Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 16** (25.11.2025)

Morphisms

(16.1) Principal open sets. Let $K \subseteq L$ be a field extension, let $n \in \mathbb{N}_0$, let $\mathcal{X} := \{X_1, \ldots, X_n\}$ be indeterminates, let $A := K[\mathcal{X}]$, and let $\mathbf{V} \subseteq L^n$ closed.

For $f \in K[\mathbf{V}]$, the associated **principal open set** is defined as

$$D_f := \mathbf{V} \setminus \mathbf{V}_L(f) = \{v \in \mathbf{V}; f(v) \neq 0\} \subseteq \mathbf{V}.$$

Then we have $D_{fg} = D_f \cap D_g$, for $g \in K[\mathbf{V}]$, and $D_0 = \emptyset$ and $D_1 = \mathbf{V}$. Moreover, $\{D_f \subseteq \mathbf{V}; f \in K[\mathbf{V}]\}$ is a **basis** of the Zariski topology on $\mathbf{V}$:

Indeed, any open subset $U \subseteq \mathbf{V}$ is a finite union of principal open sets: Let $U = \mathbf{V} \setminus \mathbf{V}_L(I)$, where $I = \langle f_1, \ldots, f_r \rangle \trianglelefteq K[\mathbf{V}]$, for some $r \in \mathbb{N}_0$; then we have $D_f \subseteq U$ for $f \in I$, and for any $u \in U$ there is $i \in \{1, \ldots, r\}$ such that $f_i(u) \neq 0$, that is $u \in D_{f_i}$; this implies that $U = \bigcup_{i=1}^r D_{f_i}$. $\sharp$

Thus the Zariski topology on $\mathbf{V}$ is **quasi-compact**, that is any open covering of $\mathbf{V}$ has a finite sub-covering: (Actually, this holds for any Noetherian topological space.)

We may assume that the covering is given as $\{D_{f_i} \subseteq U; i \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set and $f_i \in K[\mathbf{V}]$. Letting $I := \langle f_i; i \in \mathcal{I} \rangle = \langle f_j; j \in \mathcal{J} \rangle \trianglelefteq K[\mathbf{V}]$, for some $\mathcal{J} \subseteq \mathcal{I}$ finite, from $U = \bigcup_{i \in \mathcal{I}} D_{f_i}$ we get $\mathbf{V}_L(I) = \emptyset$, and thus $U = \bigcup_{j \in \mathcal{J}} D_{f_j}$. $\sharp$
