Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 17** (26.11.2025)

(17.1) Principal open subsets. We keep the earlier notation. Any principal open subset can be identified (as a set) with an affine closed set (but it remains to be shown that this identification is a homeomorphism):

Let $I := \mathbf{I}_K(\mathbf{V}) \trianglelefteq A$, and let T be an indeterminate. Then for $f \in K[\mathbf{V}] = A/I$ let $\widehat{I}_f := \langle I, fT - 1\rangle \trianglelefteq A[T]$, and let

$$\widehat{\mathbf{D}}_f := \mathbf{V}_L(\widehat{I}_f) = \{[v,t] \in \mathbf{V} \times L; f(v) \cdot t = 1\} \subseteq L^n \times L \cong L^{n+1}.$$

Let $\widehat{\pi}\colon \mathbf{V} \times L \to \mathbf{V}\colon [v,t] \mapsto v$ be the natural projection. Then $\widehat{\pi}$ restricts to a bijection $\pi\colon \widehat{\mathbf{D}}_f \to D_f$, having inverse $\pi^{-1}\colon D_f \to \widehat{\mathbf{D}}_f\colon v \mapsto [v, \frac{1}{f(v)}]$.

Moreover, $\widehat{\mathbf{D}}_f \subseteq L^{n+1}$ is affine closed, such that $\mathbf{I}_K(\widehat{\mathbf{D}}_f) = \widehat{I}_f$:

We have $\widehat{I}_f \subseteq \mathbf{I}_K(\widehat{\mathbf{D}}_f)$. To show the converse inclusion, let $0 \neq g = \sum_{i=0}^r g_i T^i \in \mathbf{I}_K(\widehat{\mathbf{D}}_f)$, where $r \geq 0$ and $g_i \in A$, and let $h := \sum_{i=0}^r g_i f^{r-i} \in A$. Then we have

$$f^r g - h = \sum_{i=0}^r (f^r T^i - f^{r-i}) g_i = \sum_{i=0}^r f^{r-i}((fT)^i - 1) g_i = 0 \in A[T]/\langle fT - 1\rangle,$$

hence $h \in \mathbf{I}_K(\widehat{\mathbf{D}}_f)$. Thus for $v \in D_f$ we have $h(v) = h(v, \frac{1}{f(v)}) = 0$. Since for $v \in \mathbf{V} \setminus D_f$ we have $f(v) = 0$, we conclude that $fh \in \mathbf{I}_K(\mathbf{V}) = I$. This entails $f^{r+1} g \in \langle I, fT - 1\rangle = \widehat{I}_f$, thus $g = (1 - (fT)^{r+1})g + T^{r+1} f^{r+1} g \in \widehat{I}_f$. ♯

Hence, recalling that $A[T]/\langle I\rangle \cong (A/I)[T]$, we have

$$K[\widehat{\mathbf{D}}_f] = A[T]/\widehat{I}_f \cong (A/I)[T]/\langle fT - 1\rangle = K[\mathbf{V}][T]/\langle fT - 1\rangle =: K[\mathbf{V}]_f.$$

(We will see later that this indeed coincides with the **localisation** of $K[\mathbf{V}]$ with respect to the multiplicatively closed set $\{f^r \in K[\mathbf{V}]; r \in \mathbb{N}_0\}$.)

(17.2) Regular functions: the affine case. Keeping the above notation, let $U \subseteq \mathbf{V}$ be open. Then the topology induced by the Zariski topology on $\mathbf{V}$ is called the **(K-)Zariski topology** on U. Then $\{D_f \subseteq U; f \in \mathbf{I}_K(\mathbf{V} \setminus U)\}$ is a basis of the Zariski topology on U.

A function $\varphi\colon U \to L$ is called **regular** at a point $v \in U$, if there are $f, g \in A$, where $g \neq 0$, that is $\frac{f}{g} \in \mathrm{Q}(A) = K(\mathcal{X})$, such that $v \in D_g \subseteq U$ and $\varphi(u) = (\frac{f}{g})(u) := \frac{f(u)}{g(u)}$, for all $u \in D_g$; note that we may indeed assume that $D_g \subseteq U$. Moreover, φ is called **regular** on U, if it is regular at any point of U.

