Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 18** (02.12.2025)

(18.1) Categories. a) A **category** $\mathcal{C}$ consists of a **class** $\mathrm{Ob}(\mathcal{C})$ of **objects** (which is not necessarily a set), together with sets of **morphisms** $\mathrm{Hom}_{\mathcal{C}}(A,B)$, also denoted by $\mathrm{Mor}_{\mathcal{C}}(A,B)$, for all $A,B\in\mathcal{C}$, such that
i) for all $A\in\mathcal{C}$ there is an **identity** $\mathrm{id}_A\in\mathrm{End}_{\mathcal{C}}(A):=\mathrm{Hom}_{\mathcal{C}}(A,A)$, and
ii) for all $A,B,C\in\mathcal{C}$ there is a **concatenation** map

$$\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{Hom}_{\mathcal{C}}(B,C)\to\mathrm{Hom}_{\mathcal{C}}(A,C)\colon[\alpha,\beta]\mapsto\alpha\beta,$$

fulfilling the following conditions for all $A,B,C,D\in\mathcal{C}$:
i) For all $\alpha\colon A\to B$ and $\beta\colon B\to A$ we have $\mathrm{id}_A\cdot\alpha=\alpha$ and $\beta\cdot\mathrm{id}_A=\beta$, and
ii) for all $\alpha\colon A\to B$, $\beta\colon B\to C$, $\gamma\colon C\to D$ we have $(\alpha\beta)\gamma=\alpha(\beta\gamma)\colon A\to D$.

Here and further on, for objects $A\in\mathrm{Ob}(\mathcal{C})$ we abbreviate by writing $A\in\mathcal{C}$, and for $\alpha\in\mathrm{Hom}_{\mathcal{C}}(A,B)$ we write an **arrow** $\alpha\colon A\to B$.

Example. The class Sets of all sets, together with the set of all maps between pairs of sets as morphisms, forms a category. Note that the class of all sets is not a set, so that we have to be careful with these constructions. (We will not go into any detail here, but we will just take this for granted.)

The following is an example of a **small** category, that is one whose class of objects is a set: Let $\mathcal{M}$ be a set. Then the set $\mathsf{Sets}(\mathcal{M})$ of all subsets of $\mathcal{M}$, together with all maps between subsets of $\mathcal{M}$ as morphisms, forms a category. This can be varied, for example by allowing only for injective maps, or only for surjective maps morphisms; or by going down to a smaller set of objects, such as the set of all finite subsets of $\mathcal{M}$; or both.

Further (large) examples are the class Top of all topological spaces, together with all continuous maps as morphisms; the class $\mathsf{Ab}=\mathsf{Mod}\text{-}\mathbb{Z}$ of abelian groups, that is $\mathbb{Z}$-modules, together with all group homomorphisms as morphisms; the class $\mathsf{Mod}\text{-}R$ of all R-modules, where R is a ring, together with all R-module homomorphisms as morphisms; or the class $\mathsf{mod}\text{-}R$ of all finitely generated R-modules, together with all R-module homomorphisms as morphisms. ♯

b) A **(covariant) functor** $\mathcal{F}\colon\mathcal{C}\to\mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ consists of a map $\mathrm{Ob}(\mathcal{C})\to\mathrm{Ob}(\mathcal{D})\colon A\to\mathcal{F}(A)$, together with maps

$$\mathcal{F}_{A,B}\colon\mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A),\mathcal{F}(B))\colon\alpha\mapsto\mathcal{F}_{A,B}(\alpha)=\mathcal{F}(\alpha),$$

fulfilling the following conditions for all $A,B,C\in\mathcal{C}$:
i) We have $\mathcal{F}(\mathrm{id}_A)=\mathrm{id}_{\mathcal{F}(A)}$, and
ii) for all $\alpha\colon A\to B$ and $\beta\colon B\to C$ we have $\mathcal{F}(\alpha)\cdot\mathcal{F}(\beta)=\mathcal{F}(\alpha\beta)$.

Similarly, a **contravariant functor** $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ consists of a map $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, together with **arrow reversing** maps

$$\mathcal{F}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(B),\mathcal{F}(A)),$$

such that $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$ and $\mathcal{F}_{B,C}(\beta) \cdot \mathcal{F}_{A,B}(\alpha) = \mathcal{F}_{A,C}(\alpha\beta) \in \mathrm{Hom}_{\mathcal{D}}(C,A)$.

For example, we have the (covariant) **identity functor** $\mathrm{Id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$, mapping $A \mapsto A$ for all $A \in \mathcal{C}$, and $\alpha \mapsto \alpha$ for all $\alpha\colon A \to B$.

Example: Algebraic-geometric correspondence. Let $K \subseteq L$ be a field extension, where L is algebraically closed. Then the set $K\text{-}\mathsf{AffVar} := \coprod_{n\in\mathbb{N}_0}\{\mathbf{V} \subseteq L^n\ K\text{-closed}\}$ of all affine K-varieties over L, together with all regular maps $\mathrm{Mor}_K(?,??)$, forms a category.

Moreover, the set K-fgAlg of quotient algebras of finitely generated polynomial K-algebras, and the subset $K\text{-}\mathsf{AffAlg} \subseteq K\text{-}\mathsf{fgAlg}$ of reduced algebras, together with all K-algebra homomorphisms $\mathrm{Hom}(?,??)$, form categories.

Then we get a contravariant functor $?^*\colon K\text{-}\mathsf{AffVar} \to K\text{-}\mathsf{AffAlg}$ by letting

$$\mathbf{V} \mapsto K[\mathbf{V}] \quad \text{and} \quad \big(\varphi\colon \mathbf{V} \to \mathbf{W}\big) \mapsto \big(\varphi^*\colon K[\mathbf{W}] \to K[\mathbf{V}]\big).$$

Then $?^*$ is **surjective** (on objects), that is any affine algebra is in the image of $?^*$, as well as **fully faithful** (on morphisms), that is $?^*\colon \mathrm{Mor}_K(?,??) \to \mathrm{Hom}(??^*,?^*)$ is surjective and injective, respectively.

