Computational Representation Theory: Remarks on Condensation

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January 27, 2004

Contents

1 Schur functors

It turns out that the functorial language is the right setting to formulate and understand some of the most powerful techniques of computational representation theory, the condensation techniques. The exposition of Section 1 is derived from [27, Sect.6]. We begin in a fairly general setting, thereby correcting an impreciseness in [31]. For the necessary notions from category theory see [3, Ch.2.1] and [2, Ch.II.1].

(1.1) Notation. Let Θ be a principal ideal domain. Let A be a Θ-algebra, which is a finitely generated Θ-free Θ-module. Let mod-A be the abelian category of finitely generated right A-modules. Let mod_Θ-A be the full additive subcategory of mod-A consisting of its Θ-free objects. In particular, if Θ is a field we have $\text{mod}_\Theta\text{-}A = \text{mod-}A$.

(1.2) Lemma. Let $V \in \text{mod-}\Theta$. Then the following are equivalent:
a) We have $V \in \text{mod}_\Theta\text{-}\Theta$, i. e. V is a finitely generated Θ-free Θ-module.
b) V is a projectve Θ-module.
c) V is a torsion-free Θ-module.

Proof. Let V be Θ-free. Then each surjection $X \to V$, for all $X \in \text{mod-}\Theta$, splits, hence V is Θ-projective. Let V be Θ-projective. Then V is a direct summand of some $X \in \text{mod}_\Theta\text{-}\Theta$, hence V is Θ-torsion-free. Let V be Θ-torsion-free. As Θ is a principal ideal domain, V is Θ-free. $\sharp$

(1.3) Definition. See [18, Ch.I.17].
Let $V \in \text{mod}_\Theta\text{-}\Theta$ and let $U \leq V$ be a Θ-submodule. Then $U \leq V$ is called **Θ-pure** in V, if V/U is a Θ-free Θ-module.

(1.4) Lemma.
a) $U \leq V$ is Θ-pure if and only if U is a Θ-direct summand of V.
b) If $U, U' \leq V$ are Θ-pure, then $U \cap U' \leq V$ is Θ-pure as well.

Proof. a) Let U be Θ-pure. Then V/U is Θ-free, hence the natural surjection $V \to V/U$ splits, thus U has a Θ-complement in V. Let $V \cong U \oplus U'$. Hence $V \geq U' \cong V/U$ is Θ-torsion-free, thus by Lemma (1.2) U' is Θ-free.
b) We have $V/(U \cap U') \leq V/U \oplus V/U'$, thus $V/(U \cap U')$ is Θ-torsion-free, hence Θ-free. $\sharp$

(1.5) Definition. Let $V \in \text{mod}_\Theta\text{-}\Theta$ and $U \leq V$ be a Θ-submodule. Then the Θ-pure Θ-submodule

$$U^V := \bigcap \{X; U \leq X \leq V \text{ is } \Theta\text{-pure}\} \leq V$$

is called the **pure closure** of U in V.

(1.6) Proposition. Es ist

$$U^V = \sum \{X; U \leq X \leq V, X/U \text{ is } \Theta\text{-torsion}\} \leq V.$$

Proof. Let $U \leq X \leq V$ such that X/U is Θ-torsion, and let $v \in X \setminus U^V$. Then there is $\vartheta \in \Theta$ such that $\vartheta v \in U \leq U^V$, hence V/U^V is not Θ-torsion-free, a contradiction. Hence $X \leq U^V$.

Let $v \in U^V \setminus U$ such that $(v+U)\Theta \leq U^V/U$ is not Θ-torsion, hence $(v+U)\Theta$ is Θ-free. Thus there is $U \leq U' < U^V$ such that $V/U' \cong V/U^V \oplus \Theta$, which is Θ-free, a contradiction. ♯

(1.7) Definition and Remark. Let $\mathbf{C}$ be an additive category, let $V, W \in \mathbf{C}$ and $\alpha \colon V \to W$ be a $\mathbf{C}$-morphism.
a) An object $K \in \mathbf{C}$ together with a monomorphism $\ker\alpha \colon K \to V$ is called a **(categorical) kernel** of α, if $(\ker\alpha)\alpha = 0$ and if for all morphisms $\xi \colon X \to V$ fulfilling $\xi\alpha = 0$ there is a morphism $\xi' \colon X \to K$ such that $\xi = \xi'(\ker\alpha)$.
b) An object $C \in \mathbf{C}$ together with an epimorphism $\operatorname{cok}\alpha \colon W \to C$ is called a **(categorical) cokernel** of α, if $\alpha(\operatorname{cok}\alpha) = 0$ and if for all morphisms $\xi \colon W \to X$ fulfilling $\alpha\xi = 0$ there is a morphism $\xi' \colon W \to C$ such that $\xi = (\operatorname{cok}\alpha)\xi'$.
c) Kernel and cokernel are uniquely determined up to isomorphism.
d) The morphism $\operatorname{im}\alpha := \ker(\operatorname{cok}\alpha)$ is called the **(categorical) image** of α. The morphism $\operatorname{coim}\alpha := \operatorname{cok}(\ker\alpha)$ is called the **(categorical) coimage** of α.
e) The morphism α induces a morphism $\hat{\alpha} \colon \operatorname{coim}\alpha \to \operatorname{im}\alpha$.
f) The category $\mathbf{C}$ is called **exact** if $\hat{\alpha}$ is an isomorphism for all $\alpha \colon V \to W$.
g) Let $\beta \colon W \to U$. The sequence $V \xrightarrow{\alpha} W \xrightarrow{\beta} U$ is called **exact**, if $\operatorname{im}\alpha \cong \ker\beta$.

(1.8) Proposition. Let $V, W \in \text{mod}_\Theta\text{-}A$ and $\alpha \in \operatorname{Hom}_A(V, W)$.
a) Then $\ker\alpha$ and $\operatorname{cok}\alpha$ exist in $\text{mod}_\Theta\text{-}A$.
b) The map $\hat{\alpha}$ induced by α is an isomorphism if and only if $V\alpha \leq W$ is a Θ-pure submodule. In particular, if Θ is not a field then $\text{mod}_\Theta\text{-}A$ fails to be an exact category.

Proof. The set theoretic kernel $K \in \text{mod-}A$ of α is a Θ-free module, and hence together with its natural embedding into V, it is a categorical kernel of α.

As $(V\alpha)^W \leq W$ is Θ-pure, we have $W/(V\alpha)^W \in \text{mod}_\Theta\text{-}A$. Let $\beta \colon W \to W/(V\alpha)^W$ denote the natural surjection. Let $X \in \text{mod}_\Theta\text{-}A$ and $\gamma \colon W \to X$ such that $\alpha\gamma = 0$. Then, by Proposition (1.6), for $w \in (V\alpha)^W$ there is $\vartheta \in \Theta$ such that $\vartheta w \in V\alpha$, hence we have $\vartheta w \cdot \gamma = 0$, and since X is Θ-free we conclude $w\gamma = 0$. Hence γ factors through β, and $W/(V\alpha)^W$ together with β is a categorical cokernel of α.

As $\ker\alpha \leq V$ is a Θ-pure submodule, we have $\operatorname{cok}(\ker\alpha) \cong V/\ker\alpha$, and as $(V\alpha)^W \leq W$ is a Θ-pure submodule, we have $\ker(\operatorname{cok}\alpha) \cong (V\alpha)^W$, while for the natural map $\hat{\alpha} \colon V/\ker\alpha \to (V\alpha)^W$ we have $(V/\ker\alpha)\hat{\alpha} = V\alpha$. ♯

We introduce the objects of interest in Section 1.

(1.9) Definition. See [11, Ch.6.2].
a) Let $e \in A$ be an idempotent. Then the additive exact functor

$$C_e \colon \text{mod-}A \to \text{mod-}eAe \colon V \mapsto Ve,$$

mapping $\alpha \in \mathrm{Hom}_A(V,W)$ to its restriction $\alpha|_{Ve} \in \mathrm{Hom}_{eAe}(Ve,We)$ to Ve, is called the **Schur functor** or **condensation functor** with respect to e. For $V \in \text{mod-}A$ the eAe-module $C_e(V) = Ve \in \text{mod-}eAe$ is called the **condensed module** of V.
b) The **uncondensation functor** with respect to e is the additive functor

$$U_e := ? \otimes_{eAe} eA \colon \text{mod-}eAe \to \text{mod-}A.$$

For $W \in \text{mod-}eAe$, the A-module $W \otimes_{eAe} eA \in \text{mod-}A$ is called the **uncondensed module** of W.

(1.10) Remark.
a) There is an equivalence $\sigma_e \colon C_e \to ? \otimes_A Ae$ of functors from mod-A to mod-eAe, given by $\sigma_e(V) \colon Ve \to V \otimes_A Ae \colon ve \mapsto v \otimes e$.

Furthermore, there is an equivalence $\tau_e \colon \mathrm{Hom}_A(eA, ?) \to ? \otimes_A Ae$ of functors from mod-A to mod-eAe, given by $\tau_e(V) \colon \mathrm{Hom}_A(eA, V) \to Ve \colon \alpha \mapsto e\alpha$, with inverse given by $\tau_e^{-1}(V) \colon Ve \to \mathrm{Hom}_A(eA, V) \colon v \mapsto (ea \mapsto v \cdot a)$.

The functor $C_e \circ U_e \colon \text{mod-}eAe \to \text{mod-}eAe$ is equivalent to the identity functor on mod-eAe, using the equivalence given by $V \otimes_{eAe} eA \cdot e \to V \colon v \otimes ea \cdot e \mapsto veae$.

b) The exactness of the Schur functor $C_e \colon \text{mod-}A \to \text{mod-}eAe$ follows from the fact that C_e is equivalent to both a covariant Hom-functor, which hence by [45, Prop.1.6.8] is left exact, and to a tensor functor, which hence by [45, Appl.2.6.2] is right exact.

In general the uncondensation functor $U_e \colon \text{mod-}eAe \to \text{mod-}A$ is not exact, see Example (1.25) and Remark (1.14).

(1.11) Proposition.
a) C_e induces an additive functor $\text{mod}_\Theta\text{-}A \to \text{mod}_\Theta\text{-}eAe$.

b) Let $V, W, U \in \text{mod}_\Theta\text{-}A$ and let $V \xrightarrow{\alpha} W \xrightarrow{\beta} U$ be an exact sequence in $\text{mod}_\Theta\text{-}A$, see Definition (1.7). Then $Ve \xrightarrow{\alpha|_{Ve}} We \xrightarrow{\beta|_{We}} Ue$ is an exact sequence in $\text{mod}_\Theta\text{-}eAe$.

Proof. If $V \in \text{mod-}A$ is Θ-free, then $Ve \in \text{mod-}eAe$ also is Θ-free.

Both $(V\alpha)^W \cdot e \leq (V\alpha)^W$ and $(V\alpha)^W \leq W$ are Θ-pure. Hence $(V\alpha)^W \cdot e \leq W$ is Θ-pure, thus this holds for $(V\alpha)^W \cdot e \leq We$ as well. Hence we have $(V\alpha \cdot e)^{We} \leq (V\alpha)^W \cdot e$. Furthermore, by Proposition (1.6), for $w \in (V\alpha)^W \cdot e = (V\alpha)^W \cap We$

there is $\theta \in \Theta$ such that $\theta w \in V\alpha \cap We = V\alpha \cdot e$. Hence we also have $(V\alpha)^W \cdot e \leq (V\alpha \cdot e)^{We}$, and thus $(V\alpha)^W \cdot e = (V\alpha \cdot e)^{We}$.

By the exactness of $V \xrightarrow{\alpha} W \xrightarrow{\beta} U$ we have $(V\alpha)^W = \operatorname{im}\alpha = \ker\beta$, see Proposition (1.8). Hence the exactness of $C_e\colon \text{mod-}A \to \text{mod-}eAe$ implies $\operatorname{im}(\alpha|_{Ve}) = (V\alpha \cdot e)^{We} = (V\alpha)^W \cdot e = (\ker\beta) \cdot e = \ker(\beta|_{We})$. ♯

The most important case, as far as computational applications are concerned, is where the base ring Θ is a field.

(1.12) Proposition. See [31, La.3.2].
Let Θ be a field and let $e \in A$ be an idempotent.
a) Let $S \in \text{mod-}A$ be simple. Then we have $Se \neq \{0\}$, if and only if S is a constituent of $eA/\operatorname{rad}(eA) \in \text{mod-}A$. If $Se \neq \{0\}$, then $Se \in \text{mod-}eAe$ is simple.
b) Let $S, S' \in \text{mod-}A$ be simple, such that $Se \neq \{0\}$. Then we have $S \cong S' \in \text{mod-}A$, if and only if $Se \cong S'e \in \text{mod-}eAe$.
c) Let $T \in \text{mod-}eAe$ be simple. Then there is a simple $S \in \text{mod-}A$, such that $T \cong Se \in \text{mod-}eAe$.

Proof. By Remark (1.10) we have $Se \cong \operatorname{Hom}_A(eA, S) \cong \operatorname{Hom}_A(eA/\operatorname{rad}(eA), S)$ as Θ-vector spaces. For $0 \neq v \in Se$, as $S \in \text{mod-}A$ is simple, we have $v \cdot eAe = vA \cdot e = Se$.

Let $Se \cong S'e \in \text{mod-}eAe$. Choose a decomposition of $e \in A$ as a sum of pairwise orthogonal primitive idempotents in A. We have $\operatorname{Hom}_A(eA, S) \cong Se \neq \{0\}$ as Θ-vector spaces, if and only if there is a summand $e_S \in eAe \subseteq A$ such that $e_S A$ is projective indecomposable with $e_S A/\operatorname{rad}(e_S A) \cong S \in \text{mod-}A$. Applying the functor $C_{e_S}\colon \text{mod-}eAe \to \text{mod-}e_S A e_S$, we obtain $Se_S \cong S'e_S \in \text{mod-}e_S A e_S$. Hence we have $\{0\} \neq S'e_S \cong \operatorname{Hom}_A(e_S A, S')$ as Θ-vector spaces, thus $S' \cong S \in \text{mod-}A$.

