# Algebraic Groups

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## Abstract

Algebraic groups are analogues of the classical Lie groups, such as the linear, orthogonal or symplectic groups, over arbitrary algebraically closed fields. Hence they are no longer classical manifolds, but varieties in the sense of algebraic geometry. In particular, they are used in the uniform description of the finite groups of Lie type, which encompass a substantial part of all finite simple groups. Subject of the lecture is an introduction to linear algebraic groups. Here, tools both from group theory as well as from algebraic geometry come into play.

## Contents

# I Algebraic geometry

## 1 Affine algebraic varieties

All rings, $R$ say, occurring will be commutative with identity $1 = 1_R$, unless otherwise specified. Let $K$ be a field, and let $\mathbb{K}$ be an algebraically closed field.

**(1.1) Theorem: Hilbert's Basissatz (1890).**
Let $R$ be Noetherian, and let $\mathcal{X} := \{X_1, \dots, X_n\}$, for $n \in \mathbb{N}_0$, be indeterminates over $R$. Then the polynomial ring $R[\mathcal{X}]$ is Noetherian as well.

**Proof.** See [3, Thm.IV.4.1] or [5, Thm.1.7]. ♯

**(1.2) Definition.** Let $\mathcal{X} := \{X_1, \dots, X_n\}$.
**a)** Let $\mathcal{S} \subseteq \mathbb{K}[\mathcal{X}]$. Then $\mathcal{V}(\mathcal{S}) := \{x \in \mathbb{K}^n; f(x) = 0 \text{ for all } f \in \mathcal{S}\}$ is called the **algebraic set** defined by $\mathcal{S}$; for polynomial functions see Exercise (11.1).

We have $\mathcal{V}(\mathcal{S}) = \mathcal{V}(\langle \mathcal{S} \rangle)$, by Hilbert's Basissatz there are $f_1, \dots, f_r \in \mathcal{S}$ such that $\mathcal{V}(\mathcal{S}) = \mathcal{V}(f_1, \dots, f_r)$, and we have $\mathcal{V}(\mathbb{K}[\mathcal{X}]) = \emptyset$ and $\mathcal{V}(0) = \mathbb{K}^n$.

**b)** Let $V \subseteq \mathbb{K}^n$. Then $\mathcal{I}(V) := \{f \in \mathbb{K}[\mathcal{X}]; f(x) = 0 \text{ for all } x \in V\} \trianglelefteq \mathbb{K}[\mathcal{X}]$ is called the **vanishing ideal** of $V$.

We have $\mathcal{I}(V) < \mathbb{K}[\mathcal{X}]$ if and only if $V \neq \emptyset$, and $\mathcal{I}(\mathbb{K}^n) = \{0\}$. Moreover, $\mathcal{I}(V) = \sqrt{\mathcal{I}(V)}$ is a **radical ideal**. Here, for any $I \lhd R$ we let $\sqrt{I} := \{f \in R; f^r \in I \text{ for some } r \in \mathbb{N}\} = \bigcap\{P \lhd R \text{ prime}; I \subseteq P\} \lhd R$ denote the **radical** of $I$, and $\sqrt{R} := R$.

**(1.3) Proposition. a)** For $V \subseteq \mathbb{K}^n$ we have $V \subseteq \mathcal{V}(\mathcal{I}(V))$.
**b)** For $I \trianglelefteq \mathbb{K}[\mathcal{X}]$ we have $I \subseteq \sqrt{I} \subseteq \mathcal{I}(\mathcal{V}(I))$.
**c)** Let $\Lambda$ be an index set. Then for $\{V_\lambda \subseteq \mathbb{K}^n; \lambda \in \Lambda\}$ we have $\mathcal{I}(\bigcup_{\lambda\in\Lambda} V_\lambda) = \bigcap_{\lambda\in\Lambda} \mathcal{I}(V_\lambda)$, and for $\{I_\lambda \trianglelefteq \mathbb{K}[\mathcal{X}]; \lambda \in \Lambda\}$ we have $\mathcal{V}(\sum_{\lambda\in\Lambda} I_\lambda) = \bigcap_{\lambda\in\Lambda} \mathcal{V}(I_\lambda)$.
**d)** For $I, I' \trianglelefteq \mathbb{K}[\mathcal{X}]$ we have $\mathcal{V}(I) \cup \mathcal{V}(I') = \mathcal{V}(I \cdot I') = \mathcal{V}(I \cap I')$.

**Proof.** See [6, Prop.I.1.1, I.1.2] or [7, Ch.I.2] or Exercise (11.2). ♯

**(1.4) Theorem: Hilbert's Nullstellensatz (1890).**
Let $I, P \lhd \mathbb{K}[\mathcal{X}]$, where $I \subseteq P$ and $P$ is maximal.
**a) Weak form.** There are $x_1, \dots, x_n \in \mathbb{K}$ such that $P = \langle X_1 - x_1, \dots, X_n - x_n \rangle$, implying $\{[x_1, \dots, x_n]\} = \mathcal{V}(P) \subseteq \mathcal{V}(I) \neq \emptyset$.
**b) Strong form.** We have $\sqrt{I} = \mathcal{I}(\mathcal{V}(I))$.

**Proof.** See [4, Thm.5.3, 5.4] or [7, Ch.I.1, Thm.I.2.1], or Exercise (11.3) on how to derive the strong form from the weak form. ♯

**(1.5) Corollary.** The map
$$\mathcal{V}\colon \{I \trianglelefteq \mathbb{K}[\mathcal{X}]; I = \sqrt{I}\} \to \{V \subseteq \mathbb{K}^n \text{ algebraic}\}\colon I \mapsto \mathcal{V}(I)$$
is an inclusion-reversing bijection with inverse map $\mathcal{I}$.

**(1.6) Definition and Remark. a)** The algebraic sets $\mathcal{V}(I) \subseteq \mathbb{K}^n$, for some $I = \sqrt{I} \trianglelefteq \mathbb{K}[\mathcal{X}]$, are the closed sets of the **Zariski topology** on $\mathbb{K}^n$.

If $V \subseteq \mathbb{K}^n$ is algebraic, then the topology on $V$ induced by the Zariski topology is also called the Zariski topology.

**b)** The **closure** of any $V \subseteq \mathbb{K}^n$ with respect to the Zariski topology is given as $\overline{V} := \bigcap\{W \subseteq \mathbb{K}^n \text{ closed}; V \subseteq W\} = \mathcal{V}(\mathcal{I}(V))$.

The Zariski topology is **Noetherian**, i. e. any strictly decreasing chain of closed subsets is finite, in particular it is **quasi-compact**, i. e. any open covering has a finite subcovering. Moreover, it is a $T_1$**-space**, i. e. singleton subsets are closed.

Algebraic sets, by the induced Zariski topology, are Noetherian and $T_1$ as well.

**c)** A non-empty Noetherian topological space is called **irreducible**, if it cannot be written as the union of two proper closed subsets. Hence in particular an irreducible topological space is **connected**, i. e. it cannot be written as the disjoint union of two proper open and closed subsets.

**(1.7) Proposition.** Let $V \neq \emptyset$ be a Noetherian topological space. Then there are $V_1, \ldots, V_r \subseteq V$, for some $r \in \mathbb{N}$, closed and irreducible such that $V = \bigcup_{i=1}^r V_i$. If we moreover have $V_i \not\subseteq V_j$, for all $i \neq j \in \{1, \ldots, r\}$, then $V_1, \ldots, V_r$ are precisely the maximal irreducible closed subsets, hence are uniquely determined, and are called the **irreducible components** of $V$.

**Proof.** See [6, Prop.I.1.5] or Exercise (11.4). $\sharp$

**(1.8) Corollary.** Let $I = \sqrt{I} \lhd \mathbb{K}[\mathcal{X}]$ be a radical ideal.
**a)** Then there are only finitely many prime ideals of $\mathbb{K}[\mathcal{X}]$ minimal over $I$.
**b)** The algebraic set $\mathcal{V}(I) \subseteq \mathbb{K}^n$ is irreducible if and only if $I$ is prime.
**c)** We have $I = \bigcap\{P \lhd \mathbb{K}[\mathcal{X}] \text{ maximal}; I \subseteq P\}$, i. e. $\mathbb{K}[\mathcal{X}]$ is a **Jacobson ring**.

**(1.9) Definition. a)** Let $V \subseteq \mathbb{K}^n$ and $W \subseteq \mathbb{K}^m$ be algebraic. Then a map $\varphi\colon V \to W$ is called **regular**, if there are $f_1, \ldots, f_m \in \mathbb{K}[\mathcal{X}] = \mathbb{K}[X_1, \ldots, X_n]$ such that $\varphi(x) = [f_1(x), \ldots, f_m(x)]$, for all $x \in V$.

In particular, a regular map is continuous with respect to the Zariski topology; see Exercise (11.5). Let $\mathrm{Hom}(V, W)$ be the set of all regular maps from $V$ to $W$.

**b)** In particular, $\mathrm{Hom}(V, \mathbb{K})$ is a $\mathbb{K}$-algebra, called the **algebra of regular functions** on $V$. We have a $\mathbb{K}$-algebra epimorphism
$$\mathbb{K}[\mathcal{X}] \to \mathrm{Hom}(V, \mathbb{K})\colon f \mapsto (f^\bullet\colon V \to \mathbb{K}\colon x \mapsto f(x)),$$

whose kernel equals $\mathcal{I}(V)$. We have $\mathbb{K}[V] := \mathbb{K}[\mathcal{X}]/\mathcal{I}(V) \cong \mathrm{Hom}(V, \mathbb{K})$ as $\mathbb{K}$-algebras, where $\mathbb{K}[V]$ is called the **affine coordinate algebra** of $V$.

Since $\mathcal{I}(V) = \sqrt{\mathcal{I}(V)}$ the $\mathbb{K}$-algebra $\mathbb{K}[V]$ is **reduced**, i. e. $\mathbb{K}[V]$ does not possess nilpotent elements, and $\mathbb{K}[V]$ is a domain if and only if $V$ is irreducible.

**c)** The algebraic set $V$ together with its Zariski topology and its affine coordinate algebra $\mathbb{K}[V]$ is called an **affine (algebraic) variety** over $\mathbb{K}$. Together with the regular maps as **morphisms** this defines the **category** of affine varieties over $\mathbb{K}$; see also Exercise (11.6).

**(1.10) Theorem. a)** For affine varieties $V, W$ there is a bijection

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}_{\mathbb{K}-\mathrm{algebra}}(\mathbb{K}[W], \mathbb{K}[V]) \colon \varphi \mapsto (\varphi^* \colon \mathbb{K}[W] \to \mathbb{K}[V] \colon f \mapsto f \circ \varphi).$$

The $\mathbb{K}$-algebra homomorphism $\varphi^*$ is called the **comorphism** associated to $\varphi$.
**b)** Assigning $V \mapsto \mathbb{K}[V]$ and $\varphi \mapsto \varphi^*$ yields an anti-equivalence from the category of affine varieties over $\mathbb{K}$ to the category of reduced finitely generated $\mathbb{K}$-algebras together with $\mathbb{K}$-algebra homomorphisms.

**Proof.** See [7, Prop. I.3.1, I.3.2]. $\sharp$

**(1.11) Example.** See Exercises (11.7) and (11.8).
**a)** Let $y = [y_1, \ldots, y_n] \in \mathbb{K}^n$. Then $\epsilon_y \colon \{y\} \to \mathbb{K}^n \colon y \mapsto y$ is a morphism, and $\epsilon_y^* \colon \mathbb{K}[X_1, \ldots, X_n] \to \mathbb{K} \colon X_i \mapsto y_i$ is the evaluation map at $y$. Similarly, $\nu_y \colon \mathbb{K}^n \to \{y\} \colon x \mapsto y$ is a morphism, and $\nu_y^* \colon \mathbb{K} \to \mathbb{K}[X_1, \ldots, X_n] \colon 1_{\mathbb{K}} \mapsto 1_{\mathbb{K}[X_1,\ldots,X_n]}$ is the natural embedding. This yields $\epsilon_y \nu_y \colon \mathbb{K}^n \to \mathbb{K}^n \colon x \mapsto y$ and $(\epsilon_y \nu_y)^* = \nu_y^* \epsilon_y^* \colon \mathbb{K}[X_1, \ldots, X_n] \to \mathbb{K}[X_1, \ldots, X_n] \colon X_i \mapsto y_i$.
**b)** Let $\varphi \colon \mathbb{K}^2 \to \mathbb{K}^2 \colon [x, y] \mapsto [xy, y]$. Then $\varphi$ is a morphism, and we have $\varphi^* \colon \mathbb{K}[X, Y] \to \mathbb{K}[X, Y] \colon X \mapsto XY, Y \mapsto Y$. Moreover, we have $\varphi(\mathbb{K}^2) = \{[0, 0]\} \,\dot\cup\, (\mathbb{K} \times (\mathbb{K} \setminus \{0\})) \subseteq \mathbb{K}^2$, which is neither open nor closed.
**c)** Let $\mathrm{char}(\mathbb{K}) = p > 0$ and $q := p^f$ for some $f \in \mathbb{N}$. Then the **geometric Frobenius map** $\Phi_q \colon \mathbb{K} \to \mathbb{K} \colon x \mapsto x^q$ is a bijective morphism, and we have $\Phi_q^* \colon \mathbb{K}[X] \to \mathbb{K}[X] \colon X \mapsto X^q$. Since $\Phi_q^*$ is not surjective, $\Phi_q$ is not an isomorphism of affine varieties.

**(1.12) Theorem.** Let $V, W$ be affine varieties over $\mathbb{K}$. Then the Cartesian product $V \times W$ again is an affine variety such that $\mathbb{K}[V \times W] \cong \mathbb{K}[V] \otimes_{\mathbb{K}} \mathbb{K}[W]$.

This is a direct product in the category of affine varieties over $\mathbb{K}$. Moreover, $V \times W$ is irreducible if and only if both $V$ and $W$ are.

**Proof.** See [7, Prop.I.6.1] or [11, Thm.1.5.4] or [10, Prop.I.1.4, I.2.4], and also Exercise (11.9). $\sharp$

**(1.13) Definition.** Let $V$ be an affine variety, and let $0 \neq f \in \mathbb{K}[V]$. Then the set $V_f := \{x \in V; f(x) \neq 0\} \neq \emptyset$ is called the associated **principal** or **elementary** open subset, which since $V_f = V \setminus \mathcal{V}(f) \subseteq V$ is indeed open in $V$.

The set $\{V_f \subseteq V; 0 \neq f \in \mathbb{K}[V]\}$ is a basis of the Zariski topology on $V$; see Exercise (11.10).

**(1.14) Remark. a)** Let $V$ be an affine variety, and let $0 \neq f \in \mathbb{K}[V]$. We consider the **localisation** $\mathbb{K}[V]_f$ of $\mathbb{K}[V]$ at the **multiplicative set** $\{f^r \in \mathbb{K}[V]; r \in \mathbb{N}_0\}$, i. e. the set of equivalence classes of **fractions** $\frac{g}{f^r}$, where $g \in \mathbb{K}[V]$ and $r \in \mathbb{N}_0$, with respect to the equivalence relation $\frac{g}{f^r} = \frac{g'}{f^s}$ if and only if there is $t \in \mathbb{N}_0$ such that $(gf^s - g'f^r)f^t = 0 \in \mathbb{K}[V]$; see Exercise (11.12).

Then $\mathbb{K}[V]_f = \mathbb{K}\langle \frac{1}{f}, \frac{g}{1}; g \in \mathbb{K}[V]\rangle$ is a finitely generated $\mathbb{K}$-algebra. Moreover, $\mathbb{K}[V]_f$ is reduced: If $(\frac{g}{f^r})^s = 0 \in \mathbb{K}[V]_f$, for some $s \in \mathbb{N}$, then we have $g^s f^t = 0 \in \mathbb{K}[V]$, for some $t \in \mathbb{N}_0$, which since $\mathbb{K}[V]$ is reduced implies $gf = 0 \in \mathbb{K}[V]$, thus $\frac{g}{1} = 0 \in \mathbb{K}[V]_f$ and $\frac{g}{f^r} = 0 \in \mathbb{K}[V]_f$.

**b)** Hence there is an affine variety $\widetilde{V}_f$ associated to $\mathbb{K}[V]_f$, and we show that $\widetilde{V}_f$ can be identified with $V_f$: We have the natural homomorphism of $\mathbb{K}$-algebras $\varphi_f^*\colon \mathbb{K}[V] \rightarrow \mathbb{K}[V]_f\colon g \mapsto \frac{g}{1}$, hence a morphism $\varphi_f\colon \widetilde{V}_f \rightarrow V$. The inclusion-preserving bijection $(\varphi_f^*)^{-1}\colon \{P \lhd \mathbb{K}[V]_f \text{ prime}\} \rightarrow \{Q \lhd \mathbb{K}[V] \text{ prime}; f \notin Q\}$, see Exercise (11.12), yields a bijection $(\varphi_f^*)^{-1}\colon \{P \lhd \mathbb{K}[V]_f \text{ maximal}\} \rightarrow \{Q \lhd \mathbb{K}[V] \text{ maximal}; f \notin Q\}$, i. e. we have a bijection $\varphi_f\colon \widetilde{V}_f \rightarrow V_f$.

We have a bijection $(\varphi_f^*)^{-1}\colon \{I = \sqrt{I} \lhd \mathbb{K}[V]_f\} \rightarrow \{J = \sqrt{J} \lhd \mathbb{K}[V]; f \notin J\}$. Since the non-empty closed subsets of $V_f$, with respect to the topology induced by the embedding $V_f \subseteq V$, are the sets $\mathcal{V}(J) \cap V_f$, for some $J = \sqrt{J} \lhd \mathbb{K}[V]$ such that $f \notin J$, we conclude that $\varphi_f\colon \widetilde{V}_f \rightarrow V_f$ is a homeomorphism. ♯

Hence $\varphi_f$ carries the structure of an affine variety from $\widetilde{V}_f$ to $V_f$, whose Zariski topology coincides with the topology induced by the embedding $V_f \subseteq V$. The set $V_f$ is also called an **affine open subset**, by definition we have an isomorphism of $\mathbb{K}$-algebras $\mathbb{K}[V]_f \rightarrow \mathbb{K}[V_f]\colon \frac{g}{1} \mapsto g|_{V_f}, \frac{1}{f} \mapsto \frac{1}{f|_{V_f}}$. The inclusion map $V_f \subseteq V$ is a morphism, whose associated comorphism is $\mathbb{K}[V] \rightarrow \mathbb{K}[V_f]\colon g \mapsto g|_{V_f}$.

Not all open subsets of an affine variety can be endowed with the structure of an affine variety compatible with the given affine variety and its affine open subsets, see Exercise (11.11).

