Algebraic Geometry (WS 2025)

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(19.1) Examples of sheaves. a) Letting $U \subseteq \mathbb{C}$ be open, with respect to the metric topology, let $\mathcal{H}(U)$ be the set of holomorphic $\mathbb{C}$-valued functions on U, which is a $\mathbb{C}$-algebra with respect to pointwise addition and multiplication. This defines a presheaf $\mathcal{H}$ of $\mathbb{C}$-algebras on $\mathbb{C}$, whose restriction maps are given by restriction of functions.

Actually, $\mathcal{H}$ is a sheaf: For any $U \subseteq \mathbb{C}$ open, any function on U is uniquely defined by its restrictions to any open covering of U. Conversely, since being holomorphic is defined 'locally', that is on open discs around any point, which form a basis of the metric topology, prescribing compatible holomorphic functions on an open covering of U defines a holomorphic function on U.

b) Let $\mathcal{V}$ be topological space, let $A \neq \emptyset$ be a set and let $0 \in A$.

i) Then letting $\mathcal{U} \mapsto A$ for $\emptyset \neq \mathcal{U} \subseteq \mathcal{V}$ open, and $\emptyset \mapsto \{0\}$, and restriction maps being given as $\rho^{\mathcal{U}}_{\mathcal{U}'} = \mathrm{id}_A$ for $\emptyset \neq \mathcal{U}' \subseteq \mathcal{U} \subseteq \mathcal{V}$ open, defines a presheaf $\mathcal{C}_0$ on $\mathcal{V}$, being called the **constant presheaf** with values in A. (In other words, $\mathcal{C}_0(\mathcal{U})$ can be considered as consisting of the constant maps from $\mathcal{U}$ to A.)

Then $\mathcal{C}_0$ is not necessarily a sheaf: Assume that $|A| \geq 2$, and that $\mathcal{V}$ is **disconnected**, that is there are $\emptyset \neq \mathcal{U}, \mathcal{W} \subseteq \mathcal{V}$ open such that $\mathcal{V} = \mathcal{U} \,\dot{\cup}\, \mathcal{W}$. Then letting $a \in \mathcal{C}_0(\mathcal{U}) = A$ and $b \in \mathcal{C}_0(\mathcal{W}) = A$, where $a \neq b$, be sections over $\mathcal{U}$ and $\mathcal{W}$, respectively, the compatibility condition is trivially fulfilled. But for any global section $c \in \mathcal{C}_0(\mathcal{V}) = A$ we get $\rho^{\mathcal{V}}_{\mathcal{U}}(c) = \mathrm{id}_A(c) = c$ and $\rho^{\mathcal{V}}_{\mathcal{W}}(c) = \mathrm{id}_A(c) = c$, thus the prescribed sections cannot be obtained by restriction from a global one.

ii) Let A be equipped with the discrete topology. For $\mathcal{U} \subseteq \mathcal{V}$ open let $\mathcal{C}(\mathcal{U})$ be the set of continuous maps from $\mathcal{U}$ to A. Together with restriction maps being given by restriction of functions this defines a presheaf $\mathcal{C}$ on $\mathcal{V}$.

Note that, due to A carrying the discrete topology, a map $\varphi\colon \mathcal{U} \to A$ is continuous if and only if $\varphi^{-1}(\{a\}) \subseteq \mathcal{U}$ is open, for all $a \in A$, which in turn is equivalent to φ being **locally constant**, that is any point in $\mathcal{U}$ has an open neighborhood on which φ is constant.

Actually, $\mathcal{C}$ is a sheaf on $\mathcal{V}$, being called the **(locally) constant sheaf** with values in A: For any $\mathcal{U} \subseteq \mathcal{V}$ open, any function on $\mathcal{U}$ is uniquely defined by its restrictions to any open covering of $\mathcal{U}$. Conversely, a map $\varphi\colon \mathcal{U} \to A$ is continuous if and only if the preimage of any open subset of A is open in $\mathcal{U}$. Now a subset of $\mathcal{U}$ is open if and only if all its intersections with some open covering of $\mathcal{U}$ are open. Hence prescribing compatible continuous maps on an open covering of $\mathcal{U}$ defines a continuous map on $\mathcal{U}$.

(19.2) Localisation. Let R be a ring, and let $\mathcal{S} \subseteq R$ be a **multiplicatively closed** subset, that is $1 \in \mathcal{S}$ and for any $f, g \in \mathcal{S}$ we have $fg \in \mathcal{S}$ as well.

a) Let M be an R-module, and for $f \in R$ let $\rho_M(f) \colon M \to M \colon m \to mf$. An R-module $M_{\mathcal{S}}$ together with a '**natural**' R-module homomorphism $\sigma \colon M \to M_{\mathcal{S}}$ is called the **localisation** of M at $\mathcal{S}$, or the **module of fractions** of M with respect to $\mathcal{S}$, if it fulfills the following **universal property** in Mod-R: **i)** The map $\rho_{M_{\mathcal{S}}}(f)$ is bijective for all $f \in \mathcal{S}$, and **ii)** for any R-module N such that $\rho_N(f)$ is bijective for all $f \in \mathcal{S}$, and any R-module homomorphism $\alpha \colon M \to N$, there is a unique R-module homomorphism $\widehat{\alpha} \colon M_{\mathcal{S}} \to N$ such that $\alpha = \sigma \cdot \widehat{\alpha}$.