By Remark (1.10) we have $\{0\} \neq T \cong C_e \circ U_e(T) \cong T \in \text{mod-}eAe$, hence $U_e(T) \neq \{0\}$. Thus there is a simple $S \in \text{mod-}A$ such that $\operatorname{Hom}_A(U_e(T), S) \neq \{0\}$. By the Adjointness Theorem [9, Thm.0.2.19] we have as Θ-vector spaces

$$\operatorname{Hom}_A(T \otimes_{eAe} eA, S) \cong \operatorname{Hom}_{eAe}(T, \operatorname{Hom}_A(eA, S)) \cong \operatorname{Hom}_{eAe}(T, Se) \neq \{0\}.$$

Thus we conclude that $\{0\} \neq Se \in \text{mod-}eAe$ is simple, hence $Se \cong T \in \text{mod-}eAe$. ♯

(1.13) Definition. Let Θ be a field and let $e \in A$ be an idempotent.
a) Let $\Sigma_e \subseteq \text{mod-}A$ be a set of representatives of the isomorphism types of simple $S \in \text{mod-}A$ such that $Se \neq \{0\}$. In particular, Σ_1 is a set of representatives of the isomorphism types of all simple A-modules.
b) For a set $\Sigma \subseteq \text{mod-}A$ of representatives of some isomorphism types of simple A-modules let $\text{mod}_\Sigma\text{-}A$ be the full subcategory of mod-A consisting of all

A-modules all of whose constituents are isomorphic to an element of Σ. In particular, let mod_e-$A := \mathrm{mod}_{\Sigma_e}$-$A$. The natural embedding induces the fully faithful exact functor $I_e \colon \mathrm{mod}_e\text{-}A \to \mathrm{mod}\text{-}A$. Let

$$C_e^{\Sigma} := C_e \circ I_e \colon \mathrm{mod}_e\text{-}A \to \mathrm{mod}\text{-}eAe.$$

c) For $V \in \mathrm{mod}$-A let $\mathcal{P}(V) \xrightarrow{\rho} V$ denote its projective cover, and let $\Omega(V) := \ker \rho \in \mathrm{mod}$-$A$ be the **Heller module** of V. Let $\mathrm{mod}_{\Omega,e}$-A be the full subcategory of mod-A consisting of all A-modules V such that both $V/\mathrm{rad}(V) \in \mathrm{mod}_e$-$A$ and $\Omega(V)/\mathrm{rad}(\Omega(V)) \in \mathrm{mod}_e$-$A$. The natural embedding induces the fully faithful exact functor $I_{\Omega,e} \colon \mathrm{mod}_{\Omega,e}\text{-}A \to \mathrm{mod}\text{-}A$. Let

$$C_e^{\Omega} := C_e \circ I_{\Omega,e} \colon \mathrm{mod}_{\Omega,e}\text{-}A \to \mathrm{mod}\text{-}eAe.$$

(1.14) Remark. Let Θ be a field and let $e \in A$ be an idempotent.
a) By Proposition (1.12), the set $\{Se; S \in \Sigma_e\} \subseteq \mathrm{mod}$-$eAe$ is a set of representatives of the isomorphism types of all simple eAe-modules.

b) If $\Sigma_e = \Sigma_1$, then by Proposition (1.12) the projective A-module $eA \in \mathrm{mod}$-A is a progenerator of mod-A. Hence in this case, by Morita's Theorem [9, Thm.0.3.54], the functor C_e induces an equivalence between mod-A and mod-eAe. Thus in particular C_e is fully faithful and essentially surjective. The inverse functor is the uncondensation functor U_e, which hence in this case is exact.

Condensation functors inducing equivalences play a prominent role in the representation theory of algebras, see [2]. In practice, such condensation functors have been examined in the group algebra case in [22].

c) If $\Sigma_e = \Sigma_1$, then we have $\mathrm{Hom}_A(eA, fA/\mathrm{rad}(fA)) \neq \{0\}$ for all primitive idempotents $f \in A$, hence the projectivity of $Ae \in \mathrm{mod}$-eAe follows from the observation in d). Thus $eA \in eAe$-mod is projective as well, and hence this also shows that in this case the uncondensation functor U_e is exact.

d) Let $f \in A$ be a primitive idempotent such that $\mathrm{Hom}_A(eA, fA/\mathrm{rad}(fA)) \neq \{0\}$. Hence we may assume that $e = f + (e - f)$ is a decomposition of $e \in A$ as a sum of orthogonal idempotents. Thus $fAe \in \mathrm{mod}$-eAe is a direct summand of $eAe \in \mathrm{mod}$-eAe and hence projective. As $f \in eAe$ is primitive as well, $fAe \in \mathrm{mod}$-eAe is indecomposable.

Motivated by Example (1.25), this leads to the **Conjecture:** If $f \in A$ is a primitive idempotent such that $\mathrm{Hom}_A(eA, fA/\mathrm{rad}(fA)) = \{0\}$, then $fAe \in \mathrm{mod}$-eAe is not projective. Moreover, as actually $fAe \in \mathrm{mod}$-eAe might be decomposable, it is even projective-free.

We discuss properties of the functor C_e in the general case, where we do not assume that C_e induces an equivalence. Proposition (1.15) shows that C_e^{Σ} is a suitable functor to examine the submodule structure of A-modules. Proposition (1.16) and Example (1.25) show that C_e^{Σ} is fully faithful, but in general is not essentially surjective. Proposition (1.18) then shows how this failure to be an equivalence can be remedied by using the functor C_e^Ω.

(1.15) Proposition. Let Θ be a field, $e \in A$ be an idempotent and let $V \in \text{mod}_e\text{-}A$. Then C_e^Σ induces a lattice isomorphism between the submodule lattices of V and $C_e^\Sigma(V) \in \text{mod-}eAe$.

Proof. Clearly C_e^Σ preserves inclusion of submodules and commutes with forming sums and intersections of submodules. Hence C_e^Σ induces a lattice homomorphism from the submodule lattice of V to the submodule lattice of $C_e^\Sigma(V)$. Since $V \in \text{mod}_e\text{-}A$ this is an injection.

Let $\alpha \colon W \to Ve$ be an injection in mod-eAe. Applying C_e to $\text{Hom}_A(U_e(W), V)$ and using the equivalences of Remark (1.10) yields a Θ-linear map

$$(C_e)_{U_e(W),V} \colon \left\{ \begin{array}{rcl} \text{Hom}_A(W \otimes_{eAe} eA, V) & \to & \text{Hom}_{eAe}(W, \text{Hom}_A(eA, V)) \colon \\ \beta & \mapsto & \big(w \mapsto (ea \mapsto (w \otimes e)^\beta \cdot a)\big) \,. \end{array} \right.$$

This coincides with the adjointness Θ-homomorphism given by [9, Thm.0.2.19], and hence is a Θ-isomorphism. Let $\beta := (C_e)^{-1}_{U_e(W),V}(\alpha) \in \text{Hom}_A(U_e(W), V)$. Then we have $U_e(W)\beta \leq V$ and thus $C_e(U_e(W)\beta) = (C_e \circ U_e(W))\alpha = W\alpha$. $\sharp$

(1.16) Proposition. Let Θ be a field and let $e \in A$ be an idempotent. Then the functor $C_e^\Sigma \colon \text{mod}_e\text{-}A \to \text{mod-}eAe$ is fully faithful.

Proof. If $\Sigma_e = \Sigma_1$, then we have $\text{mod}_e\text{-}A = \text{mod-}A$, and by Remark (1.14) the functor $C_e^\Sigma = C_e \colon \text{mod-}A \to \text{mod-}eAe$ is an equivalence of categories, in particular C_e is fully faithful. Hence we may assume $\Sigma_e \neq \Sigma_1$. Let $e' \in A$ be an idempotent orthogonal to e, such that $Se' \neq \{0\}$ if and only if $S \in \text{mod-}A$ is simple isomorphic to an element of $\Sigma_1 \setminus \Sigma_e$, and let $f := e + e' \in A$. Hence $\Sigma_f = \Sigma_1$ and thus the functor $C_f \colon \text{mod-}A \to \text{mod-}fAf$ is an equivalence of categories, in particular C_f is fully faithful.

We have the Pierce decomposition $fAf = eAe \oplus eAe' \oplus e'Ae \oplus e'Ae'$ as a Θ-vector space. Hence, for $V \in \text{mod-}eAe$ and $v \in V$ and $a \in A$, let $v \cdot eae' = v \cdot e'ae = v \cdot e'ae := 0$. It is straightforward to check that this defines an fAf-module structure on V. Thus we obtain a functor $I_e^f \colon \text{mod-}eAe \to \text{mod-}fAf$. For $V, W \in \text{mod-}eAe$ we have $\text{Hom}_{fAf}(I_e^f(V), I_e^f(W)) = \text{Hom}_{eAe}(V, W)$, hence the functor I_e^f is fully faithful. By the choice of $e' \in A$ we furthermore conclude $I_e^f \circ C_e \circ I_e = C_f \circ I_e \colon \text{mod}_e\text{-}A \to \text{mod-}fAf$. As I_e and I_e^f as well as C_f are fully faithful, C_e also is fully faithful. $\sharp$

(1.17) Corollary. Let Θ be a field and let $e \in A$ be an idempotent.
a) For $V \in \text{mod}_e\text{-}A$ we have $\text{End}_A(V) \cong \text{End}_{eAe}(Ve)$.
b) In particular, if $S \in \text{mod}_e\text{-}A$ is simple, then S is absolutely simple if and only if $Se \in \text{mod-}eAe$ is.

(1.18) Proposition. See [2, Prop.II.2.5].
Let Θ be a field and $e \in A$ be an idempotent. Then the functor

$$C_e^{\Omega} \colon \text{mod}_{\Omega,e}\text{-}A \to \text{mod-}eAe$$

is an equivalence of categories, with inverse $U_e \colon \text{mod-}eAe \to \text{mod}_{\Omega,e}\text{-}A$.

Proof. Let $V \in \text{mod-}eAe$ and let $S \in \text{mod-}A$ be simple. By the Adjointness Theorem [9, Thm.0.2.19] we have $\text{Hom}_A(U_e(V), S) \cong \text{Hom}_{eAe}(V, \text{Hom}_A(eA, S))$ as Θ-vector spaces. As $\text{Hom}_A(eA, S) = \{0\}$ if $S \notin \Sigma_e$, we conclude that $U_e(V)/\text{rad}(U_e(V)) \in \text{mod}_e\text{-}A$.

If $P \in \text{mod-}eAe$ is projective, and hence is a direct summand of a free eAe-module, then $U_e(P) \cong P \otimes_{eAe} eA \in \text{mod-}A$ is projective as well. Let $P_1 \to P_0 \to V \to \{0\}$ be the beginning of a projective resolution of $V \in \text{mod-}eAe$. By the right exactness of the tensor functor $U_e = \, ? \otimes_{eAe} eA$, see [45, Appl.2.6.2], the sequence $U_e(P_1) \to U_e(P_0) \to U_e(V) \to \{0\}$ is the beginning of a projective resolution of $U_e(V) \in \text{mod-}A$. Hence we have $\text{Hom}_A(\Omega(U_e(V)), S) \leq \text{Hom}_A(U_e(P_1), S) \cong \text{Hom}_{eAe}(P_1, \text{Hom}_A(eA, S))$ as Θ-vectorspaces. Hence we also have $\Omega(U_e(V))/\text{rad}(\Omega(U_e(V))) \in \text{mod}_e\text{-}A$.

Thus U_e restricts to a functor $U_e \colon \text{mod-}eAe \to \text{mod}_{\Omega,e}\text{-}A$. By Remark (1.10) $C_e^{\Omega} \circ U_e$ is equivalent to the identity functor on mod-eAe. Conversely, for $V \in \text{mod}_{\Omega,e}\text{-}A$ we have $U_e \circ C_e(V) \cong \text{Hom}_A(eA, V) \otimes_{\text{End}_A(eA)^{\circ}} eA \in \text{mod}_{\Omega,e}\text{-}A$. Hence it is sufficient to show that the natural evaluation map

$$\nu \colon \text{Hom}_A(eA, V) \otimes_{\text{End}_A(eA)^{\circ}} eA \to V \colon \alpha \otimes ea \mapsto (ea)\alpha$$

is an isomorphism of A-modules.

Assume that ν is not surjective. Then there is $S \in \Sigma_e$ and $0 \neq \beta \in \text{Hom}_A(V, S)$ such that $\text{im}\,\nu \leq \ker\beta \leq V$. As β is surjective, $eA \in \text{mod-}A$ is projective and $\text{Hom}_A(eA, S) \neq \{0\}$, there is $\alpha \in \text{Hom}_A(eA, V)$ such that $\alpha\beta \neq 0$. Hence $\text{im}\,\alpha \not\leq \ker\beta \leq V$, which is a contradiction. Hence ν is surjective, and we thus have an exact sequence

$$\{0\} \to \ker\nu \to \text{Hom}_A(eA, V) \otimes_{\text{End}_A(eA)^{\circ}} eA \xrightarrow{\nu} V \to \{0\}$$

of A-modules. Since $C_e \circ U_e$ is equivalent to the identity functor on mod-eAe, applying C_e yields the exact sequence $\{0\} \to (\ker\nu)e \to Ve \xrightarrow{\text{id}} Ve \to \{0\}$ in mod-eAe. Hence we conclude $(\ker\nu)e = \{0\}$.