**(1.15) Proposition.** Let $V$ be an affine variety, let $0 \neq f \in \mathbb{K}[V]$ and let $\widehat{V}_f := \{[v, y] \in V_f \times \mathbb{K}; f(v)y = 1\}$. Then the projection map $\pi_f\colon \widehat{V}_f \rightarrow V_f\colon [v, y] \mapsto v$ is an isomorphism of affine varieties, whose associated comorphism yields

$$\mathbb{K}[V_f] \cong \mathbb{K}[\widehat{V}_f] \cong \mathbb{K}[V][Y]/\langle fY - 1\rangle =: \mathbb{K}[V]\langle f^{-1}\rangle.$$

**Proof.** Let $V \subseteq \mathbb{K}^n$. Then $\widehat{V}_f = \{[v,y] \in V \times \mathbb{K}; f(v)y - 1 = 0\} \subseteq \mathbb{K}^n \times \mathbb{K} = \mathbb{K}^{n+1}$ is algebraic. We have $\langle fY - 1\rangle \subseteq \mathcal{I}(\widehat{V}_f) \lhd \mathbb{K}[V][Y]$. Conversely, let $g = \sum_{i=0}^r g_i Y^i \in \mathcal{I}(\widehat{V}_f)$, where $g_i \in \mathbb{K}[V]$. Letting $h := \sum_{i=0}^r g_i f^{r-i} \in \mathbb{K}[V]$, we obtain $gf^{r+1} = f \cdot \sum_{i=0}^r g_i f^{r-i}(fY)^i \equiv fh \pmod{\langle fY-1\rangle}$. Since $gf^{r+1} \in \mathcal{I}(\widehat{V}_f)$, this implies $fh \in \mathcal{I}(\widehat{V}_f) \cap \mathbb{K}[V]$. For $v \in V \setminus V_f$ we have $f(v) = 0$, while for $v \in V_f$ we have $f(v) \neq 0$ and hence $h(v) = 0$. Thus we have $fh = 0 \in \mathbb{K}[V] \subseteq K[V][Y]$, implying $g \equiv gf^{r+1}Y^{r+1} \equiv fhY^{r+1} \equiv 0 \pmod{\langle fY-1\rangle}$, and $\mathbb{K}[\widehat{V}_f] \cong \mathbb{K}[V][Y]/\langle fY-1\rangle$.

Since for $g \in \mathbb{K}[V]$ we have $\pi_f^*(g|_{V_f})\colon [v,y] \mapsto g(v)$, and $\pi_f^*(\frac{1}{f|_{V_f}})\colon [v,y] \mapsto \frac{1}{f(v)} = y$, we conclude that indeed $\pi_f^*\colon \mathbb{K}[V_f] \to \mathbb{K}[\widehat{V}_f]$, i. e. $\pi_f$ is a morphism. Moreover, $\pi_f$ is bijective, hence $\pi_f^*$ is injective, and from $Y = \frac{1}{f} \in \mathbb{K}[\widehat{V}_f]$ we conclude that $\pi^*$ is surjective, thus $\pi$ is an isomorphism of affine varieties. ♯

## 2 Morphisms

**(2.1) Definition. a)** Let $R \neq \{0\}$ be a ring, and let $P \lhd R$ be prime. The supremum of the lengths $r \in \mathbb{N}_0$ of chains $P_0 \subset P_1 \subset \cdots \subset P_r = P$ of prime ideals $P_i \lhd R$ is called the **height** $\mathrm{ht}(P) \in \mathbb{N}_0 \,\dot\cup\, \{\infty\}$ of $P$. If $R$ is Noetherian, by (2.2) we have $\mathrm{ht}(P) \in \mathbb{N}_0$.

Moreover, $\dim(R) := \sup\{\mathrm{ht}(P); P \lhd R \text{ prime}\} \in \mathbb{N}_0 \,\dot\cup\, \{\infty\}$ is called the **(Krull) dimension** of $R$. For a Noetherian ring having infinite dimension see Exercise (11.14).

**b)** For $I \lhd R$ let the **dimension** $\dim(I) := \dim(R/I) \in \mathbb{N}_0 \,\dot\cup\, \{\infty\}$ and the **height** $\mathrm{ht}(I) := \min\{\mathrm{ht}(P); I \subseteq P \lhd R \text{ prime}\} \in \mathbb{N}_0 \,\dot\cup\, \{\infty\}$.

We have $\dim(I) \leq \dim(R)$ and $\dim(I) + \mathrm{ht}(I) \leq \dim(R)$.

**(2.2) Theorem: Krull's Hauptidealsatz (1928).**
Let $R$ be Noetherian, let $f_1, \ldots, f_r \in R$, for some $r \in \mathbb{N}$, and let $P \lhd R$ be a minimal prime over $\langle f_1, \ldots, f_r\rangle$. Then we have $\mathrm{ht}(P) \leq r$.

**Proof.** See [4, Thm.13.5] or [5, Thm.6.8]. ♯

**(2.3) Lemma: Prime avoidance.**
Let $R$ be a ring, let $P_1, \ldots, P_n \lhd R$ be prime, for some $n \in \mathbb{N}$, and let $I \lhd R$ such that $I \subseteq \bigcup_{i=1}^n P_i$. Then there is $i \in \{1, \ldots, n\}$ such that $I \subseteq P_i$.

**Proof.** See [4, Exc.1.6] or [5, La.6.3]. ♯

**(2.4) Theorem.** Let $R$ be Noetherian and $P \lhd R$ be prime such that $\mathrm{ht}(P) = r \in \mathbb{N}$. Then there are $f_1, \ldots, f_r \in R$ such that $P$ is a minimal prime over $I := \langle f_1, \ldots, f_r\rangle \lhd R$, and for any minimal prime $P' \lhd R$ over $I$ we have $\mathrm{ht}(P') = r$.

**Proof.** By induction, quotiening out $\langle f_1, \ldots, f_{r-1}\rangle \lhd R$, we may assume that $r = 1$. Since the **Zariski topology** on the **prime spectrum** $\mathrm{Spec}(R) := \{P \lhd R \text{ prime}\}$ is Noetherian, there are only finitely many minimal prime ideals $Q_1, \ldots, Q_s \lhd R$, for some $s \in \mathbb{N}$. Since $\mathrm{ht}(P) = 1$ we have $P \notin \{Q_1, \ldots, Q_s\}$, thus by prime avoidance there is $f \in P \setminus \bigcup_{i=1}^{s} Q_i$. Hence $P$ is a minimal prime over $\langle f\rangle \lhd R$, and for any minimal prime $P' \lhd R$ over $\langle f\rangle$ we have $P' \notin \{Q_1, \ldots, Q_s\}$, by Krull's Hauptidealsatz implying $\mathrm{ht}(P') = 1$. ♯

**(2.5) Theorem: Cohen-Seidenberg (1946).**
Let $R \subseteq S$ be an integral ring extension.
**a)** Let $P \lhd R$ be prime. Then there is a prime ideal $Q \lhd S$ **(lying over)** such that $Q \cap R = P$. Moreover, if $J \lhd S$ is any ideal such that $J \cap R \subseteq P$, then $Q$ can be chosen **(going up)** such that $J \subseteq Q$.
**b)** Let $Q \neq Q' \lhd S$ be prime such that $Q \cap R = Q' \cap R$. Then we have **(incomparability)** $Q \not\subseteq Q' \not\subseteq Q$.

**Proof.** See [4, Thm.9.3] or [5, Thm.6.9]. ♯

**(2.6) Theorem.** Let $R := K\langle f_1, \ldots, f_r\rangle$ for some $r \in \mathbb{N}_0$.
**a)** We have $\dim(R) \leq r$, and $\dim(R) = r$ holds if and only if $\{f_1, \ldots, f_r\} \subseteq R$ is algebraically independent over $\mathbb{K}$.
**b)** Let additionally $R$ be a domain. Then we have $\dim(R) = \mathrm{trdeg}_K(\mathrm{Q}(R))$, where $\mathrm{Q}(R)$ denotes the **field of fractions** of $R$.

**Proof.** See [4, Thm.5.6] or [5, Cor.7.3, 7.5]. ♯

**(2.7) Theorem: Refined Noether normalisation.**
Let $R := K\langle f_1, \ldots, f_r\rangle$, for some $r \in \mathbb{N}_0$, and let $I_1 \subset \cdots \subset I_s \subset R$, for some $s \in \mathbb{N}_0$, be ideals such that $n > n_1 > \cdots > n_s \geq 0$, where $n := \dim(R)$ and $n_k := \dim(I_k)$. Then there is $\{X_1, \ldots, X_n\} \subseteq R$ algebraically independent over $K$, such that $S := K[X_1, \ldots, X_n] \subseteq R$ is a **finite ring extension**, i. e. $R$ is a finitely generated $S$-algebra and **integral** over $S$, and such that $I_k \cap S = \langle X_{n_k+1}, \ldots, X_n\rangle \lhd S$, for $k \in \{1, \ldots, s\}$. Moreover, if $K$ is infinite then we may choose $X_i \in \langle f_1, \ldots, f_r\rangle_K$, for $i \in \{1, \ldots, n\}$.

**Proof.** For infinite fields by Noether (1926), for finite fields by Zariski (1943), and the refined version, actually involving only a single ideal, by Nagata (1962).

See [1, Thm.II.13.3] or [5, Thm.7.4]. ♯

**(2.8) Theorem.** Let $R := K\langle f_1, \ldots, f_r\rangle$ be a domain, where $r \in \mathbb{N}_0$, and let $I \lhd R$. Then we have $\dim(I) + \mathrm{ht}(I) = \dim(R)$.

This implies that $R$ is **catenary**, i. e. given prime ideals $P \subseteq Q \lhd R$, then for all maximal chains $P = P_0 \subset \cdots \subset P_r = Q$ of prime ideals we have

$r = \mathrm{ht}(Q) - \mathrm{ht}(P)$, see Exercise (11.16). The assumption $R$ being a domain is cannot be dispensed of, see Exercise (11.15).

**Proof.** It suffices to show the assertion for $Q \lhd R$ prime, and we may assume $Q \neq \{0\}$, implying that $\dim(Q) < \dim(R)$. Let $S := K[X_1, \ldots, X_n] \subseteq R$ be a Noether normalisation such that $P := Q \cap S = \langle X_{m+1}, \ldots, X_n \rangle \lhd S$, where $n := \dim(R) = \dim(S)$ and $m := \dim(Q)$. By the Cohen-Seidenberg Theorem, see also Exercise (11.13), we have $\dim(P) = \dim(Q)$ and $\mathrm{ht}(P) = \mathrm{ht}(Q)$. Moreover, we have $\dim(P) = \dim(S/P) = \dim(K[X_1, \ldots, X_m]) = m$, and it is immediate that $\mathrm{ht}(P) \geq n-m$, hence $n = \dim(S) \geq \dim(P)+\mathrm{ht}(P) \geq n$ implies $\dim(P) + \mathrm{ht}(P) = n$. ♯

**(2.9) Definition and Remark. a)** Let $V \neq \emptyset$ be an affine variety. Then $\dim(V) := \dim(\mathbb{K}[V]) \in \mathbb{N}_0$ is called the **dimension** of $V$.

We have $\dim(V) = \max\{\dim(W); W \subseteq V \text{ irreducible component}\}$. For the dimension 0 case, and the dimension of direct products, see Exercise (11.17).

**b)** Let $V$ be irreducible. Then for any $\emptyset \neq W \subset V$ closed we have $\mathrm{ht}(\mathcal{I}(W)) > 0$ and thus $\dim(W) = \dim(\mathbb{K}[W]) = \dim(\mathbb{K}[V]/\mathcal{I}(W)) < \dim(\mathbb{K}[V]) = \dim(V)$.

If $W \subset V$ is closed and irreducible such that $\dim(W) = \dim(V) - 1$, then there is $0 \neq f \in \mathbb{K}[V] \setminus \mathbb{K}[V]^*$ such that $W$ is an irreducible component of the **hypersurface** $\mathcal{V}(f) \subseteq V$; for the question when $W$ is a hypersurface see Exercise (11.17). Conversely, for any $0 \neq f \in \mathbb{K}[V] \setminus \mathbb{K}[V]^*$ Krull's Hauptidealsatz implies that $\mathcal{V}(f) \subseteq V$ has **equidimension** $\dim(V)-1$, i. e. all irreducible components of $\mathcal{V}(f)$ have dimension $\dim(V) - 1$.

Let $0 \neq f \in \mathbb{K}[V]$. We have $\mathbb{K}[V] \subseteq \mathbb{K}[V]_f \subseteq \mathbb{K}(V)$, where the field of fractions $\mathbb{K}(V) := \mathrm{Q}(\mathbb{K}[V])$ is called the **field of rational functions** on $V$. Hence we have $\dim(V) = \dim(\mathbb{K}[V]) = \mathrm{trdeg}_{\mathbb{K}}(\mathbb{K}(V)) = \dim(\mathbb{K}[V]_f) = \dim(V_f)$.

**c)** Let again $V \neq \emptyset$ be arbitrary. A morphism of affine varieties $\varphi\colon V \to W$ is called **finite**, if $\varphi^*(\mathbb{K}[W]) \subseteq \mathbb{K}[V]$ is a finite ring extension; see also Exercise (11.18) and the example in Exercise (11.19).

E. g. let $\mathbb{K}[\mathcal{X}] = \mathbb{K}[X_1, \ldots, X_n] \subseteq \mathbb{K}[V]$ be a Noether normalisation, where $n := \dim(V)$, and let $\varphi^*\colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[V]$ be the natural embedding of $\mathbb{K}$-algebras. Hence the associated morphism $\varphi\colon V \to \mathbb{K}^n$ is finite and **dominant**, i. e. $\varphi(V) \subseteq \mathbb{K}^n$ is dense; see Exercise (11.7).

**(2.10) Proposition.** Let $V, W$ be affine varieties and let $\varphi\colon V \to W$ be a finite morphism. Let $Z \subseteq W$ be closed, and let $\emptyset \neq U \subseteq \varphi^{-1}(Z)$ be closed. Then $\varphi|_U : U \to Z$ is finite and $\varphi(U) \subseteq Z$ is closed.

**Proof.** Let $R := \mathbb{K}[V]$ and $S := \mathbb{K}[W]$, hence $\varphi^*(S) \subseteq R$ is a finite ring extension. Letting $I := \mathcal{I}(U) = \bigcap\{\mathcal{I}(x); x \in U\} \lhd R$ we get $J := (\varphi^*)^{-1}(I) = \bigcap\{(\varphi^*)^{-1}(\mathcal{I}(x)); x \in U\} = \bigcap\{\mathcal{I}(\varphi(x)); x \in U\} = \mathcal{I}(\varphi(U)) \lhd S$. Hence we have $\varphi(U) \subseteq \overline{\varphi(U)} = \mathcal{V}(J) \subseteq Z \subseteq W$, and the comorphism associated to $\varphi|_U : U \to Z$ is the homomorphism of $\mathbb{K}$-algebras $S/\mathcal{I}(Z) \to S/J \xrightarrow{\varphi^*} R/I$, where $S/\mathcal{I}(Z) \to S/J$ is the natural epimorphism and $\varphi^* : S/J \to R/I$ is injective. Since $\varphi^*(S/J) \subseteq R/I$ is a finite ring extension, we conclude that $\varphi|_U : U \to Z$ is finite. Given $Q \lhd S/J$ maximal, by the Cohen-Seidenberg Theorem there is a maximal ideal of $R/I$ lying over $\varphi^*(Q)$. Hence $\varphi|_U : U \to \mathcal{V}(J)$ is surjective. ♯

**(2.11) Theorem.** Let $V, W$ be irreducible affine varieties and let $\varphi : V \to W$ be a dominant morphism. Let $Z \subseteq W$ be closed and irreducible, and let $U \subseteq \varphi^{-1}(Z) \neq \emptyset$ be an irreducible component such that $\varphi|_U : U \to Z$ is dominant. Then we have $\dim(U) - \dim(Z) \geq \dim(V) - \dim(W) \geq 0$.

In particular, for any $x \in V$ any irreducible component of the **fibre** $\varphi^{-1}(\varphi(x)) \subseteq V$ has dimension $\geq \dim(V) - \dim(W)$; see also the example in Exercise (11.19).

**Proof.** We have an injective homomorphism of $\mathbb{K}$-algebras $\varphi^* : \mathbb{K}[W] \to \mathbb{K}[V]$, and hence $\dim(\mathbb{K}[W]) = \mathrm{trdeg}_{\mathbb{K}}(\mathbb{K}(W)) \leq \mathrm{trdeg}_{\mathbb{K}}(\mathbb{K}(V)) = \dim(\mathbb{K}[V])$.

We may assume that $Z \neq W$, hence $r := \dim(W) - \dim(Z) \in \mathbb{N}$. Let $f_1, \ldots, f_r \in \mathbb{K}[W]$ such that $Z$ is an irreducible component of $\mathcal{V}(f_1, \ldots, f_r) \subseteq W$. Letting $g_i := \varphi^*(f_i) \in \mathbb{K}[V]$, for $i \in \{1, \ldots, r\}$, we conclude that $U \subseteq \mathcal{V}(g_1, \ldots, g_r) \subseteq V$, and there is an irreducible component $U_0 \subseteq \mathcal{V}(g_1, \ldots, g_r)$ such that $U \subseteq U_0$. Thus we have $Z = \overline{\varphi(U)} \subseteq \overline{\varphi(U_0)} \subseteq \overline{\varphi(\mathcal{V}(g_1, \ldots, g_r))} \subseteq \overline{\mathcal{V}(f_1, \ldots, f_r)} = \mathcal{V}(f_1, \ldots, f_r)$, implying $Z = \overline{\varphi(U_0)}$, hence $U \subseteq U_0 \subseteq \varphi^{-1}(Z)$ and thus $U = U_0$. Since $V$ is irreducible by Krull's Hauptidealsatz we get $\dim(U) = \dim(U_0) = \dim(V) - \mathrm{ht}(\mathcal{I}(U_0)) \geq \dim(V) - r = \dim(V) - \dim(W) + \dim(Z)$. ♯

**(2.12) Proposition.** Let $V, W$ be irreducible affine varieties and let $\varphi : V \to W$ be a dominant morphism. Then there is $0 \neq f \in \mathbb{K}[W]$ such that we have

$$\varphi|_{V_{\varphi^*(f)}} : V_{\varphi^*(f)} \xrightarrow{\varphi_0} W_f \times \mathbb{K}^r \xrightarrow{\pi_1} W_f,$$

where $\varphi_0 : V_{\varphi^*(f)} \to W_f \times \mathbb{K}^r$ is a finite dominant morphism, hence is surjective, $r := \dim(V) - \dim(W) \in \mathbb{N}_0$, and $\pi_1 : W_f \times \mathbb{K}^r \to W_f$ is the natural projection.

**Proof.** We may consider $S := \mathbb{K}[W] \subseteq \mathbb{K}[V] =: R$ as an extensions of domains. Let $\mathbb{K} \subseteq F := \mathrm{Q}(S) \subseteq E := \mathrm{Q}(R)$ and $F \subseteq T := \{\frac{g}{h} \in E; g \in R, 0 \neq h \in S\} \subseteq E$. Since $R$ is a finitely generated $S$-algebra, we conclude that $T$ is a finitely generated $F$-algebra, and we have $\mathrm{trdeg}_F(\mathrm{Q}(T)) = \mathrm{trdeg}_F(E) = \mathrm{trdeg}_{\mathbb{K}}(E) - \mathrm{trdeg}_{\mathbb{K}}(F) = r$. By Noether normalisation there is $\mathcal{Z} = \{z_1, \ldots, z_r\} \subseteq R$ and $z \in S$, such that $\mathcal{Z} \subseteq T$ is algebraically independent over $F$ and such that $F[\frac{1}{z} \cdot \mathcal{Z}] \subseteq T$ is integral. Letting $g \in R$, then $g$ is integral over $F[\frac{1}{z} \cdot \mathcal{Z}]$. From this it is immediate that there is $0 \neq h \in S$ such that $g$ is integral over $S_h[\mathcal{Z}]$. Hence it is also immediate that there is $0 \neq f \in S$ such that all elements of a finite $S$-algebra generating set of $R$ are integral over $S_f[\mathcal{Z}]$, which implies that $R$ and thus $R_f$ are integral over $S_f[\mathcal{Z}]$.

