Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 32** (03.02.2026)

(32.1) Abstract varieties. a) A prevariety V is called an **(abstract) variety** if the **Hausdorff axiom** holds, that is for any prevariety U and any morphisms $\varphi, \psi \in \mathrm{Mor}(U, V)$ the associated **(difference) kernel** $\ker(\varphi, \psi) := \{u \in U; \varphi(u) = \psi(u)\} \subseteq U$ is closed. (This naming is reminiscent of the Hausdorff property of topological spaces, where products carry the product topology.)

Morphisms of varieties are their morphisms as prevarieties. This gives rise to the category of varieties, being a full subcategory of the category of prevarieties. There are prevarieties which are not varieties, as the example below shows.

b) We present a characterisation of varieties amongst prevarieties:

Let U and V be prevarieties, and let $\varphi \in \mathrm{Mor}(U, V)$. Then let the **graph morphism** $\gamma_\varphi := \mathrm{id}_U \times \varphi \colon U \to U \times V \colon u \mapsto [u, \varphi(u)]$ be the product morphism arising from id_U and φ, and let $\Gamma_\varphi := \gamma_\varphi(U) \subseteq U \times V$ be the **graph** of φ.

In particular, for the identity morphism id_V let $\delta_V := \gamma_{\mathrm{id}_V} = \mathrm{id}_V \times \mathrm{id}_V \colon V \to V \times V \colon v \mapsto [v, v]$ be the **diagonal morphism**, and let $\Delta_V := \delta_V(V) \subseteq V \times V$ be the **diagonal** of $V \times V$.

Proposition. a) Let V be a prevariety. Then V is a variety if and only if V is **separated**, that is the diagonal $\Delta_V \subseteq V \times V$ is closed.

b) Let V be a variety, let U be a prevariety, and let $\varphi \in \mathrm{Mor}(U, V)$. Then $\gamma_\varphi \colon U \to U \times V$ is a closed embedding. In particular, so is $\delta_V \colon V \to V \times V$.

Proof. a) Let V be a variety. Since $\Delta_V = \ker(\pi_1, \pi_2) \subseteq V \times V$, where $\pi_i \in \mathrm{Mor}(V \times V, V)$ are the projections on the first and second factors, respectively, we conclude that $\Delta_V \subseteq V \times V$ is closed.

Now let V be separated. Moreover, let U be a prevariety, let $\varphi_1, \varphi_2 \in \mathrm{Mor}(U, V)$, and let $\varphi := \varphi_1 \times \varphi_2 \colon U \to V \times V$. Then we have $\ker(\varphi_1, \varphi_2) = \ker(\varphi\pi_1, \varphi\pi_2) = \{u \in U; \varphi(u) \in \Delta_V\} = \varphi^{-1}(\Delta_V) \subseteq U$. Since $\Delta_V \subseteq V \times V$ is closed, we infer that $\varphi^{-1}(\Delta_V) \subseteq U$ is closed as well.

b) Letting $P := U \times V$, we consider the morphisms $\pi_V \colon P \to V$ and $\pi_U\varphi \colon P \to V$. Thus, since V is a variety, the kernel $\ker(\pi_V, \pi_U\varphi) = \{[u, v] \in P; \pi_V(u, v) = \varphi(\pi_U(u, v))\} = \{[u, v] \in P; v = \varphi(u)\} = \Gamma_\varphi \subseteq P$ is closed. In particular, Γ_φ together with the structure sheaf $\mathcal{O}_P|_{\Gamma_\varphi}$ is a prevariety.

Moreover, the map $\gamma'_\varphi \colon U \to \Gamma_\varphi \colon u \mapsto \gamma_\varphi(u)$, obtained from γ_φ by restricting its range, is a morphism as well: Let $W \subseteq \Gamma_\varphi$ be open, and let $f \in \mathcal{O}_P|_{\Gamma_\varphi}(W)$; then by the definition of a restricted sheaf we may assume that there is $\widehat{W} \subseteq P$ open and $\widehat{f} \in \mathcal{O}_P(\widehat{W})$ such that $\widehat{W} \cap \Gamma_\varphi = W$ and $\widehat{f}|_W = f$; then we have $(\gamma'_\varphi)^*(f) = \gamma^*_\varphi(\widehat{f}) \in \mathcal{O}_U(\gamma_\varphi^{-1}(\widehat{W})) = \mathcal{O}_U((\gamma'_\varphi)^{-1}(W))$.

Conversely, we have the morphism $\pi_U|_{\Gamma_\varphi} = \iota^P_{\Gamma_\varphi} \cdot \pi_U \colon \Gamma_\varphi \to U$; recall that the inclusion map $\iota^P_{\Gamma_\varphi} \colon \Gamma_\varphi \to P$ is a morphism with respect to the restricted sheaf. Then we get $\gamma_\varphi \cdot (\pi_U|_{\Gamma_\varphi}) = \mathrm{id}_U$ and $(\pi_U|_{\Gamma_\varphi}) \cdot \gamma_\varphi = \mathrm{id}_{\Gamma_\varphi}$. ♯

Example: The affine line with one point doubled. We consider the fibre sum $V := L \sqcup_U L$ of two disjoint copies of the affine line, with respect to the natural inclusion $U := L \setminus \{0\} \subseteq L$. Thus we have injective morphisms $\iota_i \colon L \to V$, such that $\iota_1|_U = \iota_2|_U$. Letting $V' := \iota_i(U) \subseteq V$ we have $V = \{0_1\} \,\dot\cup\, V' \,\dot\cup\, \{0_2\}$, where $V_i := \iota_i(L) = \{0_i\} \,\dot\cup\, V'$. Then V becomes a prevariety, having $\{V_1, V_2\}$ as an affine open covering. But $\ker(\iota_1, \iota_2) = \{a \in L; \iota_1(a) = \iota_2(a)\} = \{a \in L; \iota_i(a) \in V'\} = U \subseteq L$ is not closed, hence V is not a variety. ♯

(32.2) Proposition. Any irreducible affine variety $\mathbf{V}$ is a variety.

Proof. Let U be any prevariety, and let $\varphi, \psi \in \mathrm{Mor}(U, \mathbf{V})$. In order to show that $\ker(\varphi, \psi) \subseteq U$ is closed, by taking an affine open covering of U, we may assume that U is an affine variety. Given $u \in U$, we have $\varphi(u) = \psi(u)$ if and only if $(\varphi^*(f))(u) = f(\varphi(u)) = f(\psi(u)) = (\psi^*(f))(u)$ for all $f \in K[\mathbf{V}]$. In other words, we have $\ker(\varphi, \psi) = \mathbf{V}_L((\varphi^* - \psi^*)(K[\mathbf{V}])) \subseteq U$. ♯