As ν is surjective, the projective cover $\mathcal{P}(V) \xrightarrow{\rho} V$ yields the existence of $\mu \in \text{Hom}_A(\mathcal{P}(V), \text{Hom}_A(eA, V) \otimes_{\text{End}_A(eA)^{\circ}} eA)$ such that $\mu\nu = \rho$. As $\Omega(V)\mu\nu = (\ker\rho)\mu\nu = \{0\}$, we conclude $\Omega(V)\mu \leq \ker\nu$. From $(\ker\nu)e = \{0\}$ and $\Omega(V)/\text{rad}(\Omega(V)) \in \text{mod}_e\text{-}A$ we conclude that $\Omega(V)\mu = \{0\}$. Hence there is $\overline{\mu} \in \text{Hom}_A(V, \text{Hom}_A(eA, V) \otimes_{\text{End}_A(eA)^{\circ}} eA)$ such that $\rho\overline{\mu} = \mu$. Thus we have $\rho\overline{\mu}\nu = \rho$. As ρ is surjective, we conclude $\overline{\mu}\nu = \text{id}_V$. Hence $\ker\nu$ is a direct summand of $\text{Hom}_A(eA, V) \otimes_{\text{End}_A(eA)^{\circ}} eA \in \text{mod}_{\Omega,e}\text{-}A$, and hence $\ker\nu/\text{rad}(\ker\nu) \in \text{mod}_e\text{-}A$. As $(\ker\nu)e = \{0\}$ we conclude $\ker\nu = \{0\}$, and thus ν is injective as well. $\sharp$

(1.19) Remark. Let $V \in \text{mod-}A$ and let $e \in A$ be an idempotent. The natural evaluation map $\nu\colon \mathrm{Hom}_A(eA, V) \otimes_{eAe} eA \to V$ used in the proof of Proposition (1.18) is the preimage of $\mathrm{id}_{\mathrm{Hom}_A(eA,V)}$ under the adjointness Θ-isomorphism, see [9, Thm.0.2.19],

$$\mathrm{Hom}_A(\mathrm{Hom}_A(eA, V) \otimes_{eAe} eA, V) \cong \mathrm{Hom}_{eAe}(\mathrm{Hom}_A(eA, V), \mathrm{Hom}_A(eA, V)).$$

This leads to the definition of relative uncondensation functors, which are of practical importance as a constructive tool, see [33, 44]

(1.20) Definition and Remark. Let $V \in \text{mod-}A$, let $e \in A$ be an idempotent, let $W \in \text{mod-}eAe$ and let $\alpha\colon W \to Ve$ be injective.
a) Then in mod-A we have

$$(\alpha \otimes \mathrm{id}) \cdot \nu\colon\; W \otimes_{eAe} eA \xrightarrow{\alpha\otimes\mathrm{id}} Ve \otimes_{eAe} eA \xrightarrow{\nu} V,$$

where $\nu\colon \mathrm{Hom}_A(eA, V)\otimes_{eAe} eA \to V$ is the natural evaluation map as in Remark (1.19). Then $\mathrm{im}\,((\alpha \otimes \mathrm{id}) \cdot \nu) \in \text{mod-}A$ is called the **uncondensed module of** W **with respect to** V.
b) As Ve can be considered as a Θ-subspace of V, using the injection α we obtain an injection $\hat{\alpha}\colon W \to V$ as Θ-vector spaces. Thus the uncondensed module $\mathrm{im}\,((\alpha \otimes \mathrm{id}) \cdot \nu) \leq V$ equals the A-submodule $W\hat{\alpha} \cdot A \leq V$ generated by $W\hat{\alpha} = \mathrm{im}\,(\hat{\alpha})$.

(1.21) We consider the relation of Schur functors and modular reduction.

Let K be an algebraic number field, and let $R \subset K$ be a discrete valuation ring in K with maximal ideal $\wp \lhd R$ and finite residue class field $F := R/\wp$ of characteristic $p > 0$. Let $\bar{\ }\colon R \to F$ denote the natural surjection. Hence for $V \in \mathrm{mod}_R\text{-}R$ we have a natural surjection $\bar{\ }\colon V \to V \otimes_R F$.

Let $A \in \mathrm{mod}_R\text{-}R$ be an R-algebra as in Notation (1.1), let $A_K := A \otimes_R K$ and $A_F := A \otimes_R F = \overline{A}$. For an idempotent $e \in A$ we have the Pierce decomposition $A = eAe \oplus (1-e)Ae \oplus eA(1-e) \oplus (1-e)A(1-e)$ in $\mathrm{mod}_R\text{-}R$. Hence we have $eAe \otimes_R K \cong eA_Ke$ as K-algebras, and $eAe \otimes_R F \cong \overline{e}A_F\overline{e}$ as F-algebras.

Let $V \xrightarrow{\alpha} W \xrightarrow{\beta} U$ be an exact sequence in $\mathrm{mod}_R\text{-}A$. Hence it follows from the Proof of Proposition (1.8) that the induced sequence $V \otimes_R K \xrightarrow{\alpha\otimes\mathrm{id}} W \otimes_R K \xrightarrow{\beta\otimes\mathrm{id}} U \otimes_R K$ is an exact sequence in mod-A_K. Note that this does not necessarily hold for the induced sequence $V \otimes_R F \xrightarrow{\alpha\otimes\mathrm{id}} W \otimes_R F \xrightarrow{\beta\otimes\mathrm{id}} U \otimes_R F$ in mod-A_F.

In the rest of Section 1 let A be as in Section (1.21).

(1.22) Definition. See [8, Ch.XII.82-83].
a) Let $S \in \text{mod-}A_K$ be simple, let $\hat{S} \in \mathrm{mod}_R\text{-}A$ such that $\hat{S} \otimes_R K \cong S \in \text{mod-}A_K$ and let $T \in \text{mod-}A_F$ be simple. Then the **decomposition number**

$d_{S,T} \in \mathbb{N}_0$ is defined as the multiplicity of the constituent T in a composition series of $\overline{\hat{S}} := \hat{S} \otimes_R F \in \text{mod-}A_F$.

Identifying the **Grothendieck groups** $G(A_K)$ and $G(A_F)$ with the free abelian groups generated by a set of representatives of the isomorphism types of the simple A_K-modules and A_F-modules, respectively, yields the **decomposition map** $D\colon G(A_K) \to G(A_F)$.

b) For $S \in \text{mod-}eA_Ke$ simple and $T \in \text{mod-}\overline{e}A_F\overline{e}$ simple we analogously define the **decomposition number** $d^e_{S,T} \in \mathbb{N}_0$. This defines the **decomposition map** $D^e\colon G(eA_Ke) \to G(\overline{e}A_F\overline{e})$.

(1.23) Proposition. Let $e \in A \subseteq A_K$ be an idempotent.
a) The additive functors $\text{Hom}_A(eA, ?) \otimes_R K$ and $\text{Hom}_{A_K}(eA_K, ? \otimes_R K)$ from $\text{mod}_R\text{-}A$ to $\text{mod-}eA_Ke$ are equivalent.
b) The additive functors $\text{Hom}_A(eA, ?) \otimes_R F$ and $\text{Hom}_{A_F}(\overline{e}A_F, ? \otimes_R F)$ from $\text{mod}_R\text{-}A$ to $\text{mod-}\overline{e}A_F\overline{e}$ are equivalent.

Proof. As $A \in \text{mod}_R\text{-}R$, this also holds for $eA \leq A$. For $V \in \text{mod}_R\text{-}A$ hence $\text{Hom}_A(eA, V) \leq \text{Hom}_R(eA, V) \in \text{mod}_R\text{-}R$. ♯

(1.24) Proposition. Let $e \in A \subseteq A_K$ be an idempotent. Let $S \in \text{mod-}A_K$ be simple and let $T \in \text{mod-}A_F$ be simple, such that $\{0\} \neq T\overline{e} \in \text{mod-}\overline{e}A_F\overline{e}$. Then we have

$$d_{S,T} = d^e_{Se,T\overline{e}}.$$

In particular, if $Se = \{0\}$ then we have $d_{ST} = 0$.

Proof. Let $\hat{S} \in \text{mod}_R\text{-}A$ such that $\hat{S} \otimes_R K \cong S \in \text{mod-}A_K$. By Proposition (1.23), for $\hat{S}e \in \text{mod}_R\text{-}eAe$ we hence have $\hat{S}e \otimes_R K \cong Se \in \text{mod-}eA_Ke$. Thus the decomposition number $d^e_{Se,T\overline{e}} \in \mathbb{N}_0$ is the multiplicity of the constituent $T\overline{e}$ in a composition series of $\overline{\hat{S}e} \in \text{mod-}\overline{e}A_F\overline{e}$. By Proposition (1.23) again we have $\overline{\hat{S}e} \cong \overline{\hat{S}}\overline{e} \in \text{mod-}\overline{e}A_F\overline{e}$. As $C_{\overline{e}}\colon \text{mod-}A_F \to \text{mod-}\overline{e}A_F\overline{e}$ is an exact functor, by Proposition (1.12) we conclude that the multiplicity of the constituent $T\overline{e}$ in a composition series of $\overline{\hat{S}}\overline{e}$ equals the multiplicity of the constituent T in a composition series of $\overline{\hat{S}} \in \text{mod-}A_F$. ♯

We conclude Section 1 by an example showing that in general $C^\Sigma_e\colon \text{mod}_e\text{-}A \to \text{mod-}eAe$ is not essentially surjective and that in general $U_e\colon \text{mod-}eAe \to \text{mod-}A$ is not exact.

(1.25) Example. Let (K, R, F) be as in Section (1.21). Let $G := \mathcal{A}_5$ be the alternating group on 5 letters, and let $A := RG$, where we assume K to be a splitting field for A_K and F to be a splitting field for A_F. The ordinary characters and 2-modular Brauer characters of G can be found in [6] and [15], respectively. Hence the 2-modular decomposition matrix of G is as follows.

	$1a$	$2a$	$2b$	$4a$
$1a$	1	.	.	.
$3a$	1	1	.	.
$3b$	1	.	1	.
$5a$	1	1	1	.
$4a$	.	.	.	1

Let $H \leq G$ be a cyclic subgroup of order 5 and let $e := \frac{1}{|H|} \cdot \sum_{h\in H} h \in RH \subseteq A$. As $\overline{e}A_F \cong (F_H)^G \in \text{mod-}A_F$, we have $\text{Hom}_{A_F}(\overline{e}A_F, F) \neq \{0\}$, where F_G denotes the trivial A_F-module. Furthermore, $\overline{e}A_F \in \text{mod-}A_F$ is projective and we have $\dim_F(\overline{e}A_F) = 12$. As $\dim_F(\mathcal{P}(F_G)) = 12$, we conclude $\overline{e}A_F \cong \mathcal{P}(F_G) \in \text{mod-}A_F$. Hence $\overline{e} \in A_F$ is a primitive idempotent, and thus $\text{Hom}_{A_F}(\overline{e}A_F, S) = \{0\}$ for all $F_G \not\cong S \in \text{mod-}A$ simple. Hence we have $\Sigma_{\overline{e}} = \{F_G\}$ and $F_G\overline{e} \in \text{mod-}\overline{e}A_F\overline{e}$ is the only simple up to isomorphism.

Using the equivalences of Remark (1.10) and a straightforward calculation, we have $\text{End}_{A_F}(\overline{e}A_F) \cong \text{Hom}_{A_F}(\overline{e}A_F, \overline{e}A_F) \cong (\overline{e}A_F\overline{e})^{\circ}$ as F-algebras. As $\overline{e}A_F \cong \mathcal{P}(F_G)$ is a non-simple projective indecomposable module for the symmetric algebra A_F, we conclude that $\text{End}_{A_F}(\overline{e}A_F)$ is a local F-algebra containing non-zero nilpotent elements. Hence $\overline{e}A_F\overline{e}$ is not semisimple and in particular we have $\text{Ext}^1_{\overline{e}A_F\overline{e}}(F_G\overline{e}, F_G\overline{e}) \neq \{0\}$. As G is a perfect group, we have $\text{Ext}^1_{A_F}(F_G, F_G) = \{0\}$. Hence all modules in $\text{mod}_{\overline{e}}\text{-}A_F$ are semisimple. Thus $C^{\Sigma}_{\overline{e}}$ is not essentially surjective.

Furthermore, $e \in A$ is a primitive idempotent. Let $f \in A$ be a primitive idempotent, such that $\overline{f}A_F \cong \mathcal{P}(S_2) \in \text{mod-}A_F$, where $S_2 \in \text{mod-}A_F$ is simple of $\dim_F(S_2) = 2$. Hence using the projective indecomposable characters given above we have $\dim_F(\overline{f}A_F\overline{e}) = \dim_F \text{Hom}(\overline{e}A_F, \overline{f}A_F) = \text{rk}_R\text{Hom}(eA, fA) = \dim_K(eA_K, fA_K) = 2$ and $\dim_F(\overline{e}A_F\overline{e}) = \dim_K(eA_K, eA_K) = 4$. As $\overline{e}A_F\overline{e} \cong \text{End}_{A_F}(\overline{e}A_F)^{\circ}$ is a local F-algebra, $\overline{e}A_F\overline{e} \in \text{mod-}\overline{e}A_F\overline{e}$ is the only projective indecomposable. Thus $\overline{f}A_F\overline{e} \in \text{mod-}\overline{e}A_F\overline{e}$ is not projective.

By [17, p.262] the Artinian ring $\overline{e}A_F\overline{e}$ is a perfect ring, see [17, Def.11.6.1]. As by [17, Thm.10.4.4] projective modules are flat anyway, by [17, Cor.11.1.6] the flat $\overline{e}A_F\overline{e}$-modules are precisely the projective $\overline{e}A_F\overline{e}$-modules. Hence the uncondensation functor $U_{\overline{e}} = ?\otimes_{\overline{e}A_F\overline{e}}\overline{e}A_F$ is exact if and only if $\overline{e}A_F \in \text{mod-}\overline{e}A_F\overline{e}$ is projective. But $\overline{e}A_F\overline{f}$ is a non-projective $\overline{e}A_F\overline{e}$-direct summand of $\overline{e}A_F$.