Hence from the ring extension $S_f \subseteq R_f$ we get a embedding of $\mathbb{K}$-algebras $\varphi_0^*\colon S_f[Z_1,\ldots,Z_r] \to R_f\colon Z_i \mapsto z_i$, where $\{Z_1,\ldots,Z_r\}$ is algebraically independent over $F$, such that $\operatorname{im}(\varphi_0^*) \subseteq R_f$ is finite. Since $R_f$ and $S_f$ as well as $S_f[Z_1,\ldots,Z_r]$ are the affine coordinate algebras of $V_{\varphi^*(f)}$ and $W_f$ as well as $W_f \times \mathbb{K}^r$, respectively, associated to $\varphi_0^*$ there is a finite dominant morphism $\varphi_0\colon V_{\varphi^*(f)} \to W_f \times \mathbb{K}^r$. Moreover, the natural embedding of $\mathbb{K}$-algebras $S_f \subseteq S_f[Z_1,\ldots,Z_r]$ yields the natural projection $\pi_1\colon W_f \times \mathbb{K}^r \to W_f$ as associated morphism. And finally the concatenation $\varphi_0^*\pi_1^*\colon S_f \to R_f$ is the embedding induced by $\varphi^*$, hence we have $\pi_1\varphi_0 = \varphi|_{V_f}$. ♯

**(2.13) Theorem.** Let $V, W$ be irreducible affine varieties and let $\varphi\colon V \to W$ be a dominant morphism.
**a)** For any $\emptyset \neq U \subseteq V$ open there is $\emptyset \neq Z \subseteq W$ open such that $Z \subseteq \varphi(U)$.
**b)** There is $\emptyset \neq Z \subseteq W$ open such that $Z \subseteq \varphi(V)$ and such that $\dim(\varphi^{-1}(y)) = \dim(V) - \dim(W)$ for all $y \in Z$.

**Proof.** Let $r := \dim(V) - \dim(W)$ and $0 \neq f \in \mathbb{K}[W]$ such that we have a factorisation $\varphi|_{V_f} = \pi_1\varphi_0\colon V_{\varphi^*(f)} \to W_f \times \mathbb{K}^r \to W_f$ as in (2.12).
**a)** For the case $U = V$, since $\varphi_0$ and $\pi_1$ are surjective, we have $Z := W_f \subseteq \varphi(V)$. Now it suffices to show the assertion for a basis of the Zariski topology on $V$, hence let $U = V_g \subseteq V$ be a principal open subset, where $0 \neq g \in \mathbb{K}[V]$. Then $\varphi|_{V_g}\colon V_g \to W$ corresponds to the embedding of $\mathbb{K}$-algebras $\varphi^*\colon \mathbb{K}[W] \to \mathbb{K}[V]_g$, hence still is a dominant morphism of irreducible affine varieties, thus $\varphi(V_g)$ contains a non-empty open subset of $W$.
**b)** We show that $Z := W_f$ is as desired: Let $y \in W_f$ and let $U_1,\ldots,U_s \subseteq \varphi^{-1}(y)$ be the irreducible components of $\varphi^{-1}(y)$. From $\varphi^{-1}(W_f) = V_{\varphi^*(f)}$ we conclude that $\varphi^{-1}(y) = \varphi_0^{-1}(\{y\} \times \mathbb{K}^r)$ and hence $\varphi_0|_{U_i}\colon U_i \to \{y\} \times \mathbb{K}^r$ is finite, for all $i \in \{1,\ldots,s\}$. Thus $\varphi_0|_{U_i}^*(\mathbb{K}[\{y\} \times \mathbb{K}^r]) \subseteq \mathbb{K}[U_i]$ being a finite ring extension we get $\dim(U_i) = \dim(\varphi_0|_{U_i}^*(\mathbb{K}[\{y\} \times \mathbb{K}^r])) \leq \dim(\{y\} \times \mathbb{K}^r) = r$. Since $\{y\} \times \mathbb{K}^r$ is irreducible such that $\varphi_0(\varphi^{-1}(y)) = \{y\} \times \mathbb{K}^r$, and $\operatorname{im}(\varphi_0|_{U_i}) \subseteq \{y\} \times \mathbb{K}^r$ is closed and irreducible, we infer that there is $j \in \{1,\ldots,s\}$ such that $\varphi_0|_{U_j}$ is surjective, implying that $\varphi_0|_{U_j}^*$ is injective and $\dim(U_j) = r$. ♯

**(2.14) Corollary.** Let $V, W$ be affine varieties and let $\varphi\colon V \to W$ be a morphism. Then there is $Z \subseteq \overline{\varphi(V)}$ open and dense such that $Z \subseteq \varphi(V)$.

**Proof.** See Exercise (11.20).
We may assume that $V \neq \emptyset$ and $\varphi$ is dominant. Let $V = \bigcup_{i=1}^r V_i$ be the irreducible components of $V$, for some $r \in \mathbb{N}$. Letting $W_i := \overline{\varphi(V_i)} \subseteq W$, for $i \in \{1,\ldots,r\}$, we have $\bigcup_{i=1}^r W_i = \overline{\varphi(V)} = W$, and there are $Z_i \subseteq W$ open such that $\emptyset \neq Z_i \cap W_i \subseteq \varphi(V_i)$, in particular $Z_i \cap W_i \subseteq W_i$ is dense. Let $U_i := Z_i \cap \bigcap_{j\neq i}(W \setminus W_j) \subseteq W$ open, hence $U_i \subseteq Z_i \cap W_i \subseteq \varphi(V_i) \subseteq W_i$. If $U_i \neq \emptyset$ then $\overline{U_i} = W_i$, while if $U_i = \emptyset$ then $Z_i \cap W_i \subseteq Z_i \subseteq \bigcup_{j\neq i} W_j$, implying

that $W_i = \overline{Z_i \cap W_i} \subseteq \bigcup_{j \neq i} W_j$ anyway. Hence $\bigcup_{i=1}^r U_i \subseteq \varphi(V) \subseteq W$ is open and $\overline{\bigcup_{i=1}^r U_i} = \bigcup_{i=1}^r \overline{U_i} = \bigcup_{i=1}^r W_i = W$. ♯

For more details on images of morphisms, in particular **Chevalley's Theorem (1955)**, see Exercises (11.20) and (11.21) as well as (11.22) and (11.23). For an application of the dimension formulas see Exercise (11.24).

## 3 Derivations

**(3.1) Definition and Remark. a)** Let $R$ be a ring, $A$ be an $R$-algebra, and $M$ be an $A$-module. An $R$-linear map $\delta\colon A \to M$ obeying the **product rule** $\delta(ab) = \delta(a)b + \delta(b)a$, for $a, b \in A$, is called an **$R$-derivation** of $A$ with values in $M$. The set $\mathrm{Der}_R(A, M) := \{\delta\colon A \to M\ R\text{-derivation}\}$ becomes an $A$-module via $\delta c\colon a \mapsto \delta(a)c$, for all $c \in A$.

In particular, $\delta \in \mathrm{Der}_R(A, M)$ is uniquely determined by its values on an $R$-algebra generating set of $A$. Moreover, from $\delta(1) = \delta(1 \cdot 1) = \delta(1) + \delta(1)$ we get $\delta(1) = 0$, and thus for $a \in A^*$ we obtain $0 = \delta(1) = \delta(aa^{-1}) = \delta(a)a^{-1} + \delta(a^{-1})a$, implying the **quotient rule** $\delta(a^{-1}) = -\delta(a)a^{-2}$.

**b)** An $R$-module $A$ together with an $R$-bilinear map $[\cdot,\cdot]\colon A \times A \to A$ such that $[a, a] = 0$, and such that the **Jacobi identity** $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ holds, for all $a, b, c \in A$, is called a **Lie ($R$-)algebra**.

E. g. any non-commutative associative $R$-algebra becomes a Lie algebra with respect to $[a, b] := ab - ba$, where we only have to check the Jacobi identity: We have $[[a, b], c] + [[b, c], a] + [[c, a], b] = (ab - ba)c - c(ab - ba) + (bc - cb)a - a(bc - cb) + (ca - ac)b - b(ca - ac) = 0$.

**c)** Considering $A$ as the regular $A$-module, $\mathrm{Der}_R(A, A)$ becomes a Lie algebra with respect to $[\delta, \delta'] := \delta\delta' - \delta'\delta$: Since $\mathrm{End}_R(A)$ is a Lie algebra, we only have to show that the Lie product restricts to $\mathrm{Der}_R(A, A) \subseteq \mathrm{End}_R(A)$: We have $[\delta, \delta'](ab) = \delta\delta'(ab) - \delta'\delta(ab) = \delta(\delta'(a)b + a\delta'(b)) - \delta'(\delta(a)b + a\delta(b)) = (\delta\delta'(a)b + \delta'(a)\delta(b)) + (a\delta\delta'(b) + \delta(a)\delta'(b)) - (\delta'\delta(a)b + \delta(a)\delta'(b)) - (a\delta'\delta(b) + \delta'(a)\delta(b)) = (\delta\delta'(a)b - \delta'\delta(a)b) + (a\delta\delta'(b) - a\delta'\delta(b)) = [\delta, \delta'](a)b + a[\delta, \delta'](b)$, for all $a, b \in A$.

Moreover, for $\delta \in \mathrm{Der}_R(A, A)$ and $n \in \mathbb{N}$ we have the **Leibniz rule** $\delta^n(ab) = \sum_{i=0}^n \binom{n}{i} \delta^i(a)\delta^{n-i}(b)$: This by definition holds for $n = 1$, and by induction $n \in \mathbb{N}$ we get $\delta^{n+1}(ab) = \sum_{i=0}^n \binom{n}{i} \delta(\delta^i(a)\delta^{n-i}(b)) = \sum_{i=0}^n \binom{n}{i} (\delta^{i+1}(a)\delta^{n-i}(b) + \delta^i(a)\delta^{n-i+1}(b)) = \sum_{i=1}^{n+1} \binom{n}{i-1} \delta^i(a)\delta^{n+1-i}(b) + \sum_{i=0}^n \binom{n}{i} \delta^i(a)\delta^{n+1-i}(b)$, which yields $\delta^{n+1}(ab) = \sum_{i=0}^{n+1} \binom{n+1}{i} \delta^i(a)\delta^{n+1-i}(b)$. The Leibniz rule implies that for $\mathrm{char}(\mathbb{K}) = p > 0$ the Lie-algebra $\mathrm{Der}_{\mathbb{K}}(A, A)$ is **restricted**, see Exercise (12.28).

**d)** Let $\mathcal{X} := \{X_1, \ldots, X_n\}$ be indeterminates over $R$. For $i \in \{1, \ldots, n\}$ let $\partial_i\colon R[\mathcal{X}] \to R[\mathcal{X}]\colon f \mapsto \frac{\partial f}{\partial X_i}$ be the **partial derivative** with respect to $X_i$. It is immediate that $\partial_i \in \mathrm{Der}_R(R[\mathcal{X}], R[\mathcal{X}])$. See also Exercise (11.26).

Let $M$ be an $R[\mathcal{X}]$-module, and let $v \in M$. For $i \in \{1, \ldots, n\}$ let $\partial_i^\bullet(v) \in \mathrm{Hom}_R(R[\mathcal{X}], M)$ be defined by $f \mapsto v\partial_i(f)$. Since $\partial_i^\bullet(v)(fg) = v\partial_i(fg) =$

$v(\partial_i(f)g + f\partial_i(g)) = \partial_i^\bullet(v)(f) \cdot g + \partial_i^\bullet(v)(g) \cdot f$, for all $f, g \in R[\mathcal{X}]$, we have $\partial_i^\bullet(v) \in \operatorname{Der}_R(R[\mathcal{X}], M)$. Since any $\delta \in \operatorname{Der}_R(R[\mathcal{X}], M)$ is uniquely determined by $\delta(X_1), \ldots, \delta(X_n) \in M$, we get $\delta = \sum_{i=1}^n \partial_i^\bullet(\delta(X_i)) \colon f \mapsto \sum_{i=1}^n \delta(X_i)\partial_i(f)$. In particular, we infer that $\{\partial_1, \ldots, \partial_n\} \subseteq \operatorname{Der}_R(R[\mathcal{X}], R[\mathcal{X}])$ is an $R[\mathcal{X}]$-basis.

For $x \in R^n$ the evaluation map $\epsilon_x^* \colon R[\mathcal{X}] \to R \colon f \mapsto f(x)$ induces a $R[\mathcal{X}]$-module structure on $R$, which is denoted by $R_x$. For $\partial_i^\bullet(x) \colon R[\mathcal{X}] \to R \colon f \mapsto \partial_i(f)(x)$, where $i \in \{1, \ldots, n\}$, it is immediate that $\partial_i^\bullet(x) \in \operatorname{Der}_R(R[\mathcal{X}], R_x)$. Letting $\partial_x \colon R[\mathcal{X}] \to R[\mathcal{X}]_1 \colon f \mapsto \sum_{i=1}^n X_i \cdot \partial_i^\bullet(x)(f) = \sum_{i=1}^n X_i \cdot \partial_i(f)(x)$ be the **total differential** at $x$, it is immediate that $\partial_x \in \operatorname{Der}_R(R[\mathcal{X}], R[\mathcal{X}]_{1,x})$. For all $t \in R^n$ we have the **Taylor expansion** $f(x+tY) = f(x) + \partial_x(f)(t) \cdot Y + g \cdot Y^2 \in R[Y]$, for some $g \in R[Y]$, where $Y$ is an indeterminate over $R$.

For a treatment using **Kähler differentials** see [11, Ch.3.2] or [8, Ch.AG.15].

**(3.2) Definition and Remark. a)** Let $V$ be an affine variety with affine coordinate algebra $\mathbb{K}[V]$, and let $P_x \lhd \mathbb{K}[V]$ be the maximal ideal associated to $x \in V$. Again the evaluation map $\epsilon_x^* \colon \mathbb{K}[V] \to \mathbb{K} \colon f \mapsto f(x)$, whose kernel is $P_x$, induces a $\mathbb{K}[V]$-module structure on $\mathbb{K}$, which is denoted by $\mathbb{K}_x$.

The localisation $\mathcal{O}_x = \mathcal{O}_{V,x} := \mathbb{K}[V]_{P_x} := \mathbb{K}[V]_{\mathbb{K}[V] \setminus P_x}$ is called the **local ring** of $V$ at $x$. We have the $\mathbb{K}$-algebra homomorphism $\nu_x = \nu_{V,x} \colon \mathbb{K}[V] \to \mathcal{O}_x \colon f \mapsto \frac{f}{1}$, see Exercise (11.12). In general, $\nu_x$ is not necessarily injective; but if $\mathbb{K}[V]$ is an integral domain, i. e. if $V$ is irreducible, we have $\mathbb{K}[V] \subseteq \mathcal{O}_x \subseteq \mathbb{K}(V)$.

We have an inclusion-preserving bijection $\{Q \lhd \mathbb{K}[V] \text{ prime}; Q \subseteq P_x\} \to \{\mathcal{Q} \lhd \mathcal{O}_x \text{ prime}\} \colon Q \mapsto Q_{P_x}$ with inverse map $\mathcal{Q} \mapsto \nu_x^{-1}(\mathcal{Q})$. Hence $\mathcal{O}_x$ is a local ring with unique maximal ideal $\mathcal{P}_x = \mathcal{P}_{V,x} := (P_x)_{P_x} \lhd \mathcal{O}_x$. Since the elements of $\mathbb{K}[V] \setminus P_x$ act invertibly on $\mathbb{K}_x$, by the universal property of localisations we get a $\mathbb{K}$-algebra epimorphism $\mathcal{O}_x \to \mathbb{K}$, hence $\mathcal{O}_x$ has residue class field $\mathcal{O}_x/\mathcal{P}_x \cong \mathbb{K}$, and $\mathbb{K}_x$ becomes an $\mathcal{O}_x$-module, where the $\mathbb{K}[V]$-action is recovered by restriction through $\nu_x$.

**b)** If $x$ is only contained in a single irreducible component $W \subseteq V$, we may reduce to the irreducible case as follows: Letting $P := \mathcal{I}(W) \lhd \mathbb{K}[V]$ be the associated prime ideal, we infer that $P_{P_x} \lhd \mathcal{O}_x$ is the unique minimal prime ideal, which since $\mathcal{O}_x$ is Noetherian is nilpotent.

Since for the $\mathbb{K}$-algebra homomorphism $\nu_{V,W,x} \colon \mathbb{K}[V] \to \mathbb{K}[W] \cong \mathbb{K}[V]/P \to (\mathbb{K}[V]/P)_{P_x/P} \cong \mathcal{O}_{W,x}$ we have $\nu_{V,W,x}(\mathbb{K}[V] \setminus P_x) \subseteq \mathcal{O}_{W,x}^*$, by the universal property of localisations we have a $\mathbb{K}$-algebra epimorphism $\widehat{\nu}_{V,W,x} \colon \mathcal{O}_x = \mathbb{K}[V]_{P_x} \to \mathcal{O}_{W,x}$. Since $\mathcal{O}_{W,x} \subseteq \mathbb{K}(W)$ is a domain, $\ker(\widehat{\nu}_{V,W,x}) \lhd \mathcal{O}_x$ is prime, which by the description of the prime ideals in $\mathcal{O}_x$ implies $\ker(\widehat{\nu}_{V,W,x}) = P_{P_x}$. Hence we conclude $\mathcal{O}_x/P_{P_x} \cong \mathcal{O}_{W,x}$, and in particular we have $\dim(\mathcal{O}_x) = \dim(\mathcal{O}_{W,x})$.

**c)** For $0 \neq f \in \mathbb{K}[V]$ let $V_f \subseteq V$ be the associated principal open subset, and let $\varphi_f^* \colon \mathbb{K}[V] \to \mathbb{K}[V]_f \cong \mathbb{K}[V_f] \colon g \mapsto \frac{g}{1}$ be the associated natural map, where $\varphi_f \colon V_f \to V$ is the inclusion map, see (1.14). For $x \in V_f$ we have $f \notin P_x \lhd \mathbb{K}[V]$, and thus $\mathcal{O}_x := \mathbb{K}[V]_{P_x} \cong (\mathbb{K}[V]_f)_{(P_x)_f} =: \mathcal{O}_{V_f,x}$, see Exercise (11.12), where the isomorphism is induced by $\varphi_f^*$.

**(3.3) Definition.** Let $V$ be an affine variety and $x \in V$. The $\mathcal{O}_x/\mathcal{P}_x$-module $T_x^*(V) := \mathcal{P}_x/\mathcal{P}_x^2$ becomes a $\mathbb{K}$-vector space, called the **cotangent-tangent space** of $V$ at $x$. Since $\mathcal{P}_x \lhd \mathcal{O}_x$ is finitely generated we have $\dim_{\mathbb{K}}(T_x^*(V)) < \infty$.

Its $\mathbb{K}$-dual space $T_x(V) := (T_x^*(V))^* = (\mathcal{P}_x/\mathcal{P}_x^2)^* := \mathrm{Hom}_{\mathbb{K}}(\mathcal{P}_x/\mathcal{P}_x^2, \mathbb{K})$ is called the **tangent space** of $V$ at $x$. The latter becomes an $\mathcal{O}_x$-module via $\mathcal{O}_x \to \mathcal{O}_x/\mathcal{P}_x \cong \mathbb{K}$, which is denoted by the subscript in $\mathrm{Hom}_{\mathbb{K}}(\mathcal{P}_x/\mathcal{P}_x^2, \mathbb{K}_x)$.

