Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 33** (04.02.2026)

(33.1) Difference kernels. We get the following consequence on difference kernels in general:

Recall first that a subset of a topological space V is called **locally closed**, if it is the intersection of an open and a closed subset of V. Then $W \subseteq V$ is locally closed if and only if it is open in its closure: If $W = U \cap Z$, where $U \subseteq V$ is open and $Z \subseteq V$ is closed, then $W \subseteq Z$ is open, hence $W \subseteq \overline{W}$ is open; conversely, if $W \subseteq \overline{W}$ is open, then there is $U \subseteq V$ open such that $W = U \cap \overline{W}$.

Corollary. a) Let U and V be prevarieties, and let $\varphi, \psi \in \mathrm{Mor}(U, V)$. Then the kernel $\ker(\varphi, \psi) \subseteq U$ is locally closed (rather than closed, if V is a variety).

b) Assume that for any $v, w \in V$ there is an affine open subset of V containing v and w. Then V is a variety.

Proof. a) Let $u \in \ker(\varphi, \psi)$, let $\varphi(u) = \psi(u) \in V' \subseteq V$ be affine open, and let $U' := \varphi^{-1}(V') \cap \psi^{-1}(V') \subseteq U$ open, hence $u \in U'$. Since V' is affine and thus is a variety, for the morphisms $\varphi|_{U'} \colon U' \to V'$ and $\psi|_{U'} \colon U' \to V'$ we infer that $\ker(\varphi, \psi) \cap U' = \ker(\varphi|_{U'}, \psi|_{U'}) \subseteq U'$ is closed. Taking the union of the subsets $U' \subseteq U$ of the above form, we conclude that $\ker(\varphi, \psi)$ is closed in an open subset of U, thus $\ker(\varphi, \psi)$ is the intersection of a closed and an open subset of U.

b) Let U be a prevariety, and let $\varphi, \psi \in \mathrm{Mor}(U, V)$. Given $u \in U$, there is an affine open subset $V' \subseteq V$ such that $\varphi(u), \psi(u) \in V'$. Let $U' := \varphi^{-1}(V') \cap \psi^{-1}(V') \subseteq U$ open, hence $u \in U'$, and $\ker(\varphi, \psi) \cap U' = \ker(\varphi|_{U'}, \psi|_{U'}) \subseteq U'$ is closed. Now U is covered by subsets as above, thus $\ker(\varphi, \psi) \subseteq U$ is closed. $\sharp$

(33.2) Proposition. The (irreducible) projective space $\mathbf{P}$ is a variety.

Proof. Let $\mathbf{P}$ have homogeneous coordinate algebra $K[X_0, \ldots, X_n]$. We consider the (right) action of the general linear group $G := \mathrm{GL}_{n+1}(K)$ on the affine cone L^{n+1} over $\mathbf{P}$. Then G acts by L-linear maps on L^{n+1}, and thus we have an induced action of G on $\mathbf{P}$, by 'linear coordinate transformations'. We show that G acts by automorphisms on $\mathbf{P}$ as a projective variety, where it suffices to show that the map indurced by $\alpha = [\alpha_{ij}] \in G$ is a morphism:

We consider the affine open subset $D_i := D_{X_i} \subseteq \mathbf{P}$, for $i \in \{0, \ldots, n\}$, where $\Gamma(\mathcal{O}_{D_i}) = K[\frac{X_0}{X_i}, \ldots, \widehat{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}]$. Then $\alpha^*(X_i) = \sum_{j=0}^{n} \alpha_{ji} X_j$ is homogeneous of degree 1, entailing a bijection $\alpha_i \colon \alpha^{-1}(D_i) = D_{\alpha^*(X_i)} \to D_i$. We have $\alpha_i^*(\frac{X_j}{X_i}) = \frac{\alpha^*(X_j)}{\alpha^*(X_i)}$, being homogeneous of degree 0, thus being regular on $D_{\alpha^*(X_i)}$. Hence α_i is a morphism, thus so is the map induced by α. $\sharp$

This shows that $D_{\alpha^*(X_i)} \subseteq \mathbf{P}$ is an affine variety isomorphic to L^n. From this we infer that any hyperplane complement $D_f \subseteq \mathbf{P}$, where $f \in K[X_0, \ldots, X_n]$ is homogeneous of degree 1, is an affine variety isomorphic to L^n. Finally, for any $v, w \in \mathbf{P}$ there is a hyperplane complement $D_f \subseteq \mathbf{P}$, such that both $v, w \in D_f$. (It suffices to consider the case $v \in D_i \setminus D_j$ and $w \in D_j \setminus D_i$, for $i \neq j$.) $\sharp$

(33.3) Persistence properties. We show that the full subcategory of varieties within the category of prevarieties is closed with respect to certain constructions:

Proposition. i) The product of two varieties is a variety again.

ii) Any subprevariety of a variety is a variety again. In particular, any irreducible quasi-affine, projective or quasi-projective variety is a variety.

Proof. i) Let V and W be varieties. Then both $\Delta_V \subseteq V \times V$ and $\Delta_W \subseteq W \times W$ are closed, thus so is $\Delta_{V \times W} = \Delta_V \times \Delta_W \subseteq (V \times V) \times (W \times W)$.

ii) Let V be a variety, and let $W \subseteq V$ be a subprevariety. Moreover, let U be a prevariety, and let $\varphi, \psi \colon U \to W$ be morphisms of prevarieties. Then, since the injective inclusion map $\iota_W^V \colon W \to V$ is a morphism, we conclude that $\ker(\varphi, \psi) = \ker(\varphi \iota_W^V, \psi \iota_W^V) \subseteq U$ is closed. $\sharp$

Proposition. Let U be a variety, and let $V, W \subseteq U$ be affine open subsets, having coordinate algebras $K[V]$ and $K[W]$, respectively. Then $V \cap W \subseteq U$ is affine open again, having coordinate algebra $K[V \cap W] = K[V] \cdot K[W]$.

Proof. We may assume that $V \neq \emptyset \neq W$. The product $V \times W \subseteq U \times U$ is an affine open subset. Since $\delta_U \colon U \to U \times U$ is a closed embedding, we infer that the morphism $\epsilon := \delta_U|_{V \cap W} \colon V \cap W \to V \times W$ induces an isomorphism onto its image $\{[x, x] \in U \times U; x \in V \cap W\} = \Delta_U \cap (V \times W)$, where $\Delta_U \cap (V \times W) \subseteq V \times W$ is closed. Noting that $V \cap W \subseteq U$ is irreducible, we conclude that $\epsilon \colon V \cap W \to V \times W$ is a closed embedding of varieties, thus $V \cap W$ is affine.

Finally, as comorphism we get the surjective homomorphism of K-algebras $\epsilon^* \colon K[V] \otimes_K K[W] \to K[V \cap W] \colon f \otimes g \mapsto f|_{V \cap W} \cdot g|_{V \cap W}$, where since $V \cap W$ is dense in V and W both restriction maps to $V \cap W$ are injective. $\sharp$