Finally, we note that for the local F-algebra $\overline{f}A_F\overline{f} \cong \text{End}_{A_F}(\overline{f}A_F)^{\circ}$ we have $\dim_F(\overline{f}A_F\overline{f}) = \dim_K(fA_K, fA_K) = 2$. Hence the only projective indecomposable $\overline{f}A_F\overline{f} \in \text{mod-}\overline{f}A_F\overline{f}$ is uniserial of composition length 2. We have $\dim_F(\overline{e}A_F\overline{f}) = \dim_K(fA_K, eA_K) = 2$, but a straightforward calculation using the submodule lattice programs [23] available in the MeatAxe [38] shows that $\overline{e}A_F\overline{f} \in \text{mod-}\overline{f}A_F\overline{f}$ is semisimple, hence is decomposable, not projective and thus projective-free.

2 Primitive idempotents

Schur functors with respect to primitive idempotents have been applied to various computational tasks, such as the determination of submodule lattices, see [23], and of socle and radical series, see [25], and the computation of homomorphism spaces, endomorphism rings and direct sum decompositions, see [42, 24].

As is common understanding it is very difficult to find primitive idempotents in a given algebra explicitly. But for our purposes this is in fact not necessary, as it is sufficient to know the action of the idempotent on the module under consideration. Hence we define certain projections, which subsequently are shown to describe the action of suitable, and in particular primitive, idempotents. The primitive idempotents for a set of isomorphism types of simple modules thus produced are not necessarily pairwise orthogonal, and hence their importance is more practical than theoretical.

We keep the notation of Notation (1.1), and let Θ be a field.

(2.1) Definition and Remark.
a) For $V \in \text{mod-}A$ let $D_V \colon A \to \text{End}_\Theta(V) \colon a \mapsto a_V$ be the corresponding representation. In particular, let $D_A \colon a \mapsto a_A$ denote the **regular representation**. The module $V \in \text{mod-}A$ is called **faithful**, if we have $\ker(D_V) = \{0\}$. In particular, the regular module $A_A \in \text{mod-}A$ is faithful.

b) For $a \in A$ let $\langle a \rangle \subseteq A$ denote the Θ-subalgebra generated by a. There is $n = n(a_V) \in \mathbb{N}$ such that

$$\{0\} \leq \ker(a_V) < \ker(a_V^2) < \ldots < \ker(a_V^n) = \ker(a_V^{n+1}) \leq V.$$

This gives rise to the **Fitting decomposition** $V = \ker(a_V^n) \oplus \text{im}\,(a_V^n) \in \text{mod-}\langle a \rangle$ and the **Fitting projection** $\varphi_{a_V} \in \text{End}_{\langle a \rangle}(V)$, where

$$(\varphi_{a_V})|_{\ker(a_V^n)} = \text{id}_{\ker(a_V^n)} \quad \text{and} \quad \ker(\varphi_{a_V}) = \text{im}\,(a_V^n).$$

(2.2) Proposition. Let $a \in A$.
a) Let $V \in \text{mod-}A$ be faithful. Then there is a unique $e \in A$ such that $e_V = \varphi_{a_V}$. Furthermore, we have $e^2 = e$ and e is independent from the choice of V.
b) Let $S \in \text{mod-}A$ be simple and let $S = \ker(a_S^{\tilde{n}}) \oplus \text{im}\,(a_S^{\tilde{n}}) \in \text{mod-}\langle a \rangle$ be the Fitting decomposition as in Definition (2.1), where $\tilde{n} = n(a_S) \in \mathbb{N}$. Then we have $Se = \ker(a_S^{\tilde{n}})$.

Proof. a) Let $n = n(a_V) \in \mathbb{N}$ and let $\mu', \mu'' \in \Theta[X]$ denote the minimum polynomials of a_V on $\ker(a_V^n) \leq V$ and $\text{im}\,(a_V^n) \leq V$, respectively. Then we have $\mu' = X^n$, and as $(a_V)|_{\text{im}\,(a_V^n)}$ is invertible, we have $\gcd(\mu', \mu'') = 1 \in \Theta[X]$. Thus there are $f, g \in \Theta[X]$ such that $1 = f \cdot \mu' + g \cdot \mu'' \in \Theta[X]$. Now let $e := g(a) \cdot \mu''(a) \in \langle a \rangle \subseteq A$. Hence we have $(e_V)|_{\text{im}\,(a_V^n)} = 0$ and $(e_V)|_{\ker(a_V^n)} = (\text{id} - f(a) \cdot \mu'(a))|_{\ker(a_V^n)} = \text{id}|_{\ker(a_V^n)}$. Thus we have $e_V = \varphi_{a_V}$. The other assertions follow from the faithfulness of V.

b) Let $\tilde{\mu}', \tilde{\mu}'' \in \Theta[X]$ denote the minimum polynomials of a_S on $\ker(a_S^{\tilde{n}}) \leq S$ and $\mathrm{im}\,(a_S^{\tilde{n}}) \leq S$, respectively. Since S is a constituent of the regular module $A_A \in \text{mod-}A$, we have $\tilde{\mu}' \mid \mu' \in \Theta[X]$ and $\tilde{\mu}'' \mid \mu'' \in \Theta[X]$. Hence we have $1 = f \cdot \frac{\mu'}{\tilde{\mu}'} \cdot \tilde{\mu}' + g \cdot \frac{\mu''}{\tilde{\mu}''} \cdot \tilde{\mu}'' \in \Theta[X]$. Thus as above we have $(e_S)|_{\mathrm{im}\,(a_S^{\tilde{n}})} = 0$ and $(e_S)|_{\ker(a_S^{\tilde{n}})} = \mathrm{id}|_{\ker(a_S^{\tilde{n}})}$, hence $\ker(a_S^{\tilde{n}}) = Se$. $\sharp$

(2.3) Remark. Let $S \in \text{mod-}A$ be simple and let $a \in A$. Then we have $\ker(a_S) \in \text{mod-End}_A(S)$. As $\mathrm{End}_A(S)$ is a skew field having Θ in its center, we have $\dim_\Theta \ker(a_S) = [\mathrm{End}_A(S) \colon \Theta] \cdot \dim_{\mathrm{End}_A(S)}(\ker(a_S))$, hence in particular we have $[\mathrm{End}_A(S) \colon \Theta] \mid \dim_\Theta \ker(a_S)$. Thus we have $\dim_{\mathrm{End}_A(S)}(\ker(a_S)) = 1$ if and only if $[\mathrm{End}_A(S) \colon \Theta] = \dim_\Theta \ker(a_S)$, thus if and only if the unit group $\mathrm{End}_A(S)^*$ acts transitively on $\ker(a_S) \setminus \{0\}$.

In practice, this yields a method to determine $\mathrm{End}_A(S)$: Choose a few elements $a_j \in A$ and calculate $d := \gcd(\dim_\Theta \ker((a_j)_S); j \geq 1)$. Let $a \in A$ such that $\dim_\Theta \ker(a_S) = d$, and let $\{e_i \in S; i \in \{1, \ldots, d\}\}$ be a Θ-basis of $\ker(a_S)$. For $i \geq 2$ check whether there is an A-endomorphism of S mapping $e_1 \mapsto e_i$; this is done using the MeatAxe standard basis algorithm, see [35]. If this holds true, then $d = [\mathrm{End}_A(S) \colon \Theta]$ and we have found a Θ-basis of $\mathrm{End}_A(S)$. Otherwise, we start from the beginning and check a few more elements $a_j \in A$.

(2.4) Definition and Remark. Let $S \in \text{mod-}A$ be simple and let $a \in A$.
a) If $\dim_\Theta \ker(a_S^2) = [\mathrm{End}_A(S) \colon \Theta]$, then we have $\dim_\Theta \ker(a_S) = [\mathrm{End}_A(S) \colon \Theta]$ as well, and by Definition (2.1) we have $\{0\} \neq \ker(a_S) = \ker(a_S^2)$, hence $n(a_S) = 1$ and the Fitting decomposition of S is $S = \ker(a_S) \oplus \mathrm{im}\,(a_S)$.
b) Let $\Sigma \subseteq \text{mod-}A$ be a set of representatives of some isomorphism types of simple A-modules and let $S \in \Sigma$. An element $a \in A$ is called an **S-peakword with respect to Σ**, if

$$\ker(a_T) = \{0\} \text{ for all } S \not\cong T \in \Sigma \quad \text{and} \quad \dim_\Theta \ker(a_S^2) = [\mathrm{End}_A(S) \colon \Theta].$$

An S-peakword with respect to $\Sigma = \Sigma_1$, see Definition (1.13), is called an **S-peakword**.

(2.5) Proposition. Let $S \in \text{mod-}A$ be simple. Then there exists an S-peakword.

Proof. As $\mathrm{rad}(A) \subseteq \ker(D_T)$ for all $T \in \text{mod-}A$ simple, by going over to $A/\mathrm{rad}(A)$ we may assume that A is semisimple. Hence for $d_T := \frac{\dim_\Theta(T)}{[\mathrm{End}_A(T) \colon \Theta]} \in \mathbb{N}$ we have

$$A \cong \bigoplus_{T \in \Sigma_1} (\mathrm{End}_A(T)^\circ)^{d_T \times d_T}.$$

For $T \not\cong S$ let $a_T \in (\mathrm{End}_A(T)^\circ)^{d_T \times d_T}$ be invertible, hence we have $\ker(a_T) = \{0\}$. Moreover, S is a $\mathrm{End}_A(S)$-vector space of $\dim_{\mathrm{End}_A(S)}(S) = d_S$. Choose a decomposition $S = S' \oplus S''$ as $\mathrm{End}_A(S)$-vector spaces, where $\dim_{\mathrm{End}_A(S)}(S') =$

1. Let $a_S \in (\mathrm{End}_A(S)^\circ)^{d_S \times d_S}$ such that S' and S'' are a_S-invariant, $S' \cdot a_S = \{0\}$ and a_S acts invertibly on S''. Thus we have $\ker(a_S^2) = S'$ and $\dim_\Theta \ker(a_S^2) = [\mathrm{End}_A(S) \colon \Theta]$. Hence $a := a_S + \sum_{S \not\cong T \in \Sigma_1} a_T \in A$ is as desired. $\sharp$

(2.6) Remark. Let $\Theta := \mathbb{F}_q$ be a finite field and $T \in$ mod-A simple. Then we have $F := \mathrm{End}_A(T) \cong \mathbb{F}_{\tilde{q}}$, where $\tilde{q} := q^{[\mathrm{End}_A(T) \colon \mathbb{F}_q]}$. This allows to determine the proportion of peakwords amongst all the elements of A, where again we may assume that A is semisimple.

a) The proportion of invertible elements in $F^{d \times d}$, where $d := d_T$, is given as

$$\frac{|GL_d(\tilde{q})|}{|F^{d \times d}|} = \tilde{q}^{-\frac{d(d+1)}{2}} \cdot \prod_{i=1}^{d} (\tilde{q}^i - 1).$$

Hence we have $\lim_{\tilde{q} \to \infty} \frac{|GL_d(\tilde{q})|}{|F^{d \times d}|} = \tilde{q}^{-\frac{d(d+1)}{2}} \cdot \tilde{q}^{\frac{d(d+1)}{2}} = 1$.

b) The elements $a \in F^{d \times d}$ whose Fitting decomposition is $F^d = \ker(a) \oplus \mathrm{im}\,(a)$ as F-vector spaces, where $\dim_F \ker(a) = 1$, are determined by a 1-dimensional F-subspace of F^d, which is annihilated by a, and a complement, on which a acts invertibly. Hence the proportion of these elements in $F^{d \times d}$ is given as

$$(*) = \tilde{q}^{-d^2} \cdot \frac{\tilde{q}^d - 1}{\tilde{q} - 1} \cdot \prod_{i=1}^{d-1} (\tilde{q}^d - \tilde{q}^i) = \tilde{q}^{-\frac{d(d+1)}{2}} \cdot \prod_{i=2}^{d} (\tilde{q}^i - 1).$$

Hence we have $\lim_{\tilde{q} \to \infty} ((*) \cdot \tilde{q}) = \tilde{q}^{-\frac{d(d+1)}{2}} \cdot \tilde{q}^{-\frac{(d-1)(d+2)}{2}} \cdot \tilde{q} = 1$.

(2.7) Theorem. Let $a \in A$ be an S-peakword with respect to Σ and let $e \in A$ be the corresponding idempotent, see Proposition (2.2). Then $e \in A$ is an idempotent such that $eA/\mathrm{rad}(eA) \cong S \oplus M \in$ mod-A, where $M \in \mathrm{mod}_{\Sigma_1 \setminus \Sigma}$-$A$.

Proof. For $T \in$ mod-A simple, by Proposition (2.2) we have $Te = \ker(a_T^{n(a_T)})$. Hence for $S \not\cong T \in \Sigma$ by Definition (2.4) we have $\mathrm{Hom}_A(eA, T) \cong Te = \{0\}$ as Θ-vector spaces, while $\mathrm{Hom}_A(eA/\mathrm{rad}(eA), S) \cong \mathrm{Hom}_A(eA, S) \cong Se = \ker(a_S^{n(a_S)}) = \ker(a_S)$ as Θ-vector spaces, and since furthermore $\dim_\Theta \ker(a_S) = [\mathrm{End}_A(S) \colon \Theta]$ the constituent S occurs in $eA/\mathrm{rad}(eA)$ with multiplicity 1. $\sharp$

(2.8) Corollary. If $\Sigma = \Sigma_1$, hence $a \in A$ is an S-peakword, then $e \in A$ is a primitive idempotent such that $eA/\mathrm{rad}(eA) \cong S$.

We show how peakwords and primitive idempotents can be used to compute submodule lattices. For the necessary notions from lattice theory see [1].

(2.9) Notation. Let $V \in$ mod-A and let $S \in$ mod-A be simple.

Let $\mathcal{M}(V)$ denote the set of A-submodules of V. It becomes a modular lattice, see Definition (2.12), of finite length by the partial ordering $\leq$ given by set theoretic inclusion.