(18.2) Sheaves. a) Let $\mathcal{V}$ be a topological space. We get a category $\mathcal{T} = \mathcal{T}_{\mathcal{V}}$, consisting of the open subsets of $\mathcal{V}$, whose morphism sets are defined as follows: Letting $\mathcal{W},\mathcal{U} \subseteq \mathcal{V}$ be open, if $\mathcal{W} \not\subseteq \mathcal{U}$ then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{W},\mathcal{U}) := \emptyset$, if $\mathcal{W} \subseteq \mathcal{U}$ then $\mathrm{Hom}_{\mathcal{T}}(\mathcal{W},\mathcal{U}) := \{\iota^{\mathcal{U}}_{\mathcal{W}}\}$, where $\iota^{\mathcal{U}}_{\mathcal{W}}\colon \mathcal{W} \to \mathcal{U}$ is the natural inclusion map; in particular we have $\iota^{\mathcal{U}}_{\mathcal{U}} = \mathrm{id}_{\mathcal{U}}$.

Let $\mathcal{A}$ be a category, such that $\mathrm{Ob}(\mathcal{A}) \subseteq \mathrm{Ob}(\mathsf{Ab})$ and $\{0\} \in \mathrm{Ob}(\mathcal{A})$. Note that for all $A \in \mathsf{Ab}$ there is a unique morphism $\{0\} \to A$, and a unique morphism $A \to \{0\}$, that is $\{0\} \in \mathsf{Ab}$ is an **initial object** and a **terminal object**, respectively, thus is a **zero object**. We assume that the latter morphisms belong to $\mathcal{A}$ as well. (In the sequel we only need a terminal object in $\mathcal{A}$. Note that Sets has an initial object and a terminal object, but not a zero object.)

Then a **presheaf** on $\mathcal{V}$ with **values** in $\mathcal{A}$ is a contravariant functor $\mathcal{F}\colon \mathcal{T} \to \mathcal{A}$ such that $\mathcal{F}(\emptyset) = \{0\}$. Typical choices $\mathcal{A}$ are Ab, or more generally $\mathsf{mod}\text{-}R$ for certain rings R, or K-fgAlg for fields K; note that all of them contain $\{0\}$.

Thus a presheaf $\mathcal{F}$ on $\mathcal{V}$ associates an object $\Gamma(\mathcal{U},\mathcal{F}) := \mathcal{F}(\mathcal{U}) \in \mathcal{A}$ to any open subset $\mathcal{U} \subseteq \mathcal{V}$, called the **sections** of $\mathcal{F}$ over $\mathcal{U}$; in particular $\Gamma(\mathcal{F}) := \mathcal{F}(\mathcal{V})$ are called the **global sections** of $\mathcal{F}$. Moreover, for $\mathcal{U}' \subseteq \mathcal{U} \subseteq \mathcal{V}$ open there is a **restriction morphism**

$$\rho^{\mathcal{U}}_{\mathcal{U}'} := \mathcal{F}(\iota^{\mathcal{U}}_{\mathcal{U}'})\colon \mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{U}')\colon f \mapsto f|_{\mathcal{U}'},$$

such that whenever $\mathcal{U}'' \subseteq \mathcal{U}' \subseteq \mathcal{U} \subseteq \mathcal{V}$ are open we have $\rho^{\mathcal{U}'}_{\mathcal{U}''} \circ \rho^{\mathcal{U}}_{\mathcal{U}'} = \rho^{\mathcal{U}}_{\mathcal{U}''}$, and where $\rho^{\mathcal{U}}_{\mathcal{U}} = \mathrm{id}_{\mathcal{F}(\mathcal{U})}$, and $\rho^{\mathcal{U}}_{\emptyset}$ is the unique morphism $\mathcal{F}(\mathcal{U}) \to \{0\}$.

b) A presheaf $\mathcal{F}$ on $\mathcal{V}$ with values in $\mathcal{A}$ is called a **sheaf**, if the following additional condition holds:

Whenever $\mathcal{U} \subseteq \mathcal{V}$ is open, and $\{\mathcal{U}_i \subseteq \mathcal{U}; i \in \mathcal{I}\}$ is an open covering, where $\mathcal{I}$ is an index set, and $\{f_i \in \mathcal{F}(\mathcal{U}_i); i \in \mathcal{I}\}$ is a collection of sections such that $(f_i)|_{\mathcal{U}_{ij}} = (f_j)|_{\mathcal{U}_{ij}}$, where $\mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j$ for all $i, j \in \mathcal{I}$, then there is a unique section $f \in \mathcal{F}(\mathcal{U})$ such that $f|_{\mathcal{U}_i} = f_i$ for all $i \in \mathcal{I}$.

Since for any section $f \in \mathcal{F}(\mathcal{U})$ we have $(f|_{\mathcal{U}_i})|_{\mathcal{U}_{ij}} = f|_{\mathcal{U}_{ij}} = (f|_{\mathcal{U}_j})|_{\mathcal{U}_{ij}}$ the compatibility condition is necessary; note that since restriction to the empty subset is the zero map $\mathcal{F}(\mathcal{U}) \to \{0\}$ anyway, the compatibility condition is vacuous for $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$. Now the above sheaf condition says that the compatibility condition is also sufficient for the existence of such a section, by 'pasting together'. Moreover, it also says that any section is uniquely defined by its restrictions to the given open 'pieces'.