It is immediate that the set $\mathcal{O}_K(U)$ of regular functions on U is a K-algebra. Then U, equipped with the Zariski topology, together with the algebra $\mathcal{O}_K(U)$

of regular functions, is called a **quasi-affine (K-)variety**. If $U = \mathbf{U} \subseteq L^n$ is closed, then $\mathbf{U}$ is called an **affine (K-)variety**. (We will show later that we actually have $\mathcal{O}_K(\mathbf{U}) \cong K[\mathbf{U}]$, so that this coincides with the earlier notion.)

(17.3) Regular functions: the projective case. Keeping the above notation, let $\mathbf{V} \subseteq \mathbf{P}$ be closed, and let $U \subseteq \mathbf{V}$ be open. Then the topology induced by the Zariski topology on $\mathbf{V}$ is called the **(K-)Zariski topology** on U.

For $f \in A^\sharp$ homogeneous let $D_f := \mathbf{P} \setminus \mathbf{V}_L^\sharp(f) \subseteq \mathbf{P}$, which is open in $\mathbf{P}$; in particular, the affine open subsets are given as $D_i = D_{X_i}$, for $i \in \{0, \ldots, n\}$. We have $D_f \cap D_0 = D_{fX_0} = D_{f'} \subseteq L^n$, which is open in L^n; similarly for any D_i. Any D_i is a quasi-compact topological space, having the affine principal open subsets as a basis. Hence $\mathbf{P} = \bigcup_{i=0}^n D_i$ implies that $\mathbf{P}$ is a quasi-compact topological space as well, having the affine principal open subsets as a basis. Thus from $(g^\sharp)' = g$, for $g \in A$, we conclude that $\{D_f \subseteq \mathbf{P}; f \in A^\sharp \text{ homogeneous}\}$ is a basis of the Zariski topology on $\mathbf{P}$.

Recall that $A^\sharp$ does not induce functions on $\mathbf{P}$, due to the use of homogeneous coordinates. But $A^\sharp$ is a graded domain, so that we may consider its **graded field of fractions**, which is immediately seen to be a field indeed:

$$K(\mathcal{X}^\sharp)_0 := \{0\} \,\dot\cup\, \{\frac{f}{g} \in K(\mathcal{X}^*); f, g \in A^\sharp \setminus \{0\} \text{ homogeneous}, \deg(f) = \deg(g)\}.$$

Let $0 \neq \frac{f}{g} \in K(\mathcal{X}^\sharp)_0$ such that $d := \deg(f) = \deg(g)$. Then for $[x_0 \colon \ldots \colon x_n] \in \mathbf{V} \cap D_g$ we have $(\frac{f}{g})(\lambda \cdot [x_0, \ldots, x_n]) = \frac{\lambda^d \cdot f(x_0, \ldots, x_n)}{\lambda^d \cdot g(x_0, \ldots, x_n)} = (\frac{f}{g})(x_0, \ldots, x_n)$, for all $\lambda \in L^*$. Hence $\frac{f}{g}$ indeed induces a function on $\mathbf{V} \cap D_g$. Having this in place:

A function $\varphi \colon U \to L$ is called **regular** at a point $v \in U$, if there is $\frac{f}{g} \in K(\mathcal{X}^\sharp)_0$ such that $v \in D_g \subseteq U$ and $\varphi(u) = (\frac{f}{g})(u)$, for all $u \in D_g$; note that we may indeed assume that $D_g \subseteq U$. Moreover, φ is called **regular** on U, if it is regular at any point of U.

It is immediate that the set $\mathcal{O}_K(U)$ of regular functions on U is a K-algebra. Then U, equipped with the Zariski topology, together with the algebra $\mathcal{O}_K(U)$ of regular functions, is called a **quasi-projective (K-)variety**. If $U = \mathbf{U} \subseteq \mathbf{P}$ is closed, then $\mathbf{U}$ is called a **projective (K-)variety**. (We will show later that we actually have $\mathcal{O}_K(\mathbf{U}) \cong K$, so that $\mathcal{O}_K(\mathbf{U}) \not\cong K[\mathbf{U}]$; but $\mathcal{O}_K(\mathbf{U})$ does not seem to be too useful either.)