Let $\mathcal{M}_S(V) := \{W \leq V; W/\mathrm{rad}(W) \cong S \oplus \cdots \oplus S\} \subseteq \mathcal{M}(V)$. Hence $\mathcal{M}_S(V)$ is closed under taking sums, and hence becomes a lattice by letting the intersection of $W, W' \in \mathcal{M}_S(V)$ be the largest element of $\mathcal{M}_S(V)$ contained in $W \cap W' \in \mathcal{M}(V)$.

Let $\mathcal{L}_S(V) := \{W \leq V; W/\mathrm{rad}(W) \cong S\} \subseteq \mathcal{M}_S(V)$ be the set of **S-mountains** (**S-local submodules**), and let $\mathcal{L}(V) := \coprod_{S \in \Sigma_1} \mathcal{L}_S$. Hence $\mathcal{L}(V) \subseteq \mathcal{M}(V)$ is the subset of all sum-irreducible elements of $\mathcal{M}(V)$, i. e. the set of all A-submodules of V which are not the sum of strictly smaller A-submodules.

Let $\mathcal{L}_{S \oplus S}(V) := \{W \leq V; W/\mathrm{rad}(W) \cong S \oplus S\} \subseteq \mathcal{M}_S(V)$.

(2.10) Theorem. Let $V \in \text{mod-}A$, and let $e \in A$ be a primitive idempotent such that $eA/\mathrm{rad}(eA) \cong S$.

a) The map $\kappa \colon \mathcal{M}_S(V) \to \mathcal{M}(Ve) \colon W \mapsto We$ is an isomorphism of lattices. Its inverse is given as $\kappa^{-1} \colon \mathcal{M}(Ve) \to \mathcal{M}_S(V) \colon W \mapsto W \cdot A$, where we consider $W \leq Ve \leq V$ as Θ-vector spaces and $W \cdot A \leq V$ is the uncondensed module with respect to V, see Definition (1.20).

b) We have $\mathcal{L}(Ve) = \{v \cdot eAe \leq Ve; 0 \neq v \in Ve\}$ as well as

$$\mathcal{L}_S(V) = \{v \cdot A \leq V; 0 \neq v \in Ve \leq V\}.$$

Proof. a) Let $W \leq Ve \in \text{mod-}eAe$. By Definition (1.20), the A-submodule $W \cdot A \leq V$ is an epimorphic image of the uncondensed module $W \otimes_{eAe} eA \in \text{mod-}A$. By the Adjointness Theorem [9, Thm.0.2.19], for $S \not\cong T \in \text{mod-}A$ simple we conclude $\mathrm{Hom}_A(W \otimes_{eAe} eA, T) \cong \mathrm{Hom}_{eAe}(W, Te) = \{0\}$ as Θ-vector spaces, thus κ^{-1} is well-defined. As $We = W \leq Ve \in \text{mod-}eAe$, we have $(W \cdot A) \cdot e = W \cdot eAe = W$. Thus $\kappa^{-1} \cdot \kappa = \mathrm{id}_{\mathcal{M}(Ve)}$, in particular κ is surjective. Moreover, for $W, W' \in \mathcal{M}_S(V)$, $W \neq W'$, either $(W + W')/W \in \text{mod-}A$ or $(W + W')/W' \in \text{mod-}A$ has S as a constituent, hence by the exactness of the condensation functor C_e, see Definition (1.9), the map κ is injective as well.

b) For $0 \neq v \in Ve$ let $W := v \cdot eAe \leq Ve \in \text{mod-}eAe$. Hence W is an epimorphic image of the regular module $eAe \in \text{mod-}eAe$. As $e \in A$ is primitive, the algebra eAe is a local ring, having $Se \in \text{mod-}eAe$ as its only simple module up to isomorphism. Thus we have $eAe/\mathrm{rad}(eAe) \cong Se \in \text{mod-}eAe$. Conversely, each element of $\mathcal{L}(Ve)$ has a singleton generating set.

We have $v \cdot A = W \cdot A \leq V \in \text{mod-}A$. By a) we have $W \cdot A \in \mathcal{M}_S(V)$, and by Definition (1.20) and the right exactness of the tensor functor $U_e = ? \otimes_{eAe} eA$, the A-module $W \cdot A \leq V$ is an epimorphic image of $eAe \otimes_{eAe} eA \cong eA \in \text{mod-}A$. Since $eA/\mathrm{rad}(eA) \cong S$, we have $W \cdot A \in \mathcal{L}_S(V)$. Conversely, by the exactness of the condensation functor C_e, for $W \in \mathcal{L}_S(V)$ we have $\mathrm{rad}(W) \cap We = \mathrm{rad}(W)e < We$, and hence for $v \in We \setminus \mathrm{rad}(W)e \subseteq Ve$ we have $v \cdot A = W$. $\sharp$

Note that in the first part of the above proof we cannot use the general statement from Remark (1.10), saying that $C_e \circ U_e$ is equivalent to the identity functor on mod-eAe, since we do not consider the uncondensation functor U_e but a truncated relative version.

(2.11) Corollary. Let $a \in A$ be an S-peakword with respect to Σ, let $V \in \mathrm{mod}_\Sigma$-$A$ and let $n := n(a_V) \in \mathbb{N}$ be as in Definition (2.1). Then by Proposition (2.2) we have $\mathcal{L}_S(V) = \{v \cdot A \leq V; 0 \neq v \in \ker(a_V^n)\}$.

The Benson-Conway Theorem (2.13), which is of purely combinatorial nature, shows how to rebuild a modular lattice $\mathcal{M}$ from the incidence structure of the subset $\mathcal{L} \subseteq \mathcal{M}$ of its sum-irreducible elements. The notions used here are taken from [4].

(2.12) Definition and Remark. Let $\mathcal{M}$ be a modular lattice of finite length, where a lattice $\mathcal{M}$ is called **modular**, if for all $X, Y, Z \in \mathcal{M}$ such that $Z \leq X$ we have $X \cap (Y + Z) = (X \cap Y) + (X \cap Z) = (X \cap Y) + Z$. Let $\mathcal{L} \subseteq \mathcal{M}$ be the subset of the sum-irreducible elements of $\mathcal{M}$.

a) A subset $\mathcal{D} \subseteq \mathcal{L}$ is called a **dotted-line**, if $|\mathcal{D}| \geq 3$ and if it is maximal subject to the following property: For all $X, Y \in \mathcal{D}$, $X \neq Y$, we have $X + Y = \sum \mathcal{D}$.

Note that if $X, Y \in \mathcal{D}$, $X \neq Y$, where $\mathcal{D}$ is a dotted-line, then $X \not\leq Y$. To see this, assume that $X \leq Y$ holds, and let $Z \in \mathcal{D}$, $X \neq Z \neq Y$. Then we have $Y = X + Y = X + Z \in \mathcal{L}$, a contradiction.

b) A subset $\mathcal{X} \subseteq \mathcal{L}$ is called a **BC-closed**, if it has the following properties:
i) If $X \in \mathcal{X}$ and $Y \in \mathcal{L}$ such that $Y \leq X$, then $Y \in \mathcal{X}$; i. e. $\mathcal{X}$ is an **ideal** of the partially ordered set $\mathcal{L}$.
ii) If $\mathcal{D} \subseteq \mathcal{L}$ is a dotted-line such that $|\mathcal{D} \cap \mathcal{X}| \geq 2$, then $\mathcal{D} \subseteq \mathcal{X}$.

c) Let $\mathcal{M}(\mathcal{L}) \subseteq \mathrm{Pot}(\mathcal{L})$ be the partially ordered set of all BC-closed subsets of $\mathcal{L}$, where the partial order is given by set theoretic inclusion on $\mathrm{Pot}(\mathcal{L})$. Hence $\mathcal{M}(\mathcal{L})$ is closed under taking intersections, and hence becomes a lattice by letting the sum of $\mathcal{X}, \mathcal{X}' \in \mathcal{M}(\mathcal{L})$ be the smallest element of $\mathcal{M}(\mathcal{L})$ containing $\mathcal{X} + \mathcal{X}' \in \mathrm{Pot}(\mathcal{L})$.

(2.13) Theorem: Benson-Conway.
We keep notation from Definition (2.12). The map

$$\tau \colon \mathcal{M} \to \mathcal{M}(\mathcal{L}) \colon X \mapsto \{Y \in \mathcal{L}; Y \leq X\}$$

is an isomorphism of lattices, with inverse $\tau^{-1} \colon \mathcal{M}(\mathcal{L}) \to \mathcal{M} \colon \mathcal{X} \mapsto \sum \mathcal{X}$.

Proof. See [4]. ♯

For the case of submodule lattices $\mathcal{M}(V)$ the isomorphism types of the simple subquotients provide additional structure on the set of BC-closed subsets and for the dotted-lines. Actually, the statements of Proposition (2.14) Theorem (2.17) can be generalized to arbitrary modular lattices, see [30].

(2.14) Proposition. We keep the notation from Notation (2.9).
a) Let $\mathcal{D} \subseteq \mathcal{L}(V)$ be a dotted-line. Then there is $S \in \text{mod-}A$ simple such that $\mathcal{D} \subseteq \mathcal{L}_S(V)$ and $W := \sum \mathcal{D} \in \mathcal{L}_{S \oplus S}(V)$. Moreover, there is a bijection

$$\delta \colon \mathcal{D} \to \{Y \lessdot W\} \colon X \mapsto X + \operatorname{rad}(W),$$

where $Y \lessdot W$ means that $Y < W$ is a maximal A-submodule.
b) Conversely, let $W \in \mathcal{L}_{S \oplus S}(V)$. Then there is a dotted-line $\mathcal{D} \subseteq \mathcal{L}_S(V)$ such that $\sum \mathcal{D} = W$.

Proof. a) Let $X', X'' \in \mathcal{D}$, $X' \neq X''$. Since by Definition (2.12) we have $X' \not\leq X''$ and $X'' \not\leq X'$, we conclude that $X' \cap X'' \leq \operatorname{rad}(X') \cap \operatorname{rad}(X'')$ and hence $(X' + X'')/\operatorname{rad}(X' + X'') \cong X'/\operatorname{rad}(X') \oplus X''/\operatorname{rad}(X'') \cong S' \oplus S''$, for $S', S'' \in \text{mod-}A$ simple. As $|\mathcal{D}| \geq 3$ we conclude that $S' \cong S''$.

For $X \in \mathcal{D}$ we have $X \not\leq \operatorname{rad}(W)$, thus $(X + \operatorname{rad}(W))/\operatorname{rad}(W) \cong S$. As $W/\operatorname{rad}(W) \cong S \oplus S$, we have $X + \operatorname{rad}(W) \lessdot W$, and thus δ is well-defined and injective. Assume, δ is not surjective, and let $W' \lessdot W$ such that $W' \notin \operatorname{im}(\delta)$. Then there is $Y \in \mathcal{L}(V)$ such that $W' = Y + \operatorname{rad}(W)$. Hence for all $X \in \mathcal{D}$ we have $W = X + Y + \operatorname{rad}(W) = X + Y$, contradicting the maximality of $\mathcal{D}$.

b) For each $W' \lessdot W$ choose $X_{W'} \in \mathcal{L}(V)$ such that $X_{W'} \leq W'$ and $X_{W'} \not\leq \operatorname{rad}(W)$. Hence $W' = X_{W'} + \operatorname{rad}(W)$. Thus for $W', W'' \lessdot W$, $W' \neq W''$ we have $W = X_{W'} + X_{W''} + \operatorname{rad}(W) = X_{W'} + X_{W''}$. Let $\mathcal{D} := \{X_{W'} \in \mathcal{L}(V); W' \lessdot W\}$. As $\mathcal{D}$ can be enlarged to a dotted-line, and by a) the map δ is injective, $\mathcal{D}$ already fulfills the maximality condition for dotted-lines. ♯

(2.15) Corollary. Let $\Theta := \mathbb{F}_q$ be a finite field and $S \in \text{mod-}A$ simple. Then we have $\operatorname{End}_A(S) \cong \mathbb{F}_{\tilde{q}}$, where $\tilde{q} := q^{[\operatorname{End}_A(S) \colon \mathbb{F}_q]}$. Let $\mathcal{D} \subseteq \mathcal{L}_S(V)$ be a dotted-line. Then we have $|\mathcal{D}| = \tilde{q} + 1$.

Proof. Let $W := \sum \mathcal{D} \in \mathcal{L}_{S \oplus S}(V)$, thus $W/\operatorname{rad}(W) \cong S \oplus S$. We have to show that $S \oplus S$ has precisely $\tilde{q} + 1$ submodules $T \lessdot S$, which hence are isomorphic to S. Consider the natural projections $\pi_i \colon S \oplus S \to S$, for $i \in \{1, 2\}$. If both $(\pi_1)|_T \neq 0 \neq (\pi_2)|_T$, then $\pi := \pi_1^{-1} \cdot \pi_2 \in \operatorname{End}_A(S)^*$, and hence $T = \{(v, v\pi) \in S \oplus S; v \in S\}$. Conversely, all subsets of $S \oplus S$ of this form are A-submodules, yielding exactly $\tilde{q} - 1$ diagonal A-submodules, next to $S \oplus \{0\}$ and $\{0\} \oplus S$. ♯

(2.16) Corollary: To Theorem (2.13).
Let $\mathcal{M}(\mathcal{L}_S(V)) \subseteq \operatorname{Pot}(\mathcal{L}_S(V))$ be the set of all BC-closed subsets of $\mathcal{L}_S(V)$. As in Definition (2.12) the set $\mathcal{M}(\mathcal{L}_S(V))$ becomes a lattice, and by Proposition (2.14) the dotted-lines in $\mathcal{L}_S(V)$ are precisely the dotted-lines in $\mathcal{L}(V)$ which are subsets of $\mathcal{L}_S(V)$.

Then the map $\tau \colon \mathcal{M}_S(V) \to \mathcal{M}(\mathcal{L}_S(V)) \colon X \mapsto \{Y \in \mathcal{L}_S(V); Y \leq X\}$ is an isomorphism of lattices with inverse $\tau^{-1} \colon \mathcal{M}(\mathcal{L}_S(V)) \to \mathcal{M}_S(V) \colon \mathcal{X} \mapsto \sum \mathcal{X}$.