In particular, if $x$ is only contained in a single irreducible component $W \subseteq V$, we have $T_x^*(V) = \mathcal{P}_x/\mathcal{P}_x^2 \cong \mathcal{P}_{W,x}/\mathcal{P}_{W,x}^2 = T_x^*(W)$, and thus $T_x(V) \cong T_x(W)$. Moreover, for $0 \neq f \in \mathbb{K}[V]$ we have $T_x^*(V) = \mathcal{P}_x/\mathcal{P}_x^2 \cong \mathcal{P}_{V_f,x}/\mathcal{P}_{V_f,x}^2 = T_x^*(V_f)$, and thus $T_x(V) \cong T_x(V_f)$.

**(3.4) Proposition.** Let $V$ be an affine variety and let $x \in V$.
**a)** Restriction to $\mathcal{P}_x$ yields an isomorphism of $\mathcal{O}_x$-modules $\mathrm{Der}_{\mathbb{K}}(\mathcal{O}_x, \mathbb{K}_x) \to \mathrm{Hom}_{\mathbb{K}}(\mathcal{P}_x/\mathcal{P}_x^2, \mathbb{K}_x)$. Similarly, restriction to $P_x$ yields an isomorphism of $\mathbb{K}[V]$-modules $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[V], \mathbb{K}_x) \to \mathrm{Hom}_{\mathbb{K}}(P_x/P_x^2, \mathbb{K}_x)$.
**b)** The map $\nu_x^* \colon \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_x, \mathbb{K}_x) \to \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[V], \mathbb{K}_x) \colon \delta \mapsto \delta\nu_x$ is an isomorphism of $\mathbb{K}$-vector spaces; see also Exercise (11.25).

**Proof. a)** We only prove the first assertion, the second one follows similarly: Let $\delta \in \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_x, \mathbb{K}_x)$. Then for $f, g \in \mathcal{P}_x$ we have $\delta(fg) = \delta(f)g(x) + f(x)\delta(g) = 0$, hence $\mathcal{P}_x^2 \subseteq \ker(\delta)$, thus $\delta|_{\mathcal{P}_x} \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{P}_x/\mathcal{P}_x^2, \mathbb{K}_x)$ is well-defined. Since $\delta(1) = 0$ and $\mathcal{O}_x = \mathbb{K} \cdot 1 \oplus \mathcal{P}_x$ as $\mathbb{K}$-vector spaces, $\delta$ is uniquely determined by $\delta|_{\mathcal{P}_x}$, implying that the restriction map is injective.

Conversely, for $\widetilde{\delta} \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{P}_x/\mathcal{P}_x^2, \mathbb{K}_x)$ let $\delta \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{O}_x, \mathbb{K}_x)$ be defined by $\delta(\alpha + f) := \widetilde{\delta}(f)$, where $\alpha \in \mathbb{K}$ and $f \in \mathcal{P}_x$. Then for $\alpha, \beta \in \mathbb{K}$ and $f, g \in \mathcal{P}_x$ we have $\delta((\alpha+f)(\beta+g)) = \delta(\alpha\beta + \alpha g + f\beta + fg) = \alpha\widetilde{\delta}(g) + \widetilde{\delta}(f)\beta = (\alpha+f)(x) \cdot \delta(\beta+g) + \delta(\alpha+f) \cdot (\beta+g)(x)$, hence $\delta \in \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_x, \mathbb{K}_x)$.

**b)** Any $\mathbb{K}$-derivation $\mathcal{O}_x \to \mathbb{K}_x$ by the quotient rule is uniquely determined by its values on $\nu_x(\mathbb{K}[V])$, hence $\nu_x^*$ is injective. Moreover, given $\delta \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[V], \mathbb{K}_x)$, we let $\widehat{\delta}(\frac{f}{g}) := \frac{\delta(f)}{g(x)} - \frac{f(x)\delta(g)}{g(x)^2} \in \mathbb{K}_x$, for $f \in \mathbb{K}[V]$ and $g \in \mathbb{K}[V] \setminus P_x$. We show that this is well-defined; it is immediate then that $\widehat{\delta} \in \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_x, \mathbb{K}_x)$ and $\nu_x^*(\widehat{\delta}) = \delta$, implying that $\nu_x^*$ is surjective as well:

Let $f' \in \mathbb{K}[V]$ and $g', h \in \mathbb{K}[V] \setminus P_x$ such that $(fg' - f'g)h = 0$. This implies $fg' - f'g \in P_x$ as well as $\delta(fg') \cdot h(x) + (fg')(x) \cdot \delta(h) = \delta(fg'h) = \delta(f'gh) = \delta(f'g) \cdot h(x) + (f'g)(x) \cdot \delta(h)$, hence $(\delta(f'g) - \delta(fg')) \cdot h(x) = (fg' - f'g)(x) \cdot \delta(h) = 0$ and thus $\delta(f')g(x) + f'(x)\delta(g) = \delta(f'g) = \delta(fg') = \delta(f)g'(x) + f(x)\delta(g')$, hence $\frac{\delta(f')}{g'(x)} - \frac{f'(x)\delta(g')}{g'(x)^2} = \frac{\delta(f')}{g'(x)} - \frac{f(x)\delta(g')}{g(x)g'(x)} = \frac{\delta(f')}{g'(x)} - \frac{\delta(f')g(x) + f'(x)\delta(g) - \delta(f)g'(x)}{g(x)g(x)'} = \frac{\delta(f)}{g(x)} - \frac{f'(x)\delta(g)}{g(x)g'(x)} = \frac{\delta(f)}{g(x)} - \frac{f(x)\delta(g)}{g(x)^2}$. ♯

**(3.5) Proposition.** Let $V \subseteq \mathbb{K}^n$ be closed and let $\mathcal{T}_x(V) = \mathcal{T}_{\mathbb{K}^n,x}(V) := \mathcal{V}(\partial_x(\mathcal{I}(V)) \subseteq \mathbb{K}^n$, for $x \in V$, be the **Zariski tangent space** of $V \subseteq \mathbb{K}^n$ at $x$. Then the map $\partial_x^\bullet \colon \mathcal{T}_x(V) \to \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}]/\mathcal{I}(V), \mathbb{K}_x) \cong \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[V], \mathbb{K}_x) \colon t \mapsto \partial_x^\bullet(t) \colon \overline{f} \mapsto \partial_x(f)(t)$ is an isomorphism of $\mathbb{K}$-vector spaces, where $\overline{\phantom{x}} \colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}]/\mathcal{I}(V) \cong \mathbb{K}[V]$ is the natural map.

**Proof.** For all $t \in \mathbb{K}^n$ and $f, g \in \mathbb{K}[\mathcal{X}]$ we get $\partial_x^\bullet(t)(fg) = \partial_x(fg)(t) = (\partial_x(f)g(x) + f(x)\partial_x(g))(t) = \partial_x^\bullet(t)(f) \cdot g(x) + f(x) \cdot \partial_x^\bullet(t)(g)$, hence $\partial_x^\bullet(t) \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}], \mathbb{K}_x)$. Since $\partial_x(f)(t) = 0$ for all $t \in \mathcal{T}_x(V)$ and $f \in \mathcal{I}(V)$, we conclude $\partial_x^\bullet(t) \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}]/\mathcal{I}(V), \mathbb{K}_x)$ for all $t \in \mathcal{T}_x(V)$. From $\mathrm{im}(\partial_x) \subseteq \mathbb{K}[\mathcal{X}]_1$ we get that $\mathcal{T}_x(V) \leq \mathbb{K}^n$ is a $\mathbb{K}$-subspace, and that $\partial_x^\bullet \colon t \mapsto \partial_x^\bullet(t)$ is a $\mathbb{K}$-linear map. For $t = [t_1, \ldots, t_n] \in \ker(\partial_x^\bullet)$ we have $0 = \partial_x(X_j)(t) = (\sum_{i=1}^n X_i \cdot \partial_i(X_j)(x))(t) = t_j$, for all $j \in \{1, \ldots, n\}$, thus $t = 0$ and $\partial_x^\bullet$ is injective.

Moreover, for $\delta \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}]/\mathcal{I}(V), \mathbb{K}_x)$ let $t := [\delta(\overline{X_1}), \ldots, \delta(\overline{X_n})] \in \mathbb{K}^n$. Lifting $\delta$ to $\widehat{\delta} \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}], \mathbb{K}_x)$, via the natural map $\overline{\phantom{x}} \colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}]/\mathcal{I}(V)$, we get $\widehat{\delta} = \sum_{i=1}^n \partial_i^\bullet(\widehat{\delta}(X_i))$. Thus for all $f \in \mathbb{K}[\mathcal{X}]$ we obtain $\widehat{\delta}(f) = \sum_{i=1}^n \widehat{\delta}(X_i) \cdot \partial_i(f)(x) = \partial_x(f)([\widehat{\delta}(X_1), \ldots, \widehat{\delta}(X_n)]) = \partial_x(f)(t)$, hence $\widehat{\delta} = \partial_x^\bullet(t)$. Finally, for all $f \in \mathcal{I}(V)$ we have $0 = \widehat{\delta}(f) = \partial_x(f)(t)$, hence $t \in \mathcal{T}_x(V)$, and thus $\partial_x^\bullet$ is surjective as well. ♯

**(3.6) Corollary.** Let $V \subseteq \mathbb{K}^n$ be closed and $\mathcal{I}(V) = \langle f_1, \ldots, f_r \rangle \lhd \mathbb{K}[\mathcal{X}]$, for some $r \in \mathbb{N}$. Then for $x \in V$ we have $\mathcal{T}_x(V) = \mathcal{V}(\partial_x(f_1), \ldots, \partial_x(f_r)) \subseteq \mathbb{K}^n$. Hence $\mathcal{T}_x(V) = \ker(J(f_1, \ldots, f_r)(x)) \leq \mathbb{K}^n$, where $J(f_1, \ldots, f_r) := [\partial_i(f_j)]_{ij} \in K[\mathcal{X}]^{n \times r}$ is the associated **Jacobian matrix**.

**Proof.** For $f \in \mathcal{I}(V)$ there are $g_1, \ldots, g_r \in \mathbb{K}[\mathcal{X}]$ such that $f = \sum_{j=1}^r f_j g_j \in \mathbb{K}[\mathcal{X}]$, implying $\partial_x(f) = \sum_{j=1}^r (\partial_x(f_j) g_j(x) + f_j(x) \partial_x(g_j)) = \sum_{j=1}^r \partial_x(f_j) g_j(x) \in \langle \partial_x(f_1), \ldots, \partial_x(f_r) \rangle_{\mathbb{K}} \subseteq \mathbb{K}[\mathcal{X}]_1$. ♯

**(3.7) Definition.** Let $V, W$ be affine varieties, let $\varphi \colon V \to W$ be a morphism, and let $x \in V$. Then for $\delta \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[V], \mathbb{K}_x)$ we have $\delta\varphi^*(fg) = \delta(\varphi^*(f)\varphi^*(g)) = \delta((f\varphi)(g\varphi)) = \delta(f\varphi) \cdot g\varphi(x) + f\varphi(x) \cdot \delta(g\varphi) = \delta\varphi^*(f) \cdot g(y) + f(y) \cdot \delta\varphi^*(g)$, for all $f, g \in \mathbb{K}[W]$, thus $\delta\varphi^* \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[W], \mathbb{K}_{\varphi(x)})$.

The $\mathbb{K}$-linear map $d_x(\varphi) \colon \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[V], \mathbb{K}_x) \to \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[W], \mathbb{K}_{\varphi(x)}) \colon \delta \mapsto \delta\varphi^*$ is called the **differential** of $\varphi$ at $x$.

For $\mathrm{id}_V$ we have $d_x(\mathrm{id}_V) = \mathrm{id}_{\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[V], \mathbb{K}_x)}$; and if $W$ is an affine variety and $\psi \colon W \to Z$ is a morphism, then we have the **chain rule** $d_x(\psi\varphi) = d_{\varphi(x)}(\psi) d_x(\varphi)$ In particular, if $\varphi$ is an isomorphism of varieties, then $d_x(\varphi)$ is an isomorphism of $\mathbb{K}$-vector spaces, for all $x \in V$. For tangent spaces of $\mathbb{K}$-vector spaces and differentials of $\mathbb{K}$-linear maps see Exercise (11.28).

**(3.8) Remark. a)** Let $V \subseteq \mathbb{K}^n$ and $W \subseteq \mathbb{K}^m$ be closed, and let $\mathcal{X} := \{X_1, \ldots, X_n\}$ as well as $\mathcal{Y} := \{Y_1, \ldots, Y_m\}$, and let the morphism $\varphi \colon V \to W$ be given by $f_1, \ldots, f_m \in \mathbb{K}[\mathcal{X}]$. Then for $x \in V$ the $\mathbb{K}$-linear map $d_x(\varphi)\colon \mathcal{T}_x(V) \to \mathcal{T}_{\varphi(x)}(W)$ between the associated Zariski tangent spaces is given by the matrix $J(f_1, \ldots, f_m)(x) \in \mathbb{K}^{n\times m}$, where $J(f_1, \ldots, f_m) := [\partial_i(f_j)]_{ij} \in \mathbb{K}[\mathcal{X}]^{n\times m}$ is the associated Jacobian matrix:

Let $t = [t_1, \ldots, t_n] \in \mathcal{T}_x(V) \leq \mathbb{K}^n$ and $d_x(\varphi)(\partial_x^\bullet(t)) = \partial_{\varphi(x)}^\bullet(u)$, for a unique $u = [u_1, \ldots, u_m] \in \mathcal{T}_\varphi(x)(W) \leq \mathbb{K}^m$. Then for $j \in \{1, \ldots, m\}$ we have $u_j = \partial_{\varphi(x)}^\bullet(u)(\overline{Y_j}) = \partial_x^\bullet(t)\varphi^*(\overline{Y_j}) = \partial_x^\bullet(t)(\overline{f_j}) = \partial_x(f_j)(t) = \sum_{i=1}^n t_i \cdot \partial_i(f_j)(x) = [t \cdot J(f_1, \ldots, f_m)(x)]_j$, where both the natural maps $\mathbb{K}[\mathcal{X}] \to \mathbb{K}[V]$ and $\mathbb{K}[\mathcal{Y}] \to \mathbb{K}[W]$ are denoted by $\overline{\phantom{x}}$, and where $[\ldots]_j$ denotes the $j$-th entry of $[\ldots] \in \mathbb{K}^m$.

**b)** Letting $f \in \mathbb{K}[\mathcal{X}]$, the element $\overline{f} \in \mathbb{K}[V]$ can be considered as a morphism $\overline{f}\colon V \to \mathbb{K}$, with comorphism $\overline{f}^*\colon \mathbb{K}[Y] \to \mathbb{K}[V]\colon Y \mapsto \overline{f}$. Since $\mathcal{I}(\mathbb{K}) = \{0\} \lhd \mathbb{K}[Y]$, the Zariski tangent space of $\mathbb{K}$ at $y \in \mathbb{K}$ is given as $\mathcal{T}_y(\mathbb{K}) = \mathcal{V}(\{0\}) = \mathbb{K}$.

Using this identification we for $x \in V$ get $d_x(\overline{f})\colon \mathcal{T}_x(V) \to \mathcal{T}_{f(x)}(\mathbb{K}) \cong \mathbb{K}$, whose matrix is given by $[\partial_i(f)(x)] \in \mathbb{K}^{n\times 1}$; in particular we have $d_x(\overline{f}) \in T_x^*(V)$. Considering $\partial_x(f)$ as a $\mathbb{K}$-linear form on $\mathcal{T}_x(V)$, i. e. $\partial_x(f)|_{\mathcal{T}_x(V)} \in \mathcal{T}_x^*(V)$, we also obtain the matrix $[\partial_i(f)(x)] \in \mathbb{K}^{n\times 1}$. Thus we have an identification of the total differential $\partial_x(f)|_{\mathcal{T}_x(V)}$ and the differential $d_x(\overline{f})$.

**c)** Let $W \subseteq V$ be closed, with associated ideal $\mathcal{I}(W) = \langle f_1, \ldots, f_r\rangle \lhd \mathbb{K}[V]$, for some $r \in \mathbb{N}$, and let $x \in W$. Letting $\varphi\colon W \to V$ be the natural embedding, the associated comorphism $\varphi^*\colon \mathbb{K}[V] \to \mathbb{K}[V]/\mathcal{I}(W) \cong \mathbb{K}[W]$ is the natural epimorphism. Hence the differential $d_x(\varphi)\colon \mathrm{Der}_\mathbb{K}(\mathbb{K}[W], \mathbb{K}_x) \to \mathrm{Der}_\mathbb{K}(\mathbb{K}[V], \mathbb{K}_x)$ is injective, having image $\mathrm{im}(d_x(\varphi)) = \{\delta \in \mathrm{Der}_\mathbb{K}(\mathbb{K}[V], \mathbb{K}_x); \delta(\mathcal{I}(W)) = \{0\}\}$. Thus we have $T_x(W) \leq T_x(V)$ as $\mathbb{K}$-vector spaces, and using the closed embedding $W \subseteq V \subseteq \mathbb{K}^n$, we deduce that $T_x(W) = \bigcap_{f\in\mathcal{I}(W)} \ker(d_x(f)) = \bigcap_{j\in\{1,\ldots,r\}} \ker(d_x(f_j)) \leq T_x(V)$.

**d)** Let $x \in V$ and $y \in W$. Then we have $\mathbb{K}_{[x,y]} \cong \mathbb{K}_x \otimes_\mathbb{K} \mathbb{K}_y \cong \mathbb{K}$ as well as $T_{[x,y]}(V \times W) \cong \mathrm{Der}_\mathbb{K}(\mathbb{K}[V] \otimes_\mathbb{K} \mathbb{K}[W], \mathbb{K}_{[x,y]}) \cong \mathrm{Der}_\mathbb{K}(\mathbb{K}[V], \mathbb{K}_x) \oplus \mathrm{Der}_\mathbb{K}(\mathbb{K}[W], \mathbb{K}_y) \cong T_x(V) \oplus T_y(W)$ as $\mathbb{K}$-vector spaces, see Exercise (11.27):

The injective comorphisms associated to the natural projections $\pi\colon V\times W \to V$ and $\pi'\colon V \times W \to W$ are given as $\pi^*\colon \mathbb{K}[V] \to \mathbb{K}[V] \otimes_\mathbb{K} \mathbb{K}[W]\colon f \mapsto f \otimes 1$ and $\pi'^*\colon \mathbb{K}[W] \to \mathbb{K}[V] \otimes_\mathbb{K} \mathbb{K}[W]\colon g \mapsto 1 \otimes g$. Hence we have an induced $\mathbb{K}$-linear map $d_x(\pi) \oplus d_y(\pi')\colon \mathrm{Der}_\mathbb{K}(\mathbb{K}[V] \otimes_\mathbb{K} \mathbb{K}[W], \mathbb{K}_{[x,y]}) \to \mathrm{Der}_\mathbb{K}(\mathbb{K}[V], \mathbb{K}_x) \oplus \mathrm{Der}_\mathbb{K}(\mathbb{K}[W], \mathbb{K}_y)\colon \gamma \mapsto [\gamma|_{\mathbb{K}[V]}, \gamma|_{\mathbb{K}[W]}]$. Conversely, the comorphisms associated to the natural embeddings $\epsilon\colon V \to V \times W\colon z \mapsto [z, y]$ and $\epsilon'\colon W \to V \times W\colon z \mapsto [x, z]$ are given as $\epsilon^*\colon \mathbb{K}[V] \otimes_\mathbb{K} \mathbb{K}[W] \to \mathbb{K}[V]\colon f \otimes g \mapsto fg(y)$ and $\epsilon'^*\colon \mathbb{K}[V] \otimes_\mathbb{K} \mathbb{K}[W] \to \mathbb{K}[W]\colon f \otimes g \mapsto f(x)g$. Hence we have an induced $\mathbb{K}$-linear map $d_x(\epsilon) \oplus d_y(\epsilon')\colon \mathrm{Der}_\mathbb{K}(\mathbb{K}[V], \mathbb{K}_x) \oplus \mathrm{Der}_\mathbb{K}(\mathbb{K}[W], \mathbb{K}_y) \to \mathrm{Der}_\mathbb{K}(\mathbb{K}[V] \otimes_\mathbb{K} \mathbb{K}[W], \mathbb{K}_{[x,y]})\colon [\delta, \delta'] \mapsto \delta \bullet \delta'\colon f \otimes g \mapsto \delta(f)g(y) + f(x)\delta'(g)$.

