T^3.$$

Then we observe that $J \subseteq \ker(\alpha) \trianglelefteq A^\sharp$. We show that equality holds:

Since α is a homomorphism of graded algebras, with respect to the grading of $\mathbb{C}[S, T]_{3\mathbb{N}_0}$ indicated above, we infer that $\ker(\alpha) \trianglelefteq A^\sharp$ is a homogeneous ideal. Since $J \trianglelefteq A^\sharp$ is homogeneous as well, both $A^\sharp/J$ and $A^\sharp/\ker(\alpha)$ are graded algebras. Thus we have $J = \ker(\alpha)$, if and only if for all $d \geq 0$ we have

$$\dim_{\mathbb{C}}((A^\sharp)_d/J_d) \leq \dim_{\mathbb{C}}((A^\sharp)_d/\ker(\alpha)_d) = \dim_{\mathbb{C}}(\mathbb{C}[S, T]_{3d}) = 3d + 1.$$

Now $(A^\sharp/J)_d$ is generated as a $\mathbb{C}$-vector space by the cosets of the monomials in $A^\sharp$ of degree d. Taking the (**binomial**) generators of J into account, it is immediate that the following cosets suffice:

$$\{W^iZ^j; i + j = d\} \,\dot\cup\, \{W^iZ^jX; i + j = d - 1\} \,\dot\cup\, \{W^iZ^jY; i + j = d - 1\}.$$

This set has cardinality $(d+1)+2d = 3d+1$, showing $\dim_{\mathbb{C}}((A^\sharp)_d/J_d) \leq 3d+1$.

From $J = \ker(\alpha) \trianglelefteq A^\sharp$, since $\mathbb{C}[S, T]_{3\mathbb{N}_0} \subseteq \mathbb{C}[S, T]$ is a domain, we conclude that $J \trianglelefteq A^\sharp$ is prime. Thus $J' = P = (P^\sharp)'$ yields $\mathbf{V}^\sharp(J) = \overline{\mathbf{V}(J')} = \overline{\mathbf{V}(P)} = \overline{\mathbf{V}((P^\sharp)')} = \mathbf{V}^\sharp(P^\sharp) = \mathbf{V}$, entailing $J = \mathbf{I}^\sharp(\mathbf{V}^\sharp(J)) = \mathbf{I}^\sharp(\mathbf{V}^\sharp(P^\sharp)) = P^\sharp$.

From this we get the **twisted cubic** (space curve)

$$\mathbf{V} = (\mathbf{V} \cap \mathbb{C}^3) \,\dot\cup\, (\mathbf{V} \cap \mathbf{H}_0) = \{[1\colon t\colon t^2\colon t^3] \in \mathbf{P}; t \in \mathbb{C}\} \,\dot\cup\, \{[0\colon 0\colon 0\colon 1]\},$$

having homogeneous coordinate algebra $\mathbb{C}[\mathbf{V}] = A^\sharp/P^\sharp \cong \mathbb{C}[S, T]_{3\mathbb{N}_0}$.

v) In conclusion, we have $\mathbf{W} = \mathbf{V} \cup \mathbf{U}$, the latter being the irreducible components of $\mathbf{W}$, where $\mathbf{V} \cap \mathbf{U} = \{[1\colon 0\colon 0\colon 0], [0\colon 0\colon 0\colon 1]\}$.

Moreover, we have $\mathbf{I}^\sharp(\mathbf{W}) = \mathbf{I}^\sharp(\overline{\mathbf{W} \cap \mathbb{C}^3}) = \mathbf{I}(\mathbf{W} \cap \mathbb{C}^3)^\sharp = (I')^\sharp$ and $\mathbf{I}^\sharp(\mathbf{W}) = \mathbf{I}^\sharp(\mathbf{V}) \cap \mathbf{I}^\sharp(\mathbf{U}) = P^\sharp \cap Q^\sharp = P^\sharp Q^\sharp$, thus we get $(I')^\sharp = P^\sharp \cap Q^\sharp = P^\sharp Q^\sharp$.

Finally, since $I \subseteq (I')^\sharp = P^\sharp \cap Q^\sharp$, computing in $A^\sharp/I = A^\sharp/\langle XZ - Y^2, YW - X^2\rangle$ yields $P^\sharp Q^\sharp = \langle Y^2 - XZ, ZW - XY, YW - X^2\rangle\langle X, Y\rangle = \langle ZW - XY\rangle\langle X, Y\rangle \subseteq A^\sharp/I$, where $XZW - X^2Y = Y^2W - Y^2W = 0 \in A^\sharp/I$ and $YZW - XY^2 = X^2Z - X^2Z = 0 \in A^\sharp/I$ yields $P^\sharp Q^\sharp = \langle XZW - X^2Y, YZW - XY^2\rangle = \{0\} \subseteq A^\sharp/I$. Thus we get $I = (I')^\sharp = P^\sharp \cap Q^\sharp = P^\sharp Q^\sharp = \mathbf{I}^\sharp(\mathbf{W})$. $\sharp$

Remark. A couple of comments concerning part (iv) is in order:

a) The ideal $\widetilde{J} := \langle YW - X^2, ZW^2 - X^3\rangle \trianglelefteq A^\sharp$ encountered as the 'first approximation' of $P^\sharp$ is indeed properly contained in $P^\sharp$: We have $\mathbf{V}^\sharp(\widetilde{J}) \cap \mathbf{H}_0 = \mathbf{V}^\sharp(YW - X^2, ZW^2 - X^3, W) = \mathbf{V}^\sharp(X, W) = \{[0\colon 0\colon y\colon z] \in \mathbf{P}; [y\colon z] \in \mathbf{P}^1\}$, while $\mathbf{V}^\sharp(P^\sharp) \cap \mathbf{H}_0 = \mathbf{V} \cap \mathbf{H}_0 = \{[0\colon 0\colon 0\colon 1]\}$. (Recall that $(\widetilde{J})' = P = (P^\sharp)'$, so that $\mathbf{V}^\sharp(\widetilde{J}) \cap \mathbb{C}^3 = \mathbf{V}(P) = \mathbf{V}^\sharp(P^\sharp) \cap \mathbb{C}^3$.)

b) In order to avoid a specially tailored argument to determine a generating set of $P^\sharp$ from a generating set of P, we may proceed computationally as follows: We compute a Gröbner basis of P with respect to a degree-driven monomial order, then its homogenisation generates $P^\sharp$; actually it is a Gröbner basis of $P^\sharp$ with respect to a certain extension of the given monomial order from $\mathcal{X}$ to $\mathcal{X}^\sharp$. Here, we get $P = \langle Y^2 - XZ, XY - Z, X^2 - Y\rangle \trianglelefteq A$, so that we again obtain $P^\sharp = \langle Y^2 - XZ, XY - ZW, X^2 - YW\rangle \trianglelefteq A^\sharp$.