(2.17) Theorem. We keep the notation from Notation (2.9).
For each $S \in \text{mod-}A$ simple and each $W \in \mathcal{L}_{S\oplus S}(V)$ choose a dotted-line $\mathcal{D}_W \subseteq \mathcal{L}_S(V)$, and let $\Delta := \{\mathcal{D}_W; S \in \ldots, W \in \ldots\} \subseteq \mathrm{Pot}(\mathcal{L}(V))$. Then an ideal $\mathcal{X} \subseteq \mathcal{L}(V)$ is BC-closed, see Definition (2.12), if and only if it has the following property: If $\mathcal{D} \in \Delta$ such that $|\mathcal{D} \cap \mathcal{X}| \geq 2$, then we have $\mathcal{D} \subseteq \mathcal{X}$.

Proof. See [23]. $\sharp$

The next aim is the computation of socle and radical series, see [25, 42]. To calculate the socle series it is sufficient to find a Θ-basis of the socle, pass to the quotient module and to proceed iteratively. Moreover, to calculate the socle it is sufficient to find successively its isotypic components. The computation of the radical and the radical, uses duality.

(2.18) Remark. We keep the notation from Corollary (2.11).

a) Let $\mathrm{soc}_S(V) := \sum\{T \leq V; T \cong S\} \leq V$ denote the S-isotypic component of the socle $\mathrm{soc}(V) \leq V$. As for $S \cong T \leq \mathrm{soc}_S(V) \leq V$ we have $\ker(a_V) \cap T \neq \{0\}$, we by Corollary (2.11) conclude that

$$\begin{aligned}\mathrm{soc}_S(V) &= \sum\{v \cdot A \leq V; 0 \neq v \in \ker(a_V) \cap \mathrm{soc}_S(V)\} \\ &= \sum\{v \cdot A \leq V; 0 \neq v \in \ker(a_V), \dim_\Theta(v \cdot A) = \dim_\Theta(S)\}.\end{aligned}$$

Hence if $\Theta = \mathbb{F}_q$ is a finite field, $\mathrm{soc}_S(V)$ can be found by initialising $\{0\} := W \leq V$ to be the submodule of $\mathrm{soc}_S(V)$ already known, running through all $0 \neq v \in \ker(a_V)$, which are Θ-linearly independent of W, calculating $v \cdot A \leq V$ using the MeatAxe spinning algorithm, see [35], which is interrupted as soon as $\dim_\Theta(S)$ is exceeded, and if $\dim_\Theta(v \cdot A) = \dim_\Theta(S)$ adding the newly found simple summand to W.

Of course this might mean unsuccessfully trying quite a lot of vectors, and actually we can do better than that. The basic idea is as follows:

b) Let $0 \neq s \in \ker(a_S) \leq S$ be fixed. Hence there is a Θ-linear map

$$\sigma\colon \mathrm{Hom}_A(S, V) = \mathrm{Hom}_A(S, \mathrm{soc}_S(V)) \to \ker(a_V) \cap \mathrm{soc}_S(V) \leq \ker(a_V)\colon \varphi \mapsto s\varphi.$$

As S is simple, σ is injective. Moreover, letting $m := m_S(\mathrm{soc}_S(V)) \in \mathbb{N}$ denote the multiplicity of S in a decomposition of $\mathrm{soc}_S(V)$ into irreducible direct A-summands, we since $\dim_\Theta \ker(a_S) = [\mathrm{End}_A(S)\colon \Theta]$, see Definition (2.4), have $\dim_\Theta(\mathrm{Hom}_A(S, V)) = [\mathrm{End}_A(S)\colon \Theta] \cdot m = \dim_\Theta(\ker(a_V) \cap \mathrm{soc}_S(V))$, and hence σ is an isomorphism. Moreover, we have $\mathrm{soc}_S(V) = \langle \ker(a_V) \cap \mathrm{soc}_S(V)\rangle_A \leq V$, and if $\mathrm{End}_A(S) = \Theta$ and $\{w_1, \ldots, w_m\} \subseteq V$ is a Θ-basis of $\ker(a_V) \cap \mathrm{soc}_S(V)$, then even $\mathrm{soc}_S(V) = \bigoplus_{k=1}^m (w_k \cdot A) \leq V$.

Let $d := \dim_\Theta(S) \in \mathbb{N}$ and let $\{a_1, \ldots, a_d\} \subseteq A$ such that $\mathcal{B}_s := \{sa_1, \ldots, sa_d\} \subseteq S$ is the MeatAxe standard basis of S with respect to $s \in S$, see [35]. Since $\dim_\Theta \ker(a_S) = [\mathrm{End}_A(S)\colon \Theta]$ the representing matrices $D_{\mathcal{B}_s}(a) \in \Theta^{d \times d}$ with respect to the Θ-basis $\mathcal{B}_s \subseteq S$, for $a \in A$, do not depend on the particular choice of $0 \neq s \in \ker(a_S)$, see Remark (2.3).

For $v \in \ker(a_V)$ we have $v \in \operatorname{im}(\sigma)$ if and only if $va_i \cdot a = \sum_{j=1}^d D_{\mathcal{B}_s}(a)_{ij} \cdot va_j$, for all $i \in \{1, \ldots, d\}$ and $a \in A$, where $D_{\mathcal{B}_s}(a)_{ij} \in \Theta$ denotes the corresponding matrix entry of $D_{\mathcal{B}_s}(a)$. Let $l := \dim_\Theta(\ker(a_V))$ and $\mathcal{B} := \{v_1, \ldots, v_l\} \subseteq V$ be a Θ-basis of $\ker(a_V)$ and $v = \sum_{k=1}^l \lambda_k v_k$, for $\lambda_k \in \Theta$. Hence the above condition translates into

$$\sum_{k=1}^{l} \lambda_k \cdot \left(v_k a_i \cdot a - \sum_{j=1}^{d} D_{\mathcal{B}_s}(a)_{ij} \cdot v_k a_j \right) = 0,$$

for all $i \in \{1, \ldots, d\}$ and $a \in \mathcal{A}$, where $\mathcal{A} \subseteq A$ is a Θ-algebra generating set of A. As the bracketed term is a fixed element of V, for all $i \in \{1, \ldots, d\}$ and $a \in \mathcal{A}$, this can be considered as a set of homogeneous Θ-linear equations for the unknowns λ_k. Let $\Lambda \leq \Theta^{1 \times l}$ denote its solution space. Hence $\Lambda \to \ker(a_V) \cap \operatorname{soc}_S(V) \colon [\lambda_1, \ldots, \lambda_l] \mapsto \sum_{k=1}^l \lambda_k v_k$ is a Θ-isomorphism.

c) Let $V^* := \operatorname{Hom}_\Theta(V, \Theta) \in A\text{-mod} \cong \text{mod-}A^\circ$ denote the dual module of V, see [8, Ch.IX.60], where for $a \in A = A^\circ$ and a pair $\mathcal{B} \subseteq V$ and $\mathcal{B}^* \subseteq V^*$ of mutually dual Θ-bases we have $D_{\mathcal{B}^*}(a) = D_{\mathcal{B}}(a)^{\mathrm{tr}}$.

For $U \leq V$ as Θ-vector spaces let $U^\perp := \{f \in V^*; Uf = \{0\}\} \leq V^*$ as Θ-vector spaces. Hence if $U \in \text{mod-}A$ then $U^\perp \in \text{mod-}A^\circ$, and moreover we have $\operatorname{rad}(V)^\perp = \operatorname{soc}(V^*)$ and $\operatorname{soc}(V)^\perp = \operatorname{rad}(V^*)$. Thus $\operatorname{rad}(V) = \operatorname{rad}(V)^{\perp\perp} = \operatorname{soc}(V^*)^\perp$. Hence the calculation of $\operatorname{rad}(V) \in \text{mod-}A$ is reduced to the calculation of $\operatorname{soc}(V^*) \in \text{mod-}A^\circ$, and a Θ-basis $\mathcal{B} \subseteq V$ reflects the radical series of $V \in \text{mod-}A$ if and only if the corresponding dual Θ-basis $\mathcal{B}^* \subseteq V^*$ reflects the socle series of $V^* \in \text{mod-}A^\circ$.

In Remark (2.21) we describe a technique, generalizing the approach from Remark (2.18), to compute homomorphism spaces, see [42, 24]. It uses the notion of a finite module presentation, see Definition (2.19). This leads to a different field of constructive computational techniques, using finitely presented objects; for further details see [41] for the case of group actions, and [19, 28] for arbitrary finitely presented modules for finitely presented algebras.

(2.19) Definition. Let $V \in \text{mod-}A$, let $F = \langle f_1, \ldots, f_r \rangle_A \in \text{mod-}A$ be free of rank $r \in \mathbb{N}_0$ such that there is an epimorphism $\Phi \colon F \to V$, and let $\ker(\Phi) = \langle g_1, \ldots, g_s \rangle_A \leq F$, for $s \in \mathbb{N}_0$. Hence we have the description $V \cong F/\ker(\Phi) = \overline{F} = \langle f_1, \ldots, f_r | g_1, \ldots, g_s \rangle_A \in \text{mod-}A$ as a **finitely presented A-module**. Let $\overline{} \colon F \to \overline{F}$ denote the natural epimorphism.

(2.20) Proposition. We keep the notation of Definition (2.19), and let A be as a Θ-algebra be generated by $\mathcal{A} \subseteq A$ finite.

a) Let $\mathcal{B} := \{v_1, \ldots, v_d\}$ be a Θ-basis of V, for $d := \dim_\Theta(V) \in \mathbb{N}_0$. Then we have a finite A-module presentation $\overline{F} \to V \colon \overline{f_i} \mapsto v_i$, where

$$\overline{F} := \langle f_1, \ldots, f_d | f_i \cdot a - \sum_{j=1}^{d} D_{\mathcal{B}}(a)_{ij} \cdot f_j, i \in \{1, \ldots, d\}, a \in \mathcal{A} \rangle_A.$$

b) Let $r \leq d$ such that $\langle v_1, \ldots, v_r \rangle_A = V$, let $a_{ik} \in A$ such that $v_i = \sum_{k=1}^{r} v_k a_{ik} \in V$, for $i \in \{1, \ldots, d\}$. Then we have a finite A-module presentation $\overline{F} \to V \colon \overline{f_i} \mapsto v_i$, where

$$\overline{F} := \langle f_1, \ldots, f_r | \sum_{k=1}^{r} f_k a_{ik} \cdot a - \sum_{j=1}^{d} \sum_{k=1}^{r} D_{\mathcal{B}}(a)_{ij} \cdot f_k a_{jk}, i \in \{1, \ldots, r\}, a \in \mathcal{A} \rangle_A.$$

Proof. a) It is easily seen that $\dim_\Theta(\overline{F}) \leq d$. As $\overline{F} \to V \colon \overline{f_i} \mapsto v_i$ is well-defined, it hence is an isomorphism. **b)** follows from a). $\sharp$

(2.21) Remark.
a) Let $\Phi \colon F \to V$ be as in Definition (2.19), and let $W \in \text{mod-}A$. Then we have a Θ-embedding $\Phi^* \colon \mathrm{Hom}_A(V, W) \to \mathrm{Hom}_A(F, W) \colon \varphi \to \Phi\varphi$, where $\mathrm{im}\,(\Phi^*) = \{\Psi \in \mathrm{Hom}_A(F, W); \Psi|_{\ker(\Phi)} = 0\}$.

Let $\mathcal{C} := \{w_1, \ldots, w_e\}$ be a Θ-basis of W, for $e = \dim_\Theta(W) \in \mathbb{N}_0$, and for $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, e\}$ let $\Psi_{ij} \in \mathrm{Hom}_A(F, W)$ defined by $f_i \Psi_{ij} := w_j$ and $f_k \Psi_{ij} := 0$ for $k \neq i$. Hence for $\Psi = \sum_{i=1}^{r} \sum_{j=1}^{e} \alpha_{ij} \Psi_{ij} \in \mathrm{Hom}_A(F, W)$, for $\alpha_{ij} \in \Theta$, we have $\Psi \in \mathrm{im}\,(\Phi^*)$ if and only if $g\Psi = 0$ for all $g = \sum_{i=1}^{r} f_i a_i(g) \in \ker(\Phi)$, where $a_i(g) \in A$. Thus $\Psi \in \mathrm{im}\,(\Phi^*)$ if and only if for all $k \in \{1, \ldots, e\}$ and $g \in \mathcal{G}$, where $\langle \mathcal{G} \rangle_A = \ker(\Phi)$, we have

$$\sum_{i=1}^{r} \sum_{j=1}^{e} \alpha_{ij} D_{\mathcal{C}}(a_i(g))_{jk} = 0.$$

Similar to Remark (2.18)b), this can be considered as a set of homogeneous Θ-linear equations for the unknowns α_{ij}.

b) To lessen the number of unknowns in the above set of equations, we proceed as follows: We keep the notation of Corollary (2.11), let $W \in \mathrm{mod}_\Sigma\text{-}A$, and for $S \in \Sigma$ and an S-peakword $a \in A$ let $\mathcal{B}_S \subseteq \ker(a_V^n)$ such that $\overline{\mathcal{B}_S}$ is a Θ-basis of $\overline{\ker(a_V^n)} = (\ker(a_V^n) + \mathrm{rad}(V))/\mathrm{rad}(V)$, where $\overline{} \colon V \to V/\mathrm{rad}(V)$ denotes the natural Θ-epimorphism. Note that $\mathrm{rad}(V) \leq V$ can be determined using Remark (2.18)c).

By Corollary (2.11) we have $\langle \overline{\mathcal{B}_S} \rangle_A = \mathrm{soc}_S(\overline{V})$. Letting $\mathcal{B} := \coprod_{S \in \Sigma} \mathcal{B}_S \subseteq V$, we have $\langle \mathcal{B}, \mathrm{rad}(V) \rangle_A = V$, and hence $\langle \mathcal{B} \rangle_A = V$. Note that $\mathcal{B}$ is Θ-linearly independent, and thus from $\mathcal{B}$ we find a Θ-basis of V as in Proposition (2.20)b).