For $\gamma \in \mathrm{Der}_\mathbb{K}(\mathbb{K}[V]\otimes_\mathbb{K}\mathbb{K}[W], \mathbb{K}_{[x,y]})$ we have $(\gamma|_{\mathbb{K}[V]} \bullet \gamma|_{\mathbb{K}[W]})(f\otimes g) = \gamma(f)g(y) + f(x)\gamma(g) = \gamma((f \otimes 1)(1 \otimes g)) = \gamma(f \otimes g)$, and for $\delta \in \mathrm{Der}_\mathbb{K}(\mathbb{K}[V], \mathbb{K}_x)$ and $\delta' \in \mathrm{Der}_\mathbb{K}(\mathbb{K}[W], \mathbb{K}_y)$ we get $(\delta \bullet \delta')|_{\mathbb{K}[V]}(f) = \delta(f)1(y) + f(x)\delta'(1) = \delta(f)$ and $(\delta \bullet \delta')|_{\mathbb{K}[W]}(g) = \delta(1)g(y) + 1(x)\delta'(g) = \delta'(g)$, for all $f \in \mathbb{K}[V]$ and $g \in \mathbb{K}[W]$. ♯

**e)** Let $\varphi\colon V \to W$ be a morphism, and let $0 \neq f \in \mathbb{K}[V]$ as well as $0 \neq g \in \mathbb{K}[W]$ such that $\varphi$ restricts to a morphism $\varphi|_{V_f}\colon V_f \to W_g$. Then for $x \in V_f$ we have $T_x(V_f) \cong T_x(V)$ and $T_{\varphi(x)}(W_f) \cong T_{\varphi(x)}(W)$, where the isomorphisms are induced by $\varphi_f^*$ and $\varphi_g^*$, respectively. Since both $d_x(\varphi|_{V_f})$ and $d_x(\varphi)$ are induced by $\varphi^*$, the differential $d_x(\varphi|_{V_f})\colon T_x(V_f) \to T_{\varphi(x)}(W_g)$ can be identified with $d_x(\varphi)\colon T_x(V) \to T_{\varphi(x)}(W)$.

More explicitly, for $x \in V_f$ we show how to define the Zariski tangent space $\mathcal{T}_x(V_f)$, and how to identify it with $\mathcal{T}_x(V)$, see Exercise (11.29):

Let $\widehat{f} \in \mathbb{K}[\mathcal{X}]$ such that $\overline{\widehat{f}} = f \in \mathbb{K}[V]$. We have a closed embedding $V_f \to \widehat{V}_f \subseteq \mathbb{K}^{n+1}\colon y \mapsto [f(y)^{-1}, y]$, with inverse $\widehat{V}_f \to V_f\colon [f(y)^{-1}, y] \mapsto y$, where $\mathbb{K}[V]_f \cong \mathbb{K}[\widehat{V}_f] \cong \mathbb{K}[\mathcal{X}, X_0]/\langle \mathcal{I}(V), \widehat{f}X_0 - 1\rangle$, see (1.15). While for $g \in \mathcal{I}(V) \lhd \mathbb{K}[\mathcal{X}]$ we have $\partial_{[f(x)^{-1},x]}(g) = \sum_{i=0}^n \partial_i(g)([f(x)^{-1}, x]) \cdot X_i = \sum_{i=1}^n \partial_i(g)(x) \cdot X_i = \partial_x(g)$, we moreover get $\partial_{[f(x)^{-1},x]}(\widehat{f}X_0 - 1) = \sum_{i=0}^n \partial_i(\widehat{f}X_0 - 1)([f(x)^{-1}, x]) \cdot X_i = f(x) \cdot X_0 + \sum_{i=1}^n \partial_i(\widehat{f})(x) \cdot f(x)^{-1} \cdot X_i$. Thus the projection map $\mathcal{T}_x(V_f) := \mathcal{T}_{[f(x)^{-1},x]}(\widehat{V}_f) \to \mathcal{T}_x(V)\colon [t_0, t_1, \ldots, t_n] \mapsto [t_1, \ldots, t_n]$ is an isomorphism. $\sharp$

**(3.9) Theorem.** Let $V$ be an irreducible affine variety.
**a)** For all $x \in V$ we have $\dim_{\mathbb{K}}(T_x(V)) \geq \dim(V)$.
**b)** The set $U := \{x \in V; \dim_{\mathbb{K}}(T_x(V)) = \dim(V)\} \subseteq V$ is non-empty and open.

The elements of $U$ are called **regular** points, the elements of $V \setminus U$ are called **singular** points, and if $V = U$ then $V$ is called **smooth**.

**Proof.** See [6, Thm.I.5.3], or Exercise (11.30) for (a) and part of (b). $\sharp$

A differential criterion for dominance is given in Exercise (11.32).

# II Algebraic groups

## 4 Affine algebraic groups

**(4.1) Definition and Remark. a)** An affine variety $\mathbb{G}$ over $\mathbb{K}$, endowed with a group structure such that multiplication $\mu = \mu_{\mathbb{G}} \colon \mathbb{G} \times \mathbb{G} \to \mathbb{G} \colon [x, y] \mapsto xy$ and inversion $\iota = \iota_{\mathbb{G}} \colon \mathbb{G} \to \mathbb{G} \colon x \mapsto x^{-1}$ are morphisms, is called an **affine** or **linear algebraic group** over $\mathbb{K}$, see (6.2), or just an **algebraic group** for short.

If $\mathbb{H}$ also is an algebraic group, then a morphism $\varphi \colon \mathbb{G} \to \mathbb{H}$ which also is a group homomorphism is called a **homomorphism of algebraic groups**.

Since the Zariski topology on $\mathbb{G} \times \mathbb{G}$ is finer that than the product topology induced by the Zariski topology on $\mathbb{G}$, see Exercise (11.9), in general $\mathbb{G}$ is not necessarily a **topological group**.

**b)** Letting $\epsilon_{\mathbb{G}} = \epsilon_{1_{\mathbb{G}}} \colon \{1_{\mathbb{G}}\} \to \mathbb{G} \colon 1_{\mathbb{G}} \mapsto 1_{\mathbb{G}}$ and $\nu_{\mathbb{G}} = \nu_{1_{\mathbb{G}}} \colon \mathbb{G} \to \{1_{\mathbb{G}}\} \colon g \mapsto 1_{\mathbb{G}}$, the group laws can be translated into commutative diagrams of affine varieties and of affine coordinate rings, respectively; see also Exercise (12.1):
**i)** Associativity: For all $x, y, z \in \mathbb{G}$ we have $(xy)z = x(yz)$.

$$\begin{tikzcd}
\mathbb{G} \times \mathbb{G} \times \mathbb{G} \arrow[r, "\mu_{\mathbb{G}} \times \mathrm{id}_{\mathbb{G}}"] \arrow[d, "\mathrm{id}_{\mathbb{G}} \times \mu_{\mathbb{G}}"'] & \mathbb{G} \times \mathbb{G} \arrow[d, "\mu_{\mathbb{G}}"] \\
\mathbb{G} \times \mathbb{G} \arrow[r, "\mu_{\mathbb{G}}"] & \mathbb{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}] \otimes \mathbb{K}[\mathbb{G}] & \mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}] \arrow[l, "\mu_{\mathbb{G}}^* \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{G}}^*"'] \\
\mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}] \arrow[u, "\mathrm{id}_{\mathbb{G}}^* \otimes \mu_{\mathbb{G}}^*"] & \mathbb{K}[\mathbb{G}] \arrow[l, "\mu_{\mathbb{G}}^*"'] \arrow[u, "\mu_{\mathbb{G}}^*"']
\end{tikzcd}$$

**ii)** Identity: For all $x \in \mathbb{G}$ we have $x \cdot 1_{\mathbb{G}} = x = 1_{\mathbb{G}} \cdot x$.

$$\begin{tikzcd}
\mathbb{G} \arrow[r, "(\epsilon_{\mathbb{G}}\nu_{\mathbb{G}}) \times \mathrm{id}_{\mathbb{G}}"] \arrow[d, "\mathrm{id} \times (\epsilon_{\mathbb{G}}\nu_{\mathbb{G}})"'] \arrow[dr, "\mathrm{id}_{\mathbb{G}}"] & \mathbb{G} \times \mathbb{G} \arrow[d, "\mu_{\mathbb{G}}"] \\
\mathbb{G} \times \mathbb{G} \arrow[r, "\mu_{\mathbb{G}}"] & \mathbb{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K}[\mathbb{G}] & \mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}] \arrow[l, "(\epsilon_{\mathbb{G}}\nu_{\mathbb{G}})^* \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{G}}^*"'] \\
\mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}] \arrow[u, "\mathrm{id}_{\mathbb{G}}^* \otimes (\epsilon_{\mathbb{G}}\nu_{\mathbb{G}})^*"] & \mathbb{K}[\mathbb{G}] \arrow[l, "\mu_{\mathbb{G}}^*"'] \arrow[u, "\mu_{\mathbb{G}}^*"'] \arrow[ul, "\mathrm{id}_{\mathbb{G}}^*"']
\end{tikzcd}$$

**iii)** Inversion: For all $x \in \mathbb{G}$ we have $x \cdot x^{-1} = 1_{\mathbb{G}} = x^{-1} \cdot x$.

$$\begin{tikzcd}
\mathbb{G} \arrow[r, "\iota_{\mathbb{G}} \times \mathrm{id}_{\mathbb{G}}"] \arrow[d, "\mathrm{id} \times \iota_{\mathbb{G}}"'] \arrow[dr, "\epsilon_{\mathbb{G}}\nu_{\mathbb{G}}"] & \mathbb{G} \times \mathbb{G} \arrow[d, "\mu_{\mathbb{G}}"] \\
\mathbb{G} \times \mathbb{G} \arrow[r, "\mu_{\mathbb{G}}"] & \mathbb{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K}[\mathbb{G}] & \mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}] \arrow[l, "\iota_{\mathbb{G}}^* \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{G}}^*"'] \\
\mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}] \arrow[u, "\mathrm{id}_{\mathbb{G}}^* \otimes \iota_{\mathbb{G}}^*"] & \mathbb{K}[\mathbb{G}] \arrow[l, "\mu_{\mathbb{G}}^*"'] \arrow[u, "\mu_{\mathbb{G}}^*"'] \arrow[ul, "(\epsilon_{\mathbb{G}}\nu_{\mathbb{G}})^*"']
\end{tikzcd}$$

**(4.2) Example: The additive and the multiplicative group.**
**a)** $\mathbb{G}_a := \mathbb{K}$ is an algebraic group, called the **additive group**, where $\mu \colon \mathbb{K}^2 \to \mathbb{K} \colon [x, y] \mapsto x + y$ and $\iota \colon \mathbb{K} \to \mathbb{K} \colon x \mapsto -x$ and $\epsilon \colon \{0\} \to \mathbb{K}$, yielding $\mu^* \colon \mathbb{K}[X] \to \mathbb{K}[X] \otimes_{\mathbb{K}} \mathbb{K}[X] \colon X \mapsto (X \otimes 1) + (1 \otimes X)$ and $\iota^* \colon \mathbb{K}[X] \to \mathbb{K}[X] \colon X \mapsto -X$ and $\epsilon^* \colon \mathbb{K}[X] \to \mathbb{K} \colon X \mapsto 0$.

Similarly, $\mathbb{K}^n$ is an additive algebraic group, where $\mu \colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n \colon [x, y] \mapsto x + y$ and $\iota \colon \mathbb{K}^n \to \mathbb{K}^n \colon x \mapsto -x$ and $\epsilon \colon \{0\} \to \mathbb{K}^n$, yielding $\mu^* \colon \mathbb{K}[\mathcal{X}] \to$

$\mathbb{K}[\mathcal{X}] \otimes_{\mathbb{K}} \mathbb{K}[\mathcal{X}] \colon X_i \mapsto (X_i \otimes 1) + (1 \otimes X_i)$ and $\iota^* \colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}] \colon X_i \mapsto -X_i$ and $\epsilon^* \colon \mathbb{K}[\mathcal{X}] \to \mathbb{K} \colon X_i \mapsto 0$, where $\mathcal{X} = \{X_1, \ldots, X_n\}$.

**b)** $\mathbb{G}_m := \mathbb{K} \setminus \{0\} = \mathbb{K}_X \subseteq \mathbb{K}$ is an affine variety with affine coordinate algebra $\mathbb{K}[\mathbb{G}_m] \cong \mathbb{K}[X]_X$. It is an algebraic group, called the **multiplicative group**, where $\mu \colon \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \colon [x, y] \mapsto xy$ and $\iota \colon \mathbb{G}_m \to \mathbb{G}_m \colon x \mapsto x^{-1}$ and $\epsilon \colon \{1\} \to \mathbb{G}_m$, yielding $\mu^* \colon \mathbb{K}[X]_X \to \mathbb{K}[X]_X \otimes_{\mathbb{K}} \mathbb{K}[X]_X \colon X \mapsto X \otimes X$ and $\iota^* \colon \mathbb{K}[X]_X \to \mathbb{K}[X]_X \colon X \mapsto X^{-1}$ and $\epsilon^* \colon \mathbb{K}[X]_X \to \mathbb{K} \colon X \mapsto 1$.

For $n \in \mathbb{Z}$ the map $\varphi_n \colon \mathbb{G}_m \to \mathbb{G}_m \colon x \mapsto x^n$, thus $\varphi_n^* \colon \mathbb{K}[X]_X \to \mathbb{K}[X]_X \colon X \mapsto X^n$, is a morphism and a group homomorphism, thus a homomorphism of algebraic groups. If $\mathrm{char}(\mathbb{K}) = p > 0$ and $q = p^f$, for some $f \in \mathbb{N}$, then the Frobenius morphism $\Phi_q$ is a group isomorphism, but since $\Phi_q^*$ is not surjective, $\Phi_q$ is not an isomorphism of algebraic groups.

For the automorphisms of $\mathbb{G}_a$ and $\mathbb{G}_m$ as algebraic groups see Exercise (12.2).

**(4.3) Example: General and special linear groups.**
**a)** We consider $\mathbb{K}^{n \times n}$, for some $n \in \mathbb{N}$, whose affine coordinate algebra is given as $\mathbb{K}[\mathcal{X}] = \mathbb{K}[X_{11}, \ldots, X_{nn}]$. Let $\det_n := \sum_{\sigma \in \mathcal{S}_n} (\mathrm{sgn}(\sigma) \cdot \prod_{i=1}^n X_{i,i^\sigma}) \in \mathbb{K}[\mathcal{X}]$ be the $n$-th **determinant polynomial**. The principal open subset $\mathbb{GL}_n = \mathbb{GL}_n(\mathbb{K}) := (\mathbb{K}^{n \times n})_{\det_n} = \{[a_{ij}] \in \mathbb{K}^{n \times n}; \det([a_{ij}]) = \det_n(a_{11}, a_{12}, \ldots, a_{nn}) \neq 0\} \subseteq \mathbb{K}^{n \times n}$ is called the **general linear group**; we have $\mathbb{GL}_1 = \mathbb{G}_m$. Its affine coordinate algebra is $\mathbb{K}[\mathbb{GL}_n] \cong \mathbb{K}[\mathcal{X}]_{\det_n}$, and it is an algebraic group:

Multiplication $\mu \colon \mathbb{GL}_n \times \mathbb{GL}_n \to \mathbb{GL}_n \colon [[a_{ij}], [b_{ij}]] \mapsto [\sum_{j=1}^n a_{ij} b_{jk}]_{ik}$ yields $\mu^* \colon \mathbb{K}[\mathcal{X}]_{\det_n} \to \mathbb{K}[\mathcal{X}]_{\det_n} \otimes_{\mathbb{K}} \mathbb{K}[\mathcal{X}]_{\det_n} \colon X_{ik} \mapsto \sum_{j=1}^n X_{ij} \otimes X_{jk}$. Moreover, using the adjoint matrix, inversion $\iota \colon \mathbb{GL}_n \to \mathbb{GL}_n \colon A \mapsto A^{-1} = \det(A)^{-1} \cdot \mathrm{adj}(A)$, where $\mathrm{adj}(A) := [(-1)^{i+j} \cdot \det([a_{kl}]_{k \neq j, l \neq i})]_{ij} \in \mathbb{K}^{n \times n}$, yields $\iota^* \colon \mathbb{K}[\mathcal{X}]_{\det_n} \to \mathbb{K}[\mathcal{X}]_{\det_n} \colon X_{ij} \mapsto (-1)^{i+j} \cdot \det_n^{-1}(\mathcal{X}) \cdot \det_{n-1}(\{X_{kl}; k \neq j, l \neq i\})$; we let $\mathrm{adj}([a_{11}]) = [1]$ and $\det_0 = 1$. Finally, $\epsilon \colon \{E_n\} \to \mathbb{GL}_n$ yields $\epsilon^* \colon \mathbb{K}[\mathcal{X}]_{\det_n} \to \mathbb{K} \colon X_{ij} \mapsto \delta_{ij}$, where $\delta$ denotes the Kronecker function.

The map $\varphi_{\det} \colon \mathbb{GL}_n \to \mathbb{G}_m \colon A \mapsto \det(A)$ is a homomorphism of algebraic groups with comorphism $\varphi_{\det}^* \colon \mathbb{K}[X]_X \to \mathbb{K}[\mathcal{X}]_{\det_n} \colon X \mapsto \det_n$.

**b)** Similarly, $\mathbb{SL}_n = \mathbb{SL}_n(\mathbb{K}) := \mathcal{V}(\det_n - 1) = \{[a_{ij}] \in \mathbb{K}^{n \times n}; \det([a_{ij}]) = \det_n(a_{11}, a_{12}, \ldots, a_{nn}) = 1\} \subseteq \mathbb{K}^{n \times n}$ is called the **special linear group**.