It is immediate that the localisation of M at $\mathcal{S}$ is unique up to isomorphism of R-modules, if it exists at all. We show that such a localisation indeed exists:

We consider the set $M \times \mathcal{S}$, and the relation $\sim$ given by $[m, f] \sim [m', f']$ if there is $g \in \mathcal{S}$ such that $(mf' - m'f)g = 0$. Then $\sim$ is an equivalence relation indeed: Reflexivity and symmetry are immediate; to show transitivity let $[m, f] \sim [m', f']$ and $[m', f'] \sim [m'', f'']$, hence there are $g, h \in \mathcal{S}$ such that $(mf' - m'f)g = 0 = (m'f'' - m''f')h$, thus we get $(mf'' - m''f)f'gh = mf'f''gh - (mf' - m'f)f''gh - (m'f'' - m''f')fgh - m''ff'gh = 0 \in M$.

The set of equivalence classes in $M \times \mathcal{S}$ with respect to $\sim$ is denoted by $M/\mathcal{S}$, and the equivalence class of $[m, f]$ is denoted by $\frac{m}{f} \in M/\mathcal{S}$. Then $M/\mathcal{S}$ becomes an R-module by letting $\frac{m}{f} + \frac{m'}{f'} := \frac{mf' + m'f}{ff'}$ and $\frac{m}{f} \cdot g := \frac{mg}{f}$, for $g \in R$; independence of the choice of representatives is immediately checked. Then $\sigma \colon M \to M/\mathcal{S} \colon m \mapsto \frac{m}{1}$ is an R-module homomorphism. We show that $M \times \mathcal{S}$ together with σ fulfills the required universal property:

Firstly, $\rho_{M/\mathcal{S}}(g) \colon M/\mathcal{S} \to M/\mathcal{S} \colon \frac{m}{f} \mapsto \frac{mg}{f}$ is bijective, for any $g \in \mathcal{S}$, with inverse $M/\mathcal{S} \to M/\mathcal{S} \colon \frac{m}{f} \mapsto \frac{m}{fg}$.

Secondly, let $\alpha \colon M \to N$ be an R-module homomorphism, where $\rho_N(f)$ is bijective for all $f \in \mathcal{S}$. Then, for any R-module homomorphism $\widehat{\alpha} \colon M/\mathcal{S} \to N$ such that $\alpha = \sigma \cdot \widehat{\alpha}$ we have $\widehat{\alpha}(\frac{m}{f}) \cdot f = \widehat{\alpha}(\frac{m}{f} \cdot f) = \widehat{\alpha}(\frac{mf}{f}) = \widehat{\alpha}(\frac{m}{1}) = \alpha(m) \in N$, so that $\widehat{\alpha}(\frac{m}{f}) = \alpha(m) \cdot \rho_N(f)^{-1} \in N$, for $f \in \mathcal{S}$. Thus $\widehat{\alpha}$ is unique, if it exists at all. Now it is immediately checked that the latter formula indeed defines an R-module homomorphism $\widehat{\alpha}$ as desired. ♯

b) In particular, for the regular R-module we get the **localisation** $R_{\mathcal{S}}$ of R at $\mathcal{S}$, or the **ring of fractions** of R with respect to $\mathcal{S}$. Indeed $R_{\mathcal{S}}$ becomes a ring, by letting $\frac{g}{f} \cdot \frac{g'}{f'} := \frac{gg'}{ff'}$, as is immediately checked.

Then $R_{\mathcal{S}}$ has the following universal property in the category of (commutative unital) rings: **i)** $\frac{f}{1} \in R_{\mathcal{S}}$ is a unit, for all $f \in \mathcal{S}$, and **ii)** for any ring homomorphism $\alpha \colon R \to T$, such that $\alpha(f) \in T$ is a unit for $f \in \mathcal{S}$, there is a unique ring homomorphism $\widehat{\alpha} \colon R_{\mathcal{S}} \to T$ such that $\alpha = \sigma \cdot \widehat{\alpha}$.

Indeed, firstly, $\rho_{R_{\mathcal{S}}}(f)$ being bijective, thus having inverse $\rho_{R_{\mathcal{S}}}(\frac{1}{f})$, is equivalent to saying that $\sigma(f) = \frac{f}{1} \in R_{\mathcal{S}}$ is a unit. Secondly, considering T as an R-module via α, the universal property of $R_{\mathcal{S}}$ as an R-module implies the existence of a unique R-module homomorphism $\widehat{\alpha}$ as desired; then it immediately checked that $\widehat{\alpha}$ even is a ring homomorphism. ♯

In combination, since $\frac{f}{1} \in R_{\mathcal{S}}$ is a unit, for any $R_{\mathcal{S}}$-module N the map $\rho_N(f)$ is bijective, having inverse $\rho_N(\frac{1}{f})$. Conversely, if $\rho_M(f)$ is bijective for all $f \in \mathcal{S}$, then M becomes an $R_{\mathcal{S}}$-module by letting $m \cdot \frac{g}{f} := \rho_M(f)^{-1}(mg)$, as is immediately checked. In particular, the localisation $M_{\mathcal{S}}$ becomes an $R_{\mathcal{S}}$-module by letting $\frac{m}{h} \cdot \frac{g}{f} := \frac{mg}{fh}$; thus $M_{\mathcal{S}}$ is generated by $\sigma(M)$ as an $R_{\mathcal{S}}$-module.