For $\varphi \in \mathrm{Hom}_A(V, W)$ we have $\mathcal{B}_S \cdot \varphi \subseteq \ker(a_W^{n(a_V)})$. Hence let $\mathcal{C}_S \subseteq W$ be a Θ-basis of $\ker(a_W^{n(a_V)})$, and thus we conclude that $\mathrm{im}\,(\Phi^*)$ is contained in the following Θ-subspace of $\mathrm{Hom}_A(F, W)$

$$\mathrm{im}\,(\Phi^*) \leq \{\Psi \in \mathrm{Hom}_A(F, W); f_i\Psi \in \mathcal{C}_S \text{ if } f_i\Phi \in \mathcal{B}_S, i \in \{1, \ldots, |\mathcal{B}|\}\}.$$

For more details, the computation of endomorphism rings and direct sum decompositions of modules see [42], where also applications to the explicit determination of Green correspondents are given.

3 Fixed point condensation

Fixed point condensation is one of the workhorses of computational representation theory. Fixed point condensation has been applied to different types of modules for group algebras over finite fields. Historically, the first application [43] has been to permutation modules. Applications to tensor product modules have been worked out in [46, 26, 32] and arbitrary induced modules have been dealt with in [33]. Great improvements for the permutation module case have been made by the invention of the direct condense technique [37, 21, 31, 27].

Many of the appplications come from the Modular Atlas project, see [15, 47, 48], hence from the problem of explicitly calculating decomposition numbers for the almost quasi-simple groups given in [6], see e. g. [16, 7, 12, 33, 29, 31]. Furthermore, there have been applications to the determination of Green correspondents, see [42], and to endomorphism rings of permutation modules and to algebraic graph theory, see [14, 20, 13, 27, 5].

We keep the notation of Notation (1.1), let (K, R, F) be as in Section (1.21), and let $\Theta \in \{K, R, F\}$.

(3.1) Definition and Remark.
a) Let G be a finite group and let $A := RG$. Moreover let $H \leq G$ such that $p = \mathrm{char}(F) \not| \; |H|$, and let

$$e := e_H := \frac{1}{|H|} \cdot \sum_{h \in H} h \in RG \subseteq KG$$

denote the centrally primitive idempotent of RH belonging to the trivial representation.

b) We have $eA \cong R_H^G \in \mathrm{mod}_R\text{-}A$, where the latter is isomorphic to the permutation A-module on the right cosets of H in G.

By the Nakayama relations, see [3, Prop.3.3.1], for $V \in \mathrm{mod}_R\text{-}A$ we obtain $C_e(V) = Ve \cong \mathrm{Hom}_A(eA, V) \cong \mathrm{Hom}_A(R_H^G, V) \cong \mathrm{Hom}_{RH}(R_H, V_H) \cong \mathrm{Fix}_V(H) \in \mathrm{mod}_R\text{-}R$, where in fact we have $Ve = \mathrm{Fix}_V(H)$, which can also be easily directly checked.

Similarly, for $V \in \mathrm{mod}\text{-}KG$ we have $Ve = \mathrm{Fix}_V(H)$, and for $V \in \mathrm{mod}\text{-}FG$ we have $V\overline{e} = \mathrm{Fix}_V(H)$. Hence these condensation functors are called **fixed point condensation functors with respect to H.**

(3.2) Remark. For $V \in \mathrm{mod}_\Theta$-$\Theta G$ let $\chi_V \in \mathbb{Z}\mathrm{IBr}_\Theta(G)$ denote its R-valued Brauer character, see [9, Ch.2.17]. Here, for $\Theta = F$ the class function χ_V on G is defined by $\chi_V(g) := \chi_V(g_{p'})$, where $g = g_p \cdot g_{p'} \in G$ denote the p-part and the p'-part of $g \in G$, respectively.

Let $d_V := \langle (\chi_V)_H, 1_H \rangle_H = \langle \chi_V, 1_H^G \rangle_G \in \mathbb{N}_0$, where $\langle \cdot, \cdot \rangle$. denotes the hermitian product on the respective set of K-valued class functions, and $1_H \in \mathbb{Z}\mathrm{IBr}_\Theta(H)$ is the trivial character. As H is a p'-group, for $\Theta \in \{K, R\}$ we have $\mathrm{rk}_\Theta(Ve) = d_V$, and for $\Theta = F$ we have $\dim_F(V\overline{e}) = d_V$. Thus the Θ-rank of a fixed point condensed module can be determined from purely character theoretic information without actually applying the fixed point condensation functor.

(3.3) Proposition. Let $V \in \mathrm{mod}_\Theta$-$\Theta G$ and $g \in G$.
a) For $\Theta \in \{K, R\}$ we have $\mathrm{tr}_{Ve}(ege) = \frac{1}{|H|} \cdot \sum_{C \in \mathcal{C}l(G)} |C \cap Hg| \cdot \chi_V(C)$, where $\mathcal{C}l(G)$ denotes the set of conjugacy classes of G.
b) For $\Theta = F$ we have $\mathrm{tr}_{V\overline{e}}(\overline{e}g\overline{e}) = \frac{1}{|H|} \cdot \sum_{C \in \mathcal{C}l(G)} |C \cap Hg| \cdot \overline{\chi_V(C)}$.

Proof. a) We have $\chi_V(ab) = \chi_V(ba)$, for $a, b \in G$. Thus for $\Theta \in \{K, R\}$ we have $\mathrm{tr}_{Ve}(ege) = \mathrm{tr}_V(ege) = \frac{1}{|H|^2} \cdot \sum_{h', h'' \in H} \chi_V(h'gh'') = \frac{|H|}{|H|^2} \cdot \sum_{h \in Hg} \chi_V(h)$.
b) is proved analogously. ♯

This has been applied to the calculation of decomposition numbers of algebraically conjugate ordinary characters, see [31, 34, 39]. Recalling that $eAe \cong \mathrm{End}_A(eA)^\circ$, where $eA \cong R_H^G \in \mathrm{mod}_R$-$A$, is the endomorphism ring of a permutation module, Proposition (3.3) generalizes to the characters of endomorphism rings of monomial representations, see [27]. A converse of Proposition (3.3) is given by Ree's formula, see [9, Thm.1.11.28].

We proceed to consider fixed point condensation of permutation modules, where Proposition (3.5) shows that the computations actually needed boil down to a counting problem, see Notation (3.4). Hence its implementation, as is available in the MeatAxe [38], is straightforward.

(3.4) Notation. Let Ω be a G-set and let $V := \Theta\Omega \in \mathrm{mod}_\Theta$-$\Theta G$ denote the corresponding ΘG-permutation module. Let $H \leq G$ be as in Definition (3.1), let $\Omega = \coprod_{i=1}^r \Omega_i$ be the partition of Ω into H-orbits, and for $i \in \mathcal{I} := \{1, \ldots, r\}$ let $\Omega_i^+ := \sum_{\omega \in \Omega_i} \omega \in \Theta\Omega$ denote the corresponding **orbit sum**. For $g \in G$ and $i, j \in \mathcal{I}$ the number

$$c_{ij}(g) := |\{\omega \in \Omega_i : \omega g \in \Omega_j\}| = |\Omega_i \cap \Omega_j^{g^{-1}}| = |\Omega_i^g \cap \Omega_j| \in \mathbb{N}_0$$

is called the corresponding **orbit counting number**. The matrix $C(g) := [c_{ij}(g); i, j \in \mathcal{I}] \in \mathbb{N}_0^{r \times r}$ is called the **orbit counting matrix**.

(3.5) Proposition. We keep the notation from Notation (3.4), and let $\Theta = F$. Then the set $\Omega^+ := \{\Omega_i^+; i \in \mathcal{I}\} \subseteq F\Omega$ is an F-basis of $V\overline{e}$, and the representing matrix of the action of $\overline{e}g\overline{e} \in eFGe$ on $V\overline{e}$ with respect to Ω^+ is given in terms of the orbit counting matrix as

$$C(g) \cdot \mathrm{diag}[|\Omega_j|^{-1}; j \in \mathcal{I}] \in F^{r \times r}.$$

Similar statements hold for $\Theta \in \{K, R\}$.

Proof. We have $F\Omega \ni v = \sum_{\omega \in \Omega} v_\omega \cdot \omega \in (F\Omega)\overline{e}$ if and only if the coefficients $v_\omega \in F$, for $\omega \in \Omega$, are constant on the G-orbits Ω_i, for all $i \in \mathcal{I}$. Moreover, we have $\Omega_i^+ \cdot \overline{e}g\overline{e} = \frac{1}{|H|} \cdot \sum_{j \in \mathcal{I}} |\{\omega \in \Omega_i; \omega g \in \Omega_j\}| \cdot \frac{|H|}{|\Omega_j|} \cdot \Omega_j^+$. ♯

Note that $C(g) \in F^{r \times r}$ is well-defined without imposing the condition $p \not| \; |H|$, see Definition (3.1), while $\mathrm{diag}[|\Omega_i|^{-1}; i \in \mathcal{I}] \in F^{r \times r}$ is well-defined if and only if $|\Omega_i| \neq 0 \in F$ for all $i \in \mathcal{I}$. **Question:** If this is the case, but $p \mid |H|$, is there an interpretation of the matrix in Proposition (3.5), and are there examples where this occurs? Moreover, let $p^d := \gcd\{\Omega_i; i \in \mathcal{I}\}_p \in \mathbb{N}$. **Question** [36]: Is there an interpretation of the matrices

$$[c_{ij}(g); i, j \in \mathcal{I}] \cdot \mathrm{diag}[\frac{p^d}{|\Omega_j|}; j \in \mathcal{I}] \in F^{r \times r} \quad ?$$

The next aim is to describe fixed point condensation of induced modules, see [33], which has been implemented for modules over finite fields in `GAP` [10]. Mackey's Theorem is used to describe the structure of a condensed induced module, which leads to a description of the necessary computations.

(3.6) Remark.
a) Let $U \leq G$ and let $H \leq G$ be as in Definition (3.1). Let $G = \coprod_{i \in \mathcal{I}} Ug_iH$ for suitable $g_i \in G$, and for $i \in \mathcal{I}$ let $H = \coprod_{j \in \mathcal{I}_i} (U^{g_i} \cap H)h_{ij}$ for suitable $h_{ij} \in H$.

Let $V \in \mathrm{mod}_\Theta\text{-}\Theta U$. By Mackey's Theorem, see [3, Thm.3.3.4], we have $V^G = V \otimes_{\Theta U} \Theta G = \bigoplus_{i \in \mathcal{I}} \bigoplus_{j \in \mathcal{I}_i} V \otimes g_ih_{ij} \in \mathrm{mod}_\Theta\text{-}\Theta$. Hence we have $(V^G)_H \cong \bigoplus_{i \in \mathcal{I}} ((V^{g_i})_{U^{g_i} \cap H})^H \in \mathrm{mod}_\Theta\text{-}\Theta H$, where $V^{g_i} \in \mathrm{mod}_\Theta\text{-}\Theta[U^{g_i}]$ is defined by $v \cdot u^{g_i} := v \cdot u$ for $v \in V$ and $u \in U$.

Thus we have $V^G \cdot e \cong \bigoplus_{i \in \mathcal{I}} \mathrm{Hom}_{\Theta H}(\Theta_H, ((V^{g_i})_{U^{g_i} \cap H})^H) \in \mathrm{mod}_\Theta\text{-}\Theta$. Moreover, another application of the Nakayama relations, see [3, Prop.3.3.1], shows $\mathrm{Hom}_{\Theta H}(\Theta_H, ((V^{g_i})_{U^{g_i} \cap H})^H) \cong \mathrm{Hom}_{\Theta[U^{g_i} \cap H]}(\Theta_{U^{g_i} \cap H}, (V^{g_i})_{U^{g_i} \cap H})$, where the Nakayama isomorphism is given by the exterior trace map, see [3, Exc.3.3], mapping $\varphi \in \mathrm{Hom}_{\Theta[U^{g_i} \cap H]}(\Theta_{U^{g_i} \cap H}, (V^{g_i})_{U^{g_i} \cap H})$ to

$$\left(\lambda \mapsto \sum_{j \in \mathcal{I}_i} (\lambda h_{ij}^{-1})\varphi \cdot h_{ij} = \lambda\varphi \cdot \sum_{j \in \mathcal{I}_i} h_{ij} \right) \in \mathrm{Hom}_{\Theta H}(\Theta_H, ((V^{g_i})_{U^{g_i} \cap H})^H).$$

As $\mathrm{Hom}_{\Theta[U^{g_i} \cap H]}(\Theta_{U^{g_i} \cap H}, (V^{g_i})_{U^{g_i} \cap H}) \cong \mathrm{Hom}_{\Theta[U \cap {}^{g_i}H]}(\Theta_{U \cap {}^{g_i}H}, V_{U \cap {}^{g_i}H})$, and letting $e_i := e_{U \cap {}^{g_i}H} \in \Theta U$, for $i \in \mathcal{I}$, denote the idempotent belonging to the

p'-subgroup $U \cap {}^{g_i}H \leq U \leq G$, see Definition (3.1), in mod_Θ-Θ we obtain

$$V^G \cdot e \cong \bigoplus_{i\in\mathcal{I}} \left(\mathrm{Fix}_V(U \cap {}^{g_i}H) \otimes g_i \sum_{j\in\mathcal{I}_i} h_{ij} \right) \cong \bigoplus_{i\in\mathcal{I}} \left(\mathrm{im}\,((e_i)_V) \otimes g_i \sum_{j\in\mathcal{I}_i} h_{ij} \right).$$

b) For $g \in G$ the action of $ege \in e\Theta Ge$ on $V^G \cdot e$ is described as follows. For $v \in V$ as well as $i \in \mathcal{I}$ and $j \in \mathcal{I}_i$ we have $(v \otimes g_i h_{ij}) \cdot g = vu' \otimes g_{i'} h_{i'j'}$, where the indices $i' \in \mathcal{I}$ and $j' \in \mathcal{I}_{i'}$ as well as $u' \in U$ are uniquely determined by $g_i h_{ij} \cdot g = u' \cdot g_{i'} h_{i'j'}$.