We show that $\det_n - 1 \in \mathbb{K}[\mathcal{X}]$ is irreducible; see Exercise (12.3): We first show by induction that $\det_n \in \mathbb{K}[\mathcal{X}]$ is irreducible, which holds for $n = 1$. For $n \geq 2$ assume to the contrary that $\det_n$ is reducible. Expansion with respect to the $n$-th row yields $\det_n = \det_{n-1} \cdot X_{nn} + \delta_n$, where $\delta_n := \sum_{i=1}^{n-1} (-1)^{n-i} \cdot \det_{n-1}(\{X_{kl}; k \neq n, l \neq i\}) \cdot X_{ni}$. Since $\deg_{X_{nn}}(\det_n) = 1$, and by induction $\det_{n-1} \in \mathbb{K}[\{X_{kl}; k \neq n, l \neq n\}]$ is irreducible, this implies that $\det_{n-1}$ divides $\delta_n$. By specifying $X_{nj} \mapsto 0$, for all $i \neq j \in \{1, \ldots, n-1\}$, this yields that $\det_{n-1}$ divides $\det_{n-1}(\{X_{kl}; k \neq n, l \neq i\})$, for all $i \in \{1, \ldots, n-1\}$, which by induction is a contradiction. Hence $\det_n$ is irreducible, and now assume to the contrary that $\det_n - 1$ is reducible. Then we conclude similarly that $\det_{n-1}$ divides $\delta_n - 1$, which by specifying $X_{ni} \mapsto 0$, for all $i \in \{1, \dots, n-1\}$, is a contradiction. $\sharp$

This implies that $\langle \det_n -1 \rangle \lhd \mathbb{K}[\mathcal{X}]$ is prime, and $\mathbb{K}[\mathbb{SL}_n] \cong \mathbb{K}[\mathcal{X}]/\langle \det_n -1 \rangle$; in particular $\mathbb{K}[\mathbb{SL}_n]$ is a domain. Since $\mathbb{SL}_n \leq \mathbb{GL}_n$ is closed, $\mathbb{SL}_n$ is an algebraic group, and the structure morphisms are carried over from $\mathbb{GL}_n$ using the inclusion morphism its associated comorphism $\mathbb{K}[\mathcal{X}]_{\det_n} \rightarrow \mathbb{K}[\mathcal{X}]/\langle \det_n -1 \rangle \colon X_{ij} \mapsto X_{ij}, \det_n^{-1} \mapsto 1$; see Exercise (12.1).

Further examples are given in Exercises (12.4) and (12.5).

**(4.4) Example: Classical groups.**
Let $b$ be a non-degenerate $\mathbb{K}$-bilinear form on $\mathbb{K}^n$, having matrix $J = J_b \in \mathbb{K}^{n \times n}$ with respect to the standard $\mathbb{K}$-basis $\{e_1, \dots, e_n\}$ of $\mathbb{K}^n$. Let $\{A \in \mathbb{K}^{n \times n}; b(xA, yA) = b(x, y) \text{ for all } x, y \in \mathbb{K}^n\} = \{A \in \mathbb{K}^{n \times n}; AJA^{\mathrm{tr}} = J\} =: \mathbb{G}_J$ be the set of **isometries** of $b$. Since $\det(J) \neq 0$ implies $\det(A) \in \{\pm 1\}$, we indeed have $\mathbb{G}_J \leq \mathbb{GL}_n$ as groups, and since $AJA^{\mathrm{tr}} = J$ translates into polynomial equations for the matrix entries of $A$, see also (8.5), we deduce that $\mathbb{G}_J \subseteq \mathbb{GL}_n \subseteq \mathbb{K}^{n \times n}$ is closed, thus an algebraic group, called a **classical group**.

Let $b'$ is a $\mathbb{K}$-bilinear form on $\mathbb{K}^n$ equivalent to $b$, having matrix $J' \in \mathbb{K}^{n \times n}$, and let $B \in \mathbb{GL}_n$ such that $J' = BJB^{\mathrm{tr}}$. Then for $A \in \mathbb{GL}_n$ we have $AJ'A^{\mathrm{tr}} = J'$ if and only if $A^B \cdot J \cdot (A^B)^{\mathrm{tr}} = J$, implying $(\mathbb{G}_{J'})^B = \mathbb{G}_J \leq \mathbb{GL}_n$. It is immediate that conjugation $\kappa_B \colon \mathbb{GL}_n \rightarrow \mathbb{GL}_n \colon A \mapsto A^B := B^{-1}AB$ is an automorphism of algebraic groups, implying that $(\mathbb{G}_{J'})^B \cong \mathbb{G}_J$ as algebraic groups.

**a)** Let $b$ be **alternating**, i. e. we have $b(x, x) = 0$ for all $x \in \mathbb{K}^n$. This implies $0 = b(x + y, x + y) = b(x, y) + b(y, x)$, hence $b(x, y) = -b(y, x)$ for all $x, y \in \mathbb{K}^n$, i. e. $b$ is **symplectic**, and thus $J = -J^{\mathrm{tr}}$. If $\mathrm{char}(\mathbb{K}) \neq 2$, then from $b(x, y) = -b(y, x)$ we conversely get $b(x, x) = -b(x, x)$, hence $b(x, x) = 0$. We show that up to equivalence there is only one such form on $\mathbb{K}^n$:

Let $0 \neq x \in \mathbb{K}^n$. There is $y \in \mathbb{K}^n$ such that $b(x, y) \neq 0$, in particular we have $y \notin \langle x \rangle_{\mathbb{K}}$. Replacing $y$ by $\frac{1}{b(x,y)} \cdot y \in \mathbb{K}^n$ we get $b(x, y) = 1 = -b(y, x)$, i. e. $[x, y]$ is a **hyperbolic pair**. The restriction of $b$ to the **hyperbolic plane** $U := \langle x, y \rangle_{\mathbb{K}}$ has matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, thus is non-degenerate, implying that $U \cap U^\perp = \{0\}$. Hence $\mathbb{K}^n \cong U \oplus U^\perp$ where $b|_{U^\perp}$ is non-degenerate as well. By induction on $n$ we deduce that $n = 2m$ is even, and that $\mathbb{K}^n$ can be written as the orthogonal direct sum of $m$ copies of the hyperbolic plane. Note that the same argument works over any field. $\sharp$

Reshuffling hyperbolic pairs suitably we deduce that up to equivalence $J = \begin{bmatrix} 0 & J_m \\ -J_m & 0 \end{bmatrix} \in \mathbb{K}^{n \times n}$, where

$$J_m := \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{K}^{m \times m}.$$

The associated classical group $\mathbb{S}_n = \mathbb{S}\mathrm{p}_n = \mathbb{S}\mathrm{p}_n(\mathbb{K})$ is called the **symplectic group**. For the case $m = 1$ we get $\mathbb{S}_2 = \mathbb{SL}_2$, see Exercise (12.6), and by (10.2) we always have $\mathbb{S}_n \leq \mathbb{SL}_n$.

**b)** Let $\mathrm{char}(\mathbb{K}) \neq 2$ and let $b$ be **symmetric**, i. e. we have $b(x, y) = b(y, x)$ for all $x, y \in \mathbb{K}^n$, thus $J = J^{\mathrm{tr}}$. Let $q\colon \mathbb{K}^n \to \mathbb{K}\colon x \mapsto \frac{1}{2}b(x, x)$. Then we have $q(cx) = c^2q(x)$ and $q(x + y) = q(x) + q(y) + b(x, y)$, for all $x, y \in \mathbb{K}^n$ and $c \in \mathbb{K}$, as well as $q \neq 0$. Up to equivalence there is only one such form on $\mathbb{K}^n$:

Let $x \in \mathbb{K}^n$ such that $q(x) \neq 0$. Hence the restriction of $b$ to $U := \langle x\rangle_{\mathbb{K}}$ is non-degenerate, and we have $\mathbb{K}^n \cong U \oplus U^{\perp}$ where $b|_{U^{\perp}}$ is non-degenerate as well. By induction on $n$ we deduce that there is an **orthogonal** $\mathbb{K}$-basis $\{v_1, \ldots, v_n\}$ of $\mathbb{K}^n$, i. e. the associated matrix of $b$ is $\mathrm{diag}[2q(v_1), \ldots, 2q(v_n)] \in \mathbb{K}^{n\times n}$. Replacing $v_i$ by $\frac{1}{\sqrt{2q(v_i)}} \cdot v_i \in \mathbb{K}^n$ yields an **orthonormal** $\mathbb{K}$-basis, i. e. the associated matrix of $b$ is $E_n$. Note that the same argument works over any field of characteristic $\neq 2$ in which any element has a square root. ♯

Considering the case $n = 2$, given an orthonormal $\mathbb{K}$-basis of $\mathbb{K}^2$, the basis change given by $B := \frac{1}{2} \cdot \begin{bmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{bmatrix} \in \mathbb{K}^{2\times 2}$ yields $B \cdot E_2 \cdot B^{\mathrm{tr}} = J_2 \in \mathbb{K}^{2\times 2}$. Hence we deduce that up to equivalence $J = J_n \in \mathbb{K}^{n\times n}$, the associated classical group $\mathbb{O}_n = \mathbb{O}_n(\mathbb{K})$ being called the **orthogonal group**. Let $\mathbb{SO}_n = \mathbb{SO}_n(\mathbb{K}) := \mathbb{O}_n \cap \mathbb{SL}_n = \{A \in \mathbb{O}_n; \det(A) = 1\}$ be the **special orthogonal group**. For $n = 1$ we have $\mathbb{SO}_1 = \{1\}$ and $\mathbb{O}_1 = \{\pm 1\}$; for $n = 2$ and $n = 3$ see Exercise (12.7). Since $J_2 \cdot J_2 \cdot (J_2)^{\mathrm{tr}} = J_2 \in \mathbb{K}^{2\times 2}$ and $\det(J_2) = -1$ we have $J_2 \in \mathbb{O}_2 \setminus \mathbb{SO}_2$, implying that for any $n \in \mathbb{N}$ we have $[\mathbb{O}_n\colon \mathbb{SO}_n] = 2$.

**(4.5) Example: Orthogonal groups in characteristic 2.**
It remains to deal with symmetric bilinear forms in characteristic 2. To this end, a **quadratic form** $q\colon \mathbb{K}^n \to \mathbb{K}$ is a map such that $q(cx) = c^2q(x)$, for all $x \in \mathbb{K}^n$ and $c \in \mathbb{K}$, and such that the associated **polar form** $b\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}\colon [x, y] \mapsto q(x + y) - q(x) - q(y)$ is $\mathbb{K}$-bilinear; hence $b$ is symmetric, but not necessarily is non-degenerate. If $\mathrm{char}(\mathbb{K}) \neq 2$ then we have $b(x, x) = q(2x) - 2q(x) = 2q(x)$, implying that $q$ is determined by $b$, thus if $b$ is non-degenerate we recover the situation in (4.4)(b).

Let now $\mathrm{char}(\mathbb{K}) = 2$. Then $b(x, x) = 2q(x) = 0$ implies that $b$ is alternating as well, and that $q$ is not completely determined by $b$. A vector $0 \neq x \in \mathbb{K}^n$ such that $q(x) = 0$ is called **singular**, and a pair $[x, y]$ of singular vectors such that $b(x, y) = 1$ is called a **hyperbolic pair**. Letting $\mathrm{rad}(b) \leq \mathbb{K}^n$ be the radical of $b$, for $x, y \in \mathrm{rad}(b)$ we have $q(cx + y) = c^2q(x) + q(y)$, showing that $q|_{\mathrm{rad}(b)}\colon \mathrm{rad}(b) \to \mathbb{K}$ is **$\Phi_2$-semilinear**, where $\Phi_2\colon \mathbb{K} \to \mathbb{K}\colon c \mapsto c^2$. The quadratic form $q$ is called **regular** if $\ker(q|_{\mathrm{rad}(b)}) = \{0\}$, i. e. $\mathrm{rad}(b)$ does not contain singular vectors; this replaces the non-degeneracy condition on $b$.

From now we assume $q$ to be regular. This implies $\dim_{\mathbb{K}}(\mathrm{rad}(b)) \leq 1$. We show that $\mathbb{K}^n$, for $n \geq 2$, contains a singular vector: If $b$ is degenerate, then let $x \in \mathbb{K}^n \setminus \mathrm{rad}(b)$ and $0 \neq y \in \mathrm{rad}(b)$. Hence we have $q(y) \neq 0$ and thus $q(x) = c^2q(y)$, for some $c \in \mathbb{K}$. This yields $x+cy \neq 0$ and $q(x+cy) = q(x)+c^2q(y)+b(x,y) = 0$. If $b$ is non-degenerate, then let first $n \geq 3$. Let $x \in \mathbb{K}^n$ such that $q(x) \neq 0$ and, since $\dim_{\mathbb{K}}(\langle x\rangle_{\mathbb{K}}^{\perp}) \geq 2$, let $y \in \langle x\rangle_{\mathbb{K}}^{\perp} \setminus \langle x\rangle_{\mathbb{K}}$. Hence we have $q(y) = c^2q(x)$, for some $c \in \mathbb{K}$, yielding $cx+y \neq 0$ and $q(cx+y) = c^2q(x)+q(y)+b(x,y) = 0$. Finally, let still $b$ be non-degenerate but $n = 2$. Let $x, y \in \mathbb{K}^2$ such that $b(x,y) = 1$ and $q(x) \neq 0$. Since $\mathbb{K}$ is algebraically closed there is $c \in \mathbb{K}$ such that $q(cx+y) = c^2q(x)+q(y)+c = 0$. Note that for $n \geq 3$ the same argument works over any perfect field of characteristic 2. $\sharp$

We show that up to equivalence, i. e. up to change of $\mathbb{K}$-bases, there is only one regular quadratic form on $\mathbb{K}^n$, proceeding similar to (4.4)(a): For $n \geq 2$ we choose a singular vector $x \in \mathbb{K}^n$, i. e. we have $q(x) = 0$. Since $x \notin \mathrm{rad}(b)$ there is $y \in \mathbb{K}^n$ such that $b(x,y) = 1$. This yields $q(q(y)x+y) = q(y)^2q(x)+q(y)+q(y)b(x,y) = 0$, and $b(x, q(y)x+y) = q(y)b(x,x)+b(x,y) = 1$. Thus $q(y)x+y$ is singular and $[x, q(y)x+y]$ form a hyperbolic pair. The restriction of $b$ to the **hyperbolic plane** $U := \langle x, q(y)x+y\rangle_{\mathbb{K}}$ is non-degenerate, implying that $U \cap U^{\perp} = \{0\}$. Since $\dim_{\mathbb{K}}(U^{\perp}) \geq n - \dim_{\mathbb{K}}(U)$, we conclude $\dim_{\mathbb{K}}(U \oplus U^{\perp}) = \dim_{\mathbb{K}}(U) + \dim_{\mathbb{K}}(U^{\perp}) \geq n$, thus $U \oplus U^{\perp} = \mathbb{K}^n$. By induction on $n \in \mathbb{N}$ we deduce that $\mathbb{K}^n$ can be written as the orthogonal direct sum of $\mathrm{rad}(b) \leq \mathbb{K}^n$, and $m$ copies of the hyperbolic plane for some $m \in \mathbb{N}_0$. $\sharp$

**a)** If $b$ is non-degenerate then $n = 2m$ for some $m \in \mathbb{N}$, and up to equivalence we have $J = J_n \in \mathbb{K}^{n\times n}$, where the standard $\mathbb{K}$-basis of $\mathbb{K}^n$ consists of singular vectors, and thus $q(x) = \sum_{i=1}^{m} x_i x_{n+1-i} \in \mathbb{K}$, for all $x = [x_1, \ldots, x_n] \in \mathbb{K}^n$.

Let $\mathbb{O}_n := \{A \in \mathbb{K}^{n\times n}; q(xA) = q(x) \text{ for all } x \in \mathbb{K}^n\}$. Since for $A \in \mathbb{O}_n$ we have $b(xA, yA) = q(xA+yA)+q(xA)+q(yA) = q(x+y)+q(x)+q(y) = b(x,y)$, for all $x, y \in \mathbb{K}^n$, we conclude that $\mathbb{O}_n \leq \mathbb{S}_n \subseteq \mathbb{K}^{n\times n}$ is a closed subgroup, see also (8.6), hence is an algebraic group, called the associated **even-dimensional orthogonal group in characteristic** 2; it is immediate that equivalent quadratic forms yield isomorphic groups; for $n = 2$ see Exercise (12.7). We have $\det(A) = 1$ for all $A \in \mathbb{O}_n$, but still there is a **special orthogonal group** $\mathbb{SO}_n$, a closed subgroup such that $[\mathbb{O}_n \colon \mathbb{SO}_n] = 2$, being defined as the kernel of **Dickson's pseudo-determinant**, see [15, Ch.11, p.160] or [14, Ch.14, p.131].

**b)** If $b$ is degenerate then $n = 2m+1$ for some $m \in \mathbb{N}_0$, and up to equivalence we have $J = \begin{bmatrix} J_{2m} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{K}^{n\times n}$, hence $\mathrm{rad}(b) = \langle e_n\rangle_{\mathbb{K}}$. The subset $\{e_1, \ldots, e_{2m}\}$ of the standard $\mathbb{K}$-basis of $\mathbb{K}^n$ consists of singular vectors, while we have $q(e_n) = 1$, and thus $q(x) = x_n^2 + \sum_{i=1}^{m} x_i x_{n-i} \in \mathbb{K}$, for all $x = [x_1, \ldots, x_n] \in \mathbb{K}^n$.

Let $\mathbb{O}_n := \{A \in \mathbb{K}^{n\times n}; q(xA) = q(x) \text{ for all } x \in \mathbb{K}^n\}$. Hence for all $A \in \mathbb{O}_n$ and $x, y \in \mathbb{K}^n$ we have $b(xA, yA) = b(x,y)$. We show that any $A \in \mathbb{O}_n$ is invertible: Assume to the contrary that $A$ is not invertible, then from $AJA^{\mathrm{tr}} = J$ we get $\dim_{\mathbb{K}}(\mathrm{im}(A)) = n-1$ and $\mathrm{im}(A) \cap \mathrm{rad}(b) = \{0\}$, implying that $\mathbb{K}^n = \mathrm{im}(A) \oplus \mathrm{rad}(b)$ is an orthogonal direct sum, and thus $b|_{\mathrm{im}(A)}$ is non-degenerate. From $b(xA, e_nA) = b(x, e_n) = 0$ for all $x \in \mathbb{K}^n$ we conclude that $e_nA \in \mathrm{rad}(b|_{\mathrm{im}(A)})$ and hence $e_nA = 0$, implying $1 = q(e_n) = q(e_nA) = 0$, a contradiction.

This implies that $\mathbb{O}_n \leq \mathbb{GL}_n$ is a closed subgroup, see also (8.6), hence is an algebraic group, called the associated **odd-dimensional orthogonal group in characteristic** 2; it is immediate that equivalent quadratic forms yield isomorphic groups, and we have $\mathbb{O}_1 = \{1\}$. We show that for $n = 2m+1 \geq 3$ there is a bijective homomorphism of algebraic groups $\mathbb{O}_n \to \mathbb{S}_{2m}$; it then follows from (5.4) that there is no notion of a 'special orthogonal group' in this case:

Let $W := \langle e_1, \ldots, e_{2m}\rangle_{\mathbb{K}}$, hence we have an orthogonal direct sum $\mathbb{K}^n = W \oplus \mathrm{rad}(b)$, where $b|_W$ is non-degenerate having matrix $J_{2m} \in \mathbb{K}^{2m\times 2m}$. Since $\mathrm{rad}(b)$ is $\mathbb{O}_n$-invariant, we have $e_nA = ce_n$ for some $c \in \mathbb{K}$, and from $1 = q(e_nA) = c^2q(e_n) = c^2$ we conclude $c = 1$. Hence we have $A = \left[\begin{array}{c|c} A' & a^{\mathrm{tr}} \\ \hline 0 & 1 \end{array}\right] \in \mathbb{K}^{(2m+1)\times(2m+1)}$, where $a = [a_1, \ldots, a_{2m}] \in \mathbb{K}^{2m}$ and $A' \in \mathbb{S}_{2m} \subseteq \mathbb{K}^{2m\times 2m}$. Hence $\varphi\colon \mathbb{O}_n \to \mathbb{S}_{2m}\colon A \mapsto A'$ is a homomorphism of algebraic groups. Let $A \in \ker(\varphi)$, then for $x \in W$ we have $xA = x + ce_n$, where $c := \sum_{i=1}^{2m} a_ix_i \in \mathbb{K}$, and from $q(x) = q(xA) = q(x) + c^2$ we deduce $c = 0$, hence $\varphi$ is injective.