Hence for $v \in \mathrm{Fix}_V(U \cap {}^{g_i}H)$ we from this obtain $(v \otimes g_i \sum_{j\in\mathcal{I}_i} h_{ij}) \cdot eg = \sum_{j\in\mathcal{I}_i} (v \otimes g_i h_{ij}) \cdot g = \sum_{j\in\mathcal{I}_i} vu' \otimes g_{i'} h_{i'j'}$, where the indices $i' \in \mathcal{I}_i$ and $j' \in \mathcal{I}_{i'}$ as well as $u' \in U$ depend on $i \in \mathcal{I}$ and $j \in \mathcal{I}_i$.

Moreover, for $i \in \mathcal{I}$ we have $e = \frac{|U^{g_i} \cap H|}{|H|} \cdot e_i^{g_i} \cdot \sum_{j\in\mathcal{I}_i} h_{ij} \in \Theta H$. Hence for $v \in V$ as well as $i \in \mathcal{I}$ and $j \in \mathcal{I}_i$ this gives $(v \otimes g_i h_{ij}) \cdot e = (v \otimes g_i) \cdot e = \frac{|U^{g_i} \cap H|}{|H|} \cdot ve_i \otimes g_i \sum_{j\in\mathcal{I}_i} h_{ij}$.

c) In practice, the permutation representation of G on the right cosets $\Omega = U|G$ is needed, which is handled by a randomized Schreier-Sims algorithm, see [40]. Hence the U-H double cosets $U|G|H$ are in bijection with the H-orbits $\Omega = \coprod_{i\in\mathcal{I}} \Omega_i$, which yields the set $\{g_i \in G; i \in \mathcal{I}\}$. Furthermore, we have $U^{g_i} \cap H = \mathrm{Stab}_H(Ug_i)$, which yields the transversal $\{h_{ij} \in H; j \in \mathcal{I}_i\}$ and a stabilizer chain for $U^{g_i} \cap H$.

To find $\mathrm{Fix}_V(U \cap {}^{g_i}H) = \mathrm{im}\,((e_i)_V)$, we factorize $e_i^{g_i} \in \Theta[U^{g_i} \cap H]$, as a product of sums over transversals, along the stabilizer chain for $U^{g_i} \cap H$. Let $V = Ve_i \oplus V(1 - e_i) \in \mathrm{mod}_\Theta$-$\Theta$, with corresponding projection $\pi_i \colon V \to Ve_i$ and injection $\iota_i \colon Ve_i \to V$. Matrices for π_i and ι_i are found a precomputation step, which is independent of the particular element $g \in G$ to be condensed.

To find the action of $ege \in e\Theta Ge$ on $V^G \cdot e$, we fix $i \in \mathcal{I}$ and for all $j \in \mathcal{I}_i$ we calculate the corresponding indices $i' \in \mathcal{I}$ and $j' \in \mathcal{I}_{i'}$ as well as the element $u' \in H$ describing the action of g on Ω using the stabilizer chains computed above. Thus u' maps $\mathrm{Fix}_V(U \cap {}^{g_i}H)$ to $\mathrm{im}\,(\iota_i) \cdot u' \leq V^G$, while the projection induced by e to the i'-th, say, component of $V^G \cdot e$ is given by $\pi_{i'}$.

We next turn to the description of fixed point condensation of tensor product modules, see [46, 26, 32], which has been implemented for modules over finite fields in the MeatAxe [38].

(3.7) Remark.
a) Let $H \leq G$ be as in Definition (3.1), let $\Theta \in \{K, F\}$ and let $\Sigma := \Sigma_1(\Theta H) \subseteq \mathrm{mod}$-$\Theta H$ be a set of representatives of the isomorphism types of all simple ΘH-modules, see Definition (1.13). For $S, T \in \Sigma$ we have $\mathrm{Hom}_{\Theta H}(\Theta_H, S \otimes_\Theta T) \cong \mathrm{Hom}_{\Theta H}(S^*, T) \neq \{0\}$ if and only if $T \cong S^* \in \mathrm{mod}$-$\Theta H$, where $S^* \in \mathrm{mod}$-ΘH

denotes the contragredient module of S, see also Remark (2.18). In this case, we have $(S \otimes_\Theta S^*) \cdot e = \mathrm{Fix}_{S \otimes_\Theta S^*}(H) \cong \mathrm{Hom}_{\Theta H}(\Theta_H, S \otimes_\Theta S^*) \cong \mathrm{End}_{\Theta H}(S)$.

Let $V, W \in \text{mod-}\Theta G$. As ΘH is a semisimple Θ-algebra, there are $m_S(V) \in \mathbb{N}_0$, for $S \in \Sigma$, such that $V_H \cong \bigoplus_{S \in \Sigma} \bigoplus_{i=1}^{m_S(V)} S \in \text{mod-}\Theta H$. Hence we have

$$(V \otimes_\Theta W) \cdot e \cong \bigoplus_{S \in \Sigma} \bigoplus_{i=1}^{m_S(V)} \bigoplus_{j=1}^{m_{S^*}(W)} (S \otimes_\Theta S^*) \cdot e \in \text{mod-}\Theta H.$$

b) A Θ-basis of V reflecting the semisimplicity of V_H and MeatAxe standard bases for the constituents S of V_H is found using the peakword technique, see Definition (2.4), which has been described for the computation of socle series, see Remark (2.18).

For $S \in \Sigma$ let $S \otimes_\Theta S^* = \mathrm{im}\,(e_{S \otimes_\Theta S^*}) \oplus \ker(e_{S \otimes_\Theta S^*}) \in \text{mod-}\Theta$ with corresponding projection $\pi_S \colon S \otimes_\Theta S^* \to (S \otimes_\Theta S^*) \cdot e$ and injection $\iota_S \colon (S \otimes_\Theta S^*) \cdot e \to S \otimes_\Theta S^*$. Matrices for π_S and ι_S are found a precomputation step, which is independent of the particular element $g \in G$ to be condensed.

The action of $g \in G$ on $V \otimes_\Theta W$, with respect to its product Θ-basis, is given by the Kronecker product $g_V \otimes g_W$, where $g_V \in \Theta^{\dim_\Theta(V) \times \dim_\Theta(V)}$ is the representing matrix of the action of g on V. In practice, calculating the image of $v \otimes w \in V \otimes_\Theta W$ under the action of $g \in G$ amounts to considering $v \otimes w$ as an element of $\Theta^{\dim_\Theta(V) \times \dim_\Theta(W)}$ and then to calculating $(g_V)^{\mathrm{tr}} \cdot (v \otimes w) \cdot g_W$.

For the very new developments concerning the direct condense technique for permutation modules and various applications to representation theory and algebraic graph theory see [37, 21, 31, 27].

4 References

[1] M. Aigner: Combinatorial theory, Classics in Mathematics, Springer, 1997.

[2] M. Auslander, I. Reiten, S. Smalø: Representation theory of Artin algebras, Cambridge Studies in Advanced Mathematics 36, Cambridge Univ. Press, 1995.

[3] D. Benson: Representations and cohomology I, Cambridge Studies in Advanced Mathematics 30, Cambridge Univ. Press, 1983.

[4] D. Benson, J. Conway: Diagrams for modular lattices, J. Pure Appl. Algebra 37 (2), 1985, 111–116.

[5] T. Breuer, J. Müller : The character tables of endomorphism rings of multiplicity-free permutation modules of the sporadic simple groups, their automorphism groups, and their cyclic central extension groups, http://www.math.rwth-aachen.de/~Juergen.Mueller/mferctbl/mferctbl.html.

[6] J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson: Atlas of finite groups, maximal subgroups and ordinary characters for simple groups, Oxford Univ. Press, 1985.

[7] G. Cooperman, G. Hiss, K. Lux, J. Müller: The Brauer tree of the principal 19-block of the sporadic simple Thompson group, Experiment. Math. 6 (4), 1997, 293–300.

[8] C. Curtis, I. Reiner: Representation theory of finite groups and associative algebras, Wiley, 1962.

[9] C. Curtis, I. Reiner: Methods of representation theory I, Wiley, 1981.

[10] The GAP Group: GAP-4.3 — Groups, Algorithms, and Programming, Aachen, St. Andrews, 2003, http://www-gap.dcs.st-and.ac.uk/gap/.

[11] J. Green: Polynomial representations of GL_n, Lecture Notes in Mathematics 830, Springer, 1980.

[12] A. Henke, G. Hiss, J. Müller: The 7-modular decomposition matrices of the sporadic O'Nan group, J. London Math. Soc. (2) 60, 1999, 58–70.

[13] I. Höhler: Vielfachheitsfreie Permutationsdarstellungen und die Invarianten zugehöriger Graphen, Examensarbeit, RWTH Aachen, 2001.

[14] A. Ivanov, S. Linton, K. Lux, J. Saxl, L. Soicher: Distance-transitive representations of the sporadic groups, Comm. Algebra 23, 1995, 3379–3427.

[15] C. Jansen, K. Lux, R. Parker, R. Wilson: An atlas of Brauer characters, London Mathematical Society Monographs, New Series 11, Oxford Univ. Press, 1995.

[16] C. Jansen, J. Müller: The 3-modular decomposition numbers of the sporadic simple Suzuki group, Comm. Algebra 25 (8), 1997, 2437–2458.

[17] F. Kasch: Moduln und Ringe, Mathematische Leitfäden, Teubner, 1977.

[18] P. Landrock: Finite group algebras and their modules, London Mathematical Society Lecture Note Series 84, Cambridge Univ. Press, 1983.

[19] S. Linton: On vector enumeration, J. Linear Algebra and its Applications 192, 1993, 235–248.

[20] S. Linton, K. Lux, L. Soicher: The primitive distance-transitive representations of the Fischer groups, Experiment. Math. 4, 1995, 235–253.

[21] F. Lübeck, M. Neunhöffer: Enumerating large orbits and direct condensation, Experiment. Math. 10, 2001, 197–205.

[22] K. Lux: Algorithmic methods in modular representation theory, Habilitationsschrift, RWTH Aachen, 1997.

[23] K. Lux, J. Müller, M. Ringe: Peakword condensation and submodule lattices: an application of the MeatAxe, J. Symbolic Comput. 17 (6), 1994, 529–544.

[24] K. Lux, M. Szőke: Computing homomorphism spaces between modules over finite dimensional algebras, Experiment. Math. 12 (1), 2003, 91–98.

[25] K. Lux, M. Wiegelmann: Determination of socle series using the condensation method, Computational algebra and number theory, Milwaukee, 1996, J. Symb. Comput. 31, 2001, 163–178.

[26] K. Lux, M. Wiegelmann: Condensing tensor product modules, *in*: The atlas of finite groups: ten years on, Birmingham, 1995, 174–190, London Math. Soc. Lecture Note Ser. 249, Cambridge Univ. Press, 1998.

[27] J. Müller: On endomorphism rings and character tables, Habilitationsschrift, RWTH Aachen, 2003.

[28] J. Müller: A note on applications of the 'Vector Enumerator' algorithm, Linear Algebra Appl. 365, 2003, 291–300.

[29] J. Müller: The 2-modular decomposition matrices of the symmetric groups S_{15}, S_{16}, and S_{17}, Comm. Algebra 28 (10), 2000, 4997–5005.

[30] J. Müller: On the Benson-Conway Theorem, unpublished.

[31] J. Müller, M. Neunhöffer, F. Röhr, R. Wilson: Completing the Brauer trees for the sporadic simple Lyons group, LMS J. Comput. Math. 5, 2002, 18–33.

[32] J. Müller, M. Ringe: Unpublished.

[33] J. Müller, J. Rosenboom: Condensation of induced representations and an application: the 2-modular decomposition numbers of Co_2, Proc. of the Euroconference on computational methods for representations of groups and algebras, Essen, 1997, 309–321, Progr. Math. 173, Birkhäuser, 1999.

[34] M. Ottensmann: Vervollständigung der Brauerbäume von $3.ON$ in Charakteristik 11, 19 und 31 mit Methoden der Kondensation, Diplomarbeit, RWTH Aachen, 2000.

[35] R. Parker: The computer calculation of modular characters: the Meat-Axe, Computational group theory, Durham, 1982, 267–274, Academic Press, 1984.

[36] R. Parker: Private communication.

[37] R. Parker, R. Wilson: Private communication.

[38] M. Ringe: The C-MeatAxe-2.4, Manual, RWTH Aachen, 2003.

[39] F. Röhr: Die Brauer-Charaktere der sporadisch einfachen Rudvalis-Gruppe in den Charakteristiken 13 und 29, Diplomarbeit, RWTH Aachen, 2000.

[40] A. Seress: Permutation group algorithms, Cambridge Tracts in Mathematics 152, Cambridge Univ. Press, 2003.

[41] C. Sims: Computation with finitely presented groups, Encyclopedia of Mathematics and its Applications 48, Cambridge Univ. Press, 1994.

[42] M. Szőke: Examining Green correspondents of simple modules, Dissertation, RWTH Aachen, 1998.

[43] J. Thackray: Modular representations of some finite groups, Ph.D. Thesis, Univ. of Cambridge, 1981.

[44] J. Thackray: Private communication.

[45] C. Weibel: An introduction to homological algebra, Cambridge Studies in Advanced Mathematics 38, Cambridge Univ. Press, 1994.

[46] M. Wiegelmann: Fixpunktkondensation von Tensorproduktmoduln, Diplomarbeit, RWTH Aachen, 1994.

[47] R. Wilson et al.: The Modular Atlas HomePage,
http://www.math.rwth-aachen.de/homes/MOC/

[48] R. Wilson et al.: Atlas of finite group representations,
http://www.mat.bham.ac.uk/atlas/.