Let $B \in \mathbb{S}_{2m}(W) \cong \mathbb{S}_{2m}$, then for $i \in \{1, \ldots, 2m\}$ let $b_i \in \mathbb{K}$ such that $q(e_iB) + q(e_i) = b_i^2$, and $b := [b_1, \ldots, b_{2m}] \in \mathbb{K}^{2m}$. We show that $\widehat{B} := \left[\begin{array}{c|c} B & b^{\mathrm{tr}} \\ \hline 0 & 1 \end{array}\right] \in \mathbb{O}_n \subseteq \mathbb{K}^{(2m+1)\times(2m+1)}$: For $x \in W$ and $d \in \mathbb{K}$ we have $q(x + de_n) = q(x) + d^2$. Letting $c := \sum_{i=1}^{2m} b_ix_i \in \mathbb{K}$, we have $q((x + de_n)\widehat{B}) = q(xB + (c + d)e_n) = q(xB) + c^2 + d^2$ and $q(xB) = \sum_{i=1}^{2m} q(x_ie_iB) + \sum_{i=1}^{2m}\sum_{j=1}^{i-1} b(x_ie_iB, x_je_jB) = \sum_{i=1}^{2m} x_i^2b_i^2 + \sum_{i=1}^{2m} q(x_ie_i) + \sum_{i=1}^{2m}\sum_{j=1}^{i-1} b(x_ie_i, x_je_j) = q(x) + c^2$, which implies $q((x+de_n)\widehat{B}) = q(x)+d^2 = q(x+de_n)$. Hence $\varphi$ is surjective as well, thus is an isomorphism of groups; but due to taking square roots $\varphi$ is not an isomorphism of algebraic groups; see also (8.6). ♯

## 5 Basic properties

**(5.1) Proposition.** Let $\mathbb{G}$ be an algebraic group.
**a)** There is a unique irreducible component $\mathbb{G}^\circ$ of $\mathbb{G}$ containing $1_{\mathbb{G}}$.

**b)** The **identity component** $\mathbb{G}^\circ \trianglelefteq \mathbb{G}$ is a closed normal subgroup of finite index, and $\mathbb{G}^\circ|\mathbb{G} := \{\mathbb{G}^\circ g; g \in \mathbb{G}\}$ consists of the connected as well as of the irreducible components of $\mathbb{G}$.

In particular, $\mathbb{G}$ is equidimensional such that $\dim(\mathbb{G}) = \dim(\mathbb{G}^\circ)$, and $\mathbb{G}$ is irreducible if and only if it is connected; in this case $\mathbb{G}$ is called a **connected** algebraic group.

**c)** The subgroup $\mathbb{G}^\circ$ is contained in any closed subgroup of $\mathbb{G}$ of finite index.

**Proof. a)** Let $V, W \subseteq \mathbb{G}$ be irreducible components such that $1_{\mathbb{G}} \in V \cap W$. Multiplication $\mu\colon \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ yields that $VW = \mu(V \times W) \subseteq \mathbb{G}$ is irreducible, hence $\overline{VW} \subseteq \mathbb{G}$ is irreducible as well. Since both $V \subseteq \overline{VW}$ and $W \subseteq \overline{VW}$, we conclude that $V = \overline{VW} = W$.

**b)** In particular, we have $\mathbb{G}^\circ\mathbb{G}^\circ = \mathbb{G}^\circ$. Since inversion $\iota\colon \mathbb{G} \to \mathbb{G}$ is an automorphism of affine varieties, $(\mathbb{G}^\circ)^{-1} = \iota(\mathbb{G}^\circ) \subseteq \mathbb{G}$ is an irreducible component containing $1_\mathbb{G}$, implying that $(\mathbb{G}^\circ)^{-1} = \mathbb{G}^\circ$. Thus $\mathbb{G}^\circ \leq \mathbb{G}$ is a subgroup. For any $g \in \mathbb{G}$ conjugation $\kappa_g\colon \mathbb{G} \to \mathbb{G}\colon x \mapsto x^g := g^{-1}xg$ is an automorphism of algebraic groups, hence $(\mathbb{G}^\circ)^g = \kappa_g(\mathbb{G}^\circ) \subseteq \mathbb{G}$ is an irreducible component containing $1_\mathbb{G}$, thus $(\mathbb{G}^\circ)^g = \mathbb{G}^\circ$, implying that $\mathbb{G}^\circ \trianglelefteq \mathbb{G}$ is a normal subgroup.

Moreover, for any $g \in \mathbb{G}$ right translation $\rho_g\colon \mathbb{G} \to \mathbb{G}\colon x \mapsto xg$ is an automorphism of affine varieties, hence $\mathbb{G}^\circ g = \rho_g(\mathbb{G}^\circ) \subseteq \mathbb{G}$ is an irreducible component, and in particular is connected. Since $\mathbb{G}$ is Noetherian, $\mathbb{G}^\circ|\mathbb{G}$ is a finite set. Since $\mathbb{G} = \coprod_{g \in \mathbb{G}^\circ|\mathbb{G}} \mathbb{G}^\circ g$ is a finite disjoint union, we conclude that all the sets $\mathbb{G}^\circ g \subseteq \mathbb{G}$ are open and closed, hence are the connected components of $\mathbb{G}$. Finally, if $V \subseteq \mathbb{G}$ is an irreducible component, then from $V = \coprod_{g \in \mathbb{G}^\circ|\mathbb{G}}(V \cap \mathbb{G}^\circ g)$ we conclude that $V = V \cap \mathbb{G}^\circ g$, hence $V = \mathbb{G}^\circ g$, for some $g \in \mathbb{G}$.

**c)** Let $\mathbb{H} \leq \mathbb{G}$ be a closed subgroup of finite index. Hence $\mathbb{G} = \coprod_{g \in \mathbb{H}|\mathbb{G}} \mathbb{H}g$ is a finite disjoint union of open and closed subsets. Thus $\mathbb{G}^\circ = \coprod_{g \in \mathbb{H}|\mathbb{G}}(\mathbb{G}^\circ \cap \mathbb{H}g)$, and since $1_\mathbb{G} \in \mathbb{G}^\circ \cap \mathbb{H}$ this implies $\mathbb{G}^\circ = \mathbb{G}^\circ \cap \mathbb{H}$, hence $\mathbb{G}^\circ \leq \mathbb{H}$. ♯

For variations on subgroups see Exercises (12.11), (12.9) and (12.10).

**(5.2) Lemma.** Let $\mathbb{G}$ be an algebraic group.
**a)** Let $V, W \subseteq \mathbb{G}$ be open and dense. Then $VW = \mathbb{G}$.
**b)** Let $H \leq \mathbb{G}$ be a subgroup. Then $\overline{H} \leq \mathbb{G}$ is a subgroup as well. If moreover $H$ contains a non-empty open subset of $\overline{H}$, then we have $H = \overline{H}$.

**Proof. a)** Let $g \in \mathbb{G}$. Then $V^{-1}g \subseteq \mathbb{G}$ is open and dense as well, hence we have $V^{-1}g \cap W \neq \emptyset$, implying that there is $v^{-1}g = w \in V^{-1}g \cap W$, for some $v \in V$ and $w \in W$, thus $g = vw \in VW$.
**b)** We have $\overline{H}^{-1} = \overline{H^{-1}} = \overline{H}$. Moreover, for any $h \in H$ we have $\overline{H}h = \overline{Hh} = \overline{H}$, implying $\overline{H}H \subseteq \overline{H}$. Thus for any $g \in \overline{H}$ we have $gH \subseteq \overline{H}$, implying $g\overline{H} = \overline{gH} \subseteq \overline{H}$, thus $\overline{H}\,\overline{H} \subseteq \overline{H}$. This shows that $\overline{H} \leq \mathbb{G}$ is a closed subgroup. Moreover, if $\emptyset \neq U \subseteq \overline{H}$ is open such that $U \subseteq H$, then $H = \bigcup\{Uh; h \in H\} \subseteq \overline{H}$ is open and dense, thus $H = HH = \overline{H}$. ♯

**(5.3) Proposition.** Let $\varphi\colon \mathbb{G} \to \mathbb{H}$ be a homomorphism of algebraic groups.
**a)** Kernel $\ker(\varphi) \trianglelefteq \mathbb{G}$ and image $\varphi(\mathbb{G}) \leq \mathbb{H}$ are closed subgroups.
**b)** We have $\varphi(\mathbb{G}^\circ) = \varphi(\mathbb{G})^\circ$.
**c)** We have $\dim(\mathbb{G}) = \dim(\ker(\varphi)) + \dim(\varphi(\mathbb{G}))$.

**Proof. a)** Since $\{1_\mathbb{H}\} \subseteq \mathbb{H}$ is closed, $\ker(\varphi) = \varphi^{-1}(\{1_\mathbb{H}\}) \subseteq \mathbb{G}$ is closed as well. Next we consider the restrictions $\varphi|_{\mathbb{G}^\circ}\colon \mathbb{G}^\circ \to \overline{\varphi(\mathbb{G}^\circ)}$. Since $\mathbb{G}^\circ$ is irreducible, $\overline{\varphi(\mathbb{G}^\circ)}$ is irreducible as well, hence there is $\emptyset \neq U \subseteq \overline{\varphi(\mathbb{G}^\circ)}$ open such that $U \subseteq \varphi(\mathbb{G}^\circ)$, implying that $\varphi(\mathbb{G}^\circ) \leq \mathbb{H}$ is closed. Now, $\mathbb{G} = \coprod_{g \in \mathbb{G}^\circ|\mathbb{G}} \mathbb{G}^\circ g$ being a finite union implies that $\varphi(\mathbb{G}) = \bigcup_{g \in \mathbb{G}^\circ|\mathbb{G}} \varphi(\mathbb{G}^\circ)\varphi(g) \leq \mathbb{H}$ is closed.

**b)** Since $\varphi(\mathbb{G}^\circ) \leq \varphi(\mathbb{G})$ is closed and irreducible, containing $1_\mathbb{H}$, we have $\varphi(\mathbb{G}^\circ) \leq \varphi(\mathbb{G})^\circ$. Conversely, since $\mathbb{G}^\circ \leq \mathbb{G}$ is a closed subgroup of finite index, $\varphi(\mathbb{G}^\circ) \leq \varphi(\mathbb{G})$ is a closed subgroup of finite index, implying $\varphi(\mathbb{G})^\circ \leq \varphi(\mathbb{G}^\circ)$.
**c)** We may assume that $\varphi$ is surjective, hence $\varphi_0 := \varphi|_{\mathbb{G}^\circ} \colon \mathbb{G}^\circ \to \mathbb{H}^\circ$ is a surjective morphism between irreducible affine varieties. Since the fibres of $\varphi_0$ are cosets of $\ker(\varphi_0)$ in $\mathbb{G}^\circ$, they all have dimension $\dim(\ker(\varphi_0))$. Moreover $\ker(\varphi_0) = \ker(\varphi) \cap \mathbb{G}^\circ \leq \ker(\varphi)$ has finite index, hence $\ker(\varphi)^\circ \leq \ker(\varphi_0) \leq \ker(\varphi)$, implying that $\dim(\ker(\varphi)^\circ) = \dim(\ker(\varphi_0)) = \dim(\ker(\varphi))$. Hence we have $\dim(\mathbb{G}) = \dim(\mathbb{G}^\circ) = \dim(\ker(\varphi_0)) + \dim(\mathbb{H}^\circ) = \dim(\ker(\varphi)) + \dim(\mathbb{H})$. ♯

**(5.4) Example. a)** Since $\mathbb{K}[X]$ is a domain, the additive group $\mathbb{G}_a = \mathbb{K}$ is connected, and $\dim(\mathbb{G}_a) = 1$. Since the multiplicative group $\mathbb{G}_m = \mathbb{K}_X \subseteq \mathbb{K}$ is open in an irreducible space it is connected, and $\dim(\mathbb{G}_m) = 1$. By [11, Thm.2.6.6] these are up to isomorphism the only connected algebraic groups of dimension 1; see also Exercise (12.21).
**b)** The general linear group $\mathbb{GL}_n = (\mathbb{K}^{n\times n})_{\det_n} \subseteq \mathbb{K}^{n\times n}$ is connected, and $\dim(\mathbb{GL}_n) = \dim(\mathbb{K}^{n\times n}) = n^2$. Since $\mathbb{K}[\mathbb{SL}_n] \cong \mathbb{K}[X_{11}, \ldots, X_{nn}]/\langle \det_n -1\rangle$ is a domain, the special linear group $\mathbb{SL}_n$ is connected as well, and since it is a hypersurface in $K^{n\times n}$ we have $\dim(\mathbb{SL}_n) = n^2 - 1$. For the examples mentioned in Exercises (12.4) and(12.5) see Exercise (12.8).
**c)** By (10.2) $\mathbb{S}_{2m}$ is connected. If $\mathrm{char}(\mathbb{K}) \neq 2$ then by (10.2) $\mathbb{SO}_n$ is connected, hence $[\mathbb{O}_n \colon \mathbb{SO}_n] = 2$ implies $\mathbb{O}_n^\circ = \mathbb{SO}_n$. Similarly, if $\mathrm{char}(\mathbb{K}) = 2$ then $\mathbb{O}_{2m}^\circ = \mathbb{SO}_{2m}$. Finally, if $\mathrm{char}(\mathbb{K}) = 2$ it follows from the bijective morphism $\mathbb{O}_{2m+1} \to \mathbb{S}_{2m}$, for $m \in \mathbb{N}$, that $\mathbb{O}_{2m+1}$ is connected.

**(5.5) Definition and Remark. a)** Let $\mathbb{G}$ be an algebraic group, and let $V \neq \emptyset$ be an affine variety. A (right) group action $\varphi \colon V \times \mathbb{G} \to V \colon [x, g] \mapsto xg$, such that $\varphi$ is a morphism, is called a **morphical** action, and $V$ is called a $\mathbb{G}$**-variety**.

In this case, for any $g \in \mathbb{G}$ we have the automorphism of affine varieties $\varphi_g \colon V \to V \colon x \mapsto xg$, and the associated automorphism of $\mathbb{K}$-algebras $\varphi_g^* \colon \mathbb{K}[V] \to \mathbb{K}[V]$, also called **translation of functions**. Since $\varphi_h \varphi_g = \varphi_{gh}$ for all $g, h \in \mathbb{G}$, we have $\varphi_g^* \varphi_h^* = \varphi_{gh}^*$, implying that $g \mapsto \varphi_g^*$ is a $\mathbb{K}$-representation of $\mathbb{G}$ on $\mathbb{K}[V]$.

Moreover, for any $x \in V$ we have the **orbit morphism** $\varphi_x \colon \mathbb{G} \to V \colon g \mapsto xg$, its image $x\mathbb{G} = \varphi_x(\mathbb{G}) \subseteq V$ is called the associated $\mathbb{G}$**-orbit**. If $\mathbb{G}$ acts transitively on $V$, i. e. we have $x\mathbb{G} = V$ for some and hence any $x \in V$, then $V$ is called a **homogeneous** $\mathbb{G}$-variety.

If $\mathbb{G}$ acts morphically on affine varieties $V$ and $W$, then a morphism $\psi \colon V \to W$ is called $\mathbb{G}$**-equivariant** if $\psi(xg) = \psi(x)g$, for all $x \in V$ and $g \in \mathbb{G}$.

**b)** E. g. $\mathbb{G}$ acts morphically on $\mathbb{G}$ by **right translation** $\rho = \mu \colon [x, g] \mapsto xg$, as well as by **left translation** $\lambda \colon [x, g] \mapsto g^{-1}x$, where $\mathbb{G}$ is homogeneous for both of these **regular** actions; and $\mathbb{G}$ acts morphically on $\mathbb{G}$ by **conjugation** $\kappa \colon [x, g] \mapsto x^g := g^{-1}xg$, the orbits being called **conjugacy classes**; for any $g \in \mathbb{G}$ we have $\kappa_g = \rho_g \lambda_g = \lambda_g \rho_g$.

**c)** Let $U \subseteq V$ be a subset, and let $W \subseteq V$ be closed. Then the **transporter** $\mathrm{Tran}_{\mathbb{G}}(U, W) := \{g \in \mathbb{G}; Ug \subseteq W\} = \bigcap_{x \in U} \varphi_x^{-1}(W) \subseteq \mathbb{G}$ is a closed subset.

In particular, for any $x \in V$ the **isotropy group** or **centraliser** or **stabiliser** $\mathbb{G}_x = C_{\mathbb{G}}(x) = \mathrm{Stab}_{\mathbb{G}}(x) := \{g \in \mathbb{G}; xg = x\} = \mathrm{Tran}_{\mathbb{G}}(\{x\}, \{x\}) \leq \mathbb{G}$ is a closed subgroup, and hence $C_{\mathbb{G}}(U) := \bigcap_{x \in U} \mathbb{G}_x \leq \mathbb{G}$ is a closed subgroup as well; see also Exercise (12.12).

For any $g \in \mathbb{G}$ the set of **fixed points** $V^g = \mathrm{Fix}_V(g) := \{x \in V; xg = x\} \subseteq V$ is closed, implying that $V^{\mathbb{G}} = \mathrm{Fix}_V(\mathbb{G}) := \bigcap_{g \in \mathbb{G}} V^g \subseteq V$ is closed as well: Let $\psi_g \colon V \to V \times V \colon x \mapsto [x, xg]$, then the **diagonal** $\Delta(V) := \{[x, x] \in V \times V; x \in V\} \subseteq V \times V$ is closed, hence $\psi_g^{-1}(\Delta(V)) = \{x \in V; x = xg\} \subseteq V$ is closed.

Each irreducible component of $V_0 \subseteq V$ is $\mathbb{G}^\circ$-invariant: The group $\mathbb{G}$ permutes the finitely many irreducible components, hence $\{g \in \mathbb{G}; V_0 g = V_0\} = \mathrm{Tran}_{\mathbb{G}}(V_0, V_0) \leq \mathbb{G}$ is a closed subgroup of finite index, thus contains $\mathbb{G}^\circ$.

**(5.6) Lemma.** Let $\mathbb{G}$ be an algebraic group, let $\mathbb{H} \leq \mathbb{G}$ be a closed subgroup, and let $\mathcal{I}(\mathbb{H}) \lhd \mathbb{K}[\mathbb{G}]$ be the associated vanishing ideal. Then we have $\mathbb{H} = \{g \in \mathbb{G}; \rho_g^*(\mathcal{I}(\mathbb{H})) \subseteq \mathcal{I}(\mathbb{H})\}$ and $\mathbb{H} = \{g \in \mathbb{G}; \lambda_g^*(\mathcal{I}(\mathbb{H})) \subseteq \mathcal{I}(\mathbb{H})\}$.

**Proof.** See Exercise (12.13).
**i)** For $g \in \mathbb{H}$ and $f \in \mathcal{I}(\mathbb{H})$ we have $(\rho_g^*(f))(x) = f(xg) = 0$, for all $x \in \mathbb{H}$, implying $\rho_g^*(f) \in \mathcal{I}(\mathbb{H})$. Conversely, let $g \in \mathbb{G}$ such that $\rho_g^*(\mathcal{I}(\mathbb{H})) \subseteq \mathcal{I}(\mathbb{H})$. Then for $f \in \mathcal{I}(\mathbb{H})$ we have $f(g) = (\rho_g^*(f))(1_{\mathbb{G}}) = 0$, implying $g \in \mathcal{V}(\mathcal{I}(\mathbb{H})) = \mathbb{H}$.
**ii)** For $g \in \mathbb{H}$ and $f \in \mathcal{I}(\mathbb{H})$ we have $(\lambda_g^*(f))(x) = f(g^{-1}x) = 0$, for all $x \in \mathbb{H}$, implying $\lambda_g^*(f) \in \mathcal{I}(\mathbb{H})$. Conversely, let $g \in \mathbb{G}$ such that $\lambda_g^*(\mathcal{I}(\mathbb{H})) \subseteq \mathcal{I}(\mathbb{H})$. Then for $f \in \mathcal{I}(\mathbb{H})$ we have $f(g^{-1}) = (\lambda_g^*(f))(1_{\mathbb{G}}) = 0$, implying $g \in \mathcal{V}(\mathcal{I}(\mathbb{H})) = \mathbb{H}$. ♯

## 6 Linearisation and Jordan decomposition

**(6.1) Proposition.** Let $\mathbb{G}$ be an algebraic group acting morphically on $V$ via $\varphi$, and let $F \leq \mathbb{K}[V]$ be a $\mathbb{K}$-subspace such that $\dim_{\mathbb{K}}(F) < \infty$.
**a)** There is a $\mathbb{K}$-subspace $F \leq E \leq \mathbb{K}[V]$ such that $\dim_{\mathbb{K}}(E) < \infty$, which is $\varphi_g^*$-invariant for all $g \in \mathbb{G}$.
**b)** $F \leq \mathbb{K}[V]$ is $\varphi_g^*$-invariant, for all $g \in \mathbb{G}$, if and only if $\varphi^*(F) \leq F \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}]$.

**Proof. a)** We may assume that $F = \langle f \rangle_{\mathbb{K}}$, for some $0 \neq f \in \mathbb{K}[V]$. Hence $\varphi^*(f) = \sum_{i=1}^r f_i \otimes g_i \in \mathbb{K}[V] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}]$, for some $r \in \mathbb{N}$ as well as $f_i \in \mathbb{K}[V]$ and $g_i \in \mathbb{K}[\mathbb{G}]$. For $g \in \mathbb{G}$ and $x \in V$ we have $(\varphi_g^*(f))(x) = f(\varphi_g(x)) = f(xg) = f(\varphi([x, g])) = (\varphi^*(f))([x, g]) = \sum_{i=1}^r f_i(x) g_i(g)$, implying $\varphi_g^*(f) = \sum_{i=1}^r f_i \cdot g_i(g) \in \mathbb{K}[V]$. Hence $E := \langle f_1, \ldots, f_r \rangle_{\mathbb{K}} \leq \mathbb{K}[V]$ is as desired.
**b)** If $\varphi^*(F) \leq F \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}]$, then the above computation shows that $\varphi_g^*(F) \leq F$, for all $g \in \mathbb{G}$. Conversely, if $F \leq \mathbb{K}[V]$ is $\varphi_g^*$-invariant, for all $g \in \mathbb{G}$, then let $\{f_1, \ldots, f_s, f_{s+1}, \ldots\} \subseteq \mathbb{K}[V]$ be a $\mathbb{K}$-basis, where $\{f_1, \ldots, f_s\} \subseteq F$ is a $\mathbb{K}$-basis and $s := \dim_{\mathbb{K}}(F)$. For $f \in F$ we have $\varphi^*(f) = \sum_{i \geq 1} f_i \otimes g_i$, for some $g_i \in \mathbb{K}[\mathbb{G}]$, implying that $\varphi_g^*(f) = \sum_{i=1}^s f_i \cdot g_i(g) + \sum_{i \geq s+1} f_i \cdot g_i(g)$. Since $\varphi_g^*(f) \in F$, for

all $g \in \mathbb{G}$, we deduce that for all $i \geq s+1$ we have $g_i = 0 \in \mathbb{K}[\mathbb{G}]$, thus $\varphi^*(f) = \sum_{i=1}^s f_i \otimes g_i \in F \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}]$. ♯

**(6.2) Theorem.** Let $\mathbb{G}$ be an algebraic group. Then $\mathbb{G}$ is isomorphic as an algebraic group to a closed subgroup of $\mathbb{GL}_n$, for some $n \in \mathbb{N}$.

**Proof.** We consider the regular action of $\mathbb{G}$ on $\mathbb{G}$ by right translation $\rho = \mu\colon \mathbb{G} \times \mathbb{G} \to \mathbb{G}$. Using the fact that $\mathbb{K}[\mathbb{G}]$ is a finitely generated $\mathbb{K}$-algebra, we choose a $\mathbb{K}$-linear independent subset $\{f_1, \ldots, f_n\} \subseteq \mathbb{K}[\mathbb{G}]$, for some $n \in \mathbb{N}$, such that $F := \langle f_1, \ldots, f_n \rangle_{\mathbb{K}} \leq \mathbb{K}[\mathbb{G}]$ is $\rho_g^*$-invariant, for all $g \in \mathbb{G}$, and such that $\mathbb{K}\langle F \rangle = \mathbb{K}[\mathbb{G}]$. Hence for all $i \in \{1, \ldots, n\}$ we have $\rho^*(f_i) = \sum_{j=1}^n f_j \otimes g_{ij} \in \mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}]$, where the $g_{ij} \in \mathbb{K}[\mathbb{G}]$ are uniquely defined, and thus $\rho_g^*(f_i) = \sum_{j=1}^n f_j \cdot g_{ij}(g) \in \mathbb{K}[\mathbb{G}]$, for all $g \in \mathbb{G}$. Since $\rho_g^*$ is injective and $\rho_g^* \rho_h^* = \rho_{gh}^*$, for all $g, h \in \mathbb{G}$, this implies that $\varphi\colon \mathbb{G} \to \mathbb{GL}_n\colon g \mapsto [g_{ij}(g)] \in \mathbb{K}^{n \times n}$ is a morphism of algebraic groups, called an **algebraic** or **rational** $\mathbb{K}$-representation of $\mathbb{G}$ on $F$. Since for $g \in \mathbb{G}$ we have $f_i(g) = f_i(1_{\mathbb{G}} \cdot g) = (\rho^*(f_i))([1_{\mathbb{G}}, g]) = \sum_{j=1}^n f_j(1_{\mathbb{G}}) \cdot g_{ij}(g)$, we get $f_i = \sum_{j=1}^n f_j(1_{\mathbb{G}}) \cdot g_{ij} \in \mathbb{K}[\mathbb{G}]$, implying that $\mathbb{K}\langle g_{ij}; i, j \in \{1, \ldots, n\} \rangle = \mathbb{K}[\mathbb{G}]$. Hence $\varphi^*\colon \mathbb{K}[\mathbb{GL}_n] \to \mathbb{K}[\mathbb{G}]$ is surjective, implying that $\varphi$ is a closed embedding, see Exercise (11.7). ♯

For linearisation of arbitrary actions see Exercise (12.15).

**(6.3) Definition.** A matrix $A \in \mathbb{K}^{n \times n}$, where $n \in \mathbb{N}$, is called **semisimple**, if its minimum polynomial is multiplicity-free, i. e. if it is diagonalisable; and it is called **nilpotent**, if there is $k \in \mathbb{N}$ such that $A^k = 0$, i. e. if 0 is its only eigenvalue. Hence $A$ is both semisimple and nilpotent if and only if $A = 0$.

Moreover, $A$ is called **unipotent**, if $A - E_n \in \mathbb{K}^{n \times n}$ is nilpotent, i. e. if 1 is its only eigenvalue. Hence if $A$ is unipotent, then $A \in \mathbb{GL}_n$, and it is both semisimple and unipotent if and only if $A = E_n$; see also Exercise (12.17).

**(6.4) Lemma. a)** Let $A \in \mathbb{K}^{n \times n}$. There are uniquely determined matrices $A_s, A_n \in \mathbb{K}^{n \times n}$, where $A_s$ is semisimple and $A_n$ is nilpotent, such that $A_s A_n = A_n A_s$ and $A = A_s + A_n$, called the **additive Jordan decomposition** of $A$, where $A_s$ and $A_n$ are called the **semisimple** and **nilpotent part**, respectively.

Moreover, there are $f, g \in \mathbb{K}[T]$ such that $A_s = f(A)$ and $A_n = g(A)$; see also Exercise (12.16). If a matrix $B \in \mathbb{K}^{n \times n}$ commutes with $A$, then $B$ also commutes with $A_s$ and $A_n$, and we have $(A+B)_s = A_s + B_s$ as well as $(A+B)_n = A_n + B_n$.

**b)** Let $A \in \mathbb{GL}_n \subseteq \mathbb{K}^{n \times n}$. Then there are uniquely determined matrices $A_s, A_u \in \mathbb{GL}_n$, where $A_s$ is semisimple and $A_u$ is unipotent, such that $A = A_s A_u = A_u A_s$, called the **(multiplicative) Jordan decomposition** of $A$, where $A_s$ and $A_u$ are called the **semisimple** and **unipotent part**, respectively.

Moreover, there are $f, g \in \mathbb{K}[T]$ such that $A_s = f(A) \in \mathbb{K}^{n\times n}$ and $A_u = g(A) \in \mathbb{K}^{n\times n}$. If a matrix $B \in \mathbb{GL}_n$ commutes with $A$, then we have $(AB)_s = A_sB_s$ and $(AB)_u = A_uB_u$ .

**Proof. a)** Let $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$ be pairwise distinct and $n_1, \ldots, n_r \in \mathbb{N}$ such that $\chi_A \mid \prod_{i=1}^r (T - \lambda_i)^{n_i} \in \mathbb{K}[T]$, where $\chi_A \in \mathbb{K}[T]$ is the characteristic polynomial of $A$. Moreover, let $E_{\lambda_i,k}(A) := \ker((A - \lambda_i E_n)^k) \leq \mathbb{K}^n$, for $k \in \mathbb{N}$, be the associated **generalised eigenspaces**, hence we have $\mathbb{K}^n \cong \bigoplus_{i=1}^r E_{\lambda_i,n_i}(A)$. By the Chinese remainder theorem there is $f \in \mathbb{K}[T]$ such that for all $i \in \{1, \ldots, r\}$ we have $f \equiv \lambda_i \pmod{(T - \lambda_i)^{n_i}}$. Letting $A_s := f(A) \in \mathbb{K}^{n\times n}$, we have $A_s|_{E_{\lambda_i,n_i}(A)} = \lambda_i E_n|_{E_{\lambda_i,n_i}(A)}$, hence $A_s$ is semisimple. Letting $A_n := A - A_s \in \mathbb{K}^{n\times n}$, we get $A_n|_{E_{\lambda_i,n_i}(A)} = (A - \lambda_i E_n)|_{E_{\lambda_i,n_i}(A)}$, hence $(A_n)^{n_i}|_{E_{\lambda_i,n_i}(A)} = 0$, implying that $A_n$ is nilpotent.

It remains to prove uniqueness: Let $A = A'_s + A'_n$ be an additive Jordan decomposition, then we have $A_s - A'_s = A'_n - A_n$, where $A_s, A_n$ are as above. Then both $A'_s$ and $A'_n$ commute with $A$, and hence with both $A_s$ and $A_n$. Thus $A_s$ and $A'_s$ are simultaneously diagonalisable, implying that $A_s - A'_s$ is semisimple, and it is immediate from the binomial formula that $A'_n - A_n$ is nilpotent. This in turn implies $A_s - A'_s = A'_n - A_n = 0$.

**b)** Let $A = A_s + A_n \in \mathbb{K}^{n\times n}$ be the additive Jordan decomposition of $A$. Since $A \in \mathbb{GL}_n$ we infer that $A_s \in \mathbb{GL}_n$ as well, and we let $A_u := E_n + (A_s)^{-1}A_n \in \mathbb{K}^{n\times n}$. Since $A_s$ and $A_n$ commute we have $A_sA_u = A_uA_s = A_s + A_n = A$, and we conclude that $A_u - E_n$ is nilpotent, i. e. $A_u \in \mathbb{GL}_n$ is unipotent.

It remains to prove uniqueness: Let $A = A'_sA'_u$ be a Jordan decomposition, then $(A'_s)^{-1}A_s = A'_u(A_u)^{-1}$, where both $A'_s$ and $A'_u$ commute with $A_s$ and $A_u$. Hence $(A'_s)^{-1}A_s$ is semisimple, and $A'_u(A_u)^{-1} - E_n = (A'_u - E_n)((A_u)^{-1} - E_n) + (A'_u - E_n) + ((A_u)^{-1} - E_n)$ is nilpotent, i. e. $A'_u(A_u)^{-1}$ is unipotent, hence $(A'_s)^{-1}A_s = A'_u(A_u)^{-1} = E_n$. ♯

**(6.5) Definition and Remark. a)** Let $E$ be an arbitrary $\mathbb{K}$-vector space. An element $A \in \mathrm{End}_{\mathbb{K}}(E)$ is called **locally finite**, if $E$ is the union of finite dimensional $A$-invariant $\mathbb{K}$-subspaces. E. g. if $\mathbb{G}$ is an algebraic group acting morphically on $V$ via $\varphi$, then $\varphi_g^*$ is locally finite on $\mathbb{K}[V]$.

A locally finite element $A \in \mathrm{End}_{\mathbb{K}}(E)$ is called **(locally) semisimple** if its restriction to any finite dimensional $A$-invariant $\mathbb{K}$-subspace is semisimple, it is called **(locally) nilpotent** if its restriction to any finite dimensional $A$-invariant $\mathbb{K}$-subspace is nilpotent, and it is called **(locally) unipotent** if $A - \mathrm{id}_E$ is locally nilpotent.

**b)** For a locally finite element $A \in \mathrm{End}_{\mathbb{K}}(E)$ we get an additive Jordan decomposition $A = A_s + A_n \in \mathrm{End}_{\mathbb{K}}(E)$ as follows: For $x \in E$ let $\langle x\rangle \leq F \leq E$ be any finite dimensional $A$-invariant $\mathbb{K}$-subspace, and let $xA_s := x(A|_F)_s \in E$ as well as $xA_n := x(A|_F)_n \in E$. This indeed yields well-defined maps: If $\langle x\rangle \leq F' \leq E$ also is a finite dimensional $A$-invariant $\mathbb{K}$-subspace, then $((A|_F)_s)_{F\cap F'} =$

$(A|_{F\cap F'})_s = ((A|_{F'})_s)_{F\cap F'}$ and $((A|_F)_n)_{F\cap F'} = (A|_{F\cap F'})_n = ((A|_{F'})_n)_{F\cap F'}$. Hence we have $A_sA_n = A_nA_s$, where $A_s$ is locally semisimple and $A_n$ is locally nilpotent, and $A_s$ and $A_n$ are uniquely determined by these properties.

Moreover, for a locally finite element $A \in \mathrm{Aut}_{\mathbb{K}}(E)$ we have $A|_F \in \mathrm{Aut}_{\mathbb{K}}(F)$ for all finite dimensional $A$-invariant $\mathbb{K}$-subspaces $F \leq E$, implying that $A_s \in \mathrm{Aut}_{\mathbb{K}}(E)$. Hence we let $A_u := \mathrm{id}_E + (A_s)^{-1}A_n \in \mathrm{End}_{\mathbb{K}}(E)$. Since $A_u|_F = (A_F)_u \in \mathrm{Aut}_{\mathbb{K}}(F)$, for all finite dimensional $A$-invariant $\mathbb{K}$-subspaces $F \leq E$, we infer $A_u \in \mathrm{Aut}_{\mathbb{K}}(E)$, and thus obtain a Jordan decomposition $A = A_sA_u \in \mathrm{Aut}_{\mathbb{K}}(E)$. Hence we have $A_sA_u = A_uA_s$, where $A_s$ is locally semisimple and $A_u$ is locally unipotent, and $A_s$ and $A_u$ are uniquely determined by these properties.

**(6.6) Theorem: Jordan decomposition.**
Let $\mathbb{G}$ be an algebraic group.
**a)** For $g \in \mathbb{G}$ there are uniquely determined elements $g_s, g_u \in \mathbb{G}$, called the **semisimple** and **unipotent part** of $g$, respectively, such that $g = g_sg_u = g_ug_s$ as well as $(\rho_g^*)_s = \rho_{g_s}^*$ and $(\rho_g^*)_u = \rho_{g_u}^*$.
**b)** For $\mathbb{G} = \mathbb{GL}_n$ the semisimple and unipotent parts coincide with (6.4).
**c)** If $\varphi\colon \mathbb{G} \to \mathbb{H}$ is a homomorphism of algebraic groups, then for all $g \in \mathbb{G}$ we have $\varphi(g_s) = \varphi(g)_s$ and $\varphi(g_u) = \varphi(g)_u$.

**Proof. a)** Let $\mu\colon \mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}] \to \mathbb{K}[\mathbb{G}]\colon h \otimes h' \mapsto hh'$. Since $\gamma := \rho_g^*\colon \mathbb{K}[\mathbb{G}] \to \mathbb{K}[\mathbb{G}]$ is a $\mathbb{K}$-algebra homomorphism, we have $\mu(\gamma \otimes \gamma) = \gamma\mu$. We show that $\mu(\gamma_s \otimes \gamma_s) = \gamma_s\mu$: It suffices to consider finite dimensional $\gamma$-invariant $\mathbb{K}$-subspaces $F, E \leq \mathbb{K}[\mathbb{G}]$ such that $\mu(F \otimes F) \leq E$, hence we have $\mu(\gamma|_F \otimes \gamma|_F) = \gamma|_E\mu$. Since there is $f \in \mathbb{K}[T]$ such that $f(\gamma|_F \otimes \gamma|_F) = (\gamma|_F \otimes \gamma|_F)_s$ and $f(\gamma|_E) = (\gamma|_E)_s$, and since we have $(\gamma|_F \otimes \gamma|_F)_s = (\gamma|_F)_s \otimes (\gamma|_F)_s$, we conclude that $\mu((\gamma|_F)_s \otimes (\gamma|_F)_s) = (\gamma|_E)_s\mu$.

Hence $\gamma_s = (\rho_g^*)_s$ is a $\mathbb{K}$-algebra homomorphism. Thus the $\mathbb{K}$-algebra homomorphism $\mathbb{K}[\mathbb{G}] \to \mathbb{K}\colon h \mapsto ((\rho_g^*)_s(h))(1_{\mathbb{G}})$ defines $g_s \in \mathbb{G}$ such that $((\rho_g^*)_s(h))(1_{\mathbb{G}}) = h(g_s)$, for all $h \in \mathbb{K}[\mathbb{G}]$. Similarly, $\gamma_u = (\rho_g^*)_u$ is a $\mathbb{K}$-algebra homomorphisms as well, yielding $g_u \in \mathbb{G}$ such that $((\rho_g^*)_u(h))(1_{\mathbb{G}}) = h(g_u)$.

Letting $\mathbb{G} \times \mathbb{G}$ act on $\mathbb{G}$ via $[z; x, y] \mapsto x^{-1}zy$ shows that $\mathbb{K}[\mathbb{G}]$ is the union of finite dimensional $(\lambda_x^*\rho_y^*)$-invariant $\mathbb{K}$-subspaces for all $x, y \in \mathbb{G}$; see Exercise (12.13). Since $\lambda_x^*$ commutes with $\rho_y^*$, we deduce that $\lambda_x^*$ also commutes with $(\rho_y^*)_s$ and $(\rho_y^*)_u$. Hence we have $(\rho_{g_s}^*(h))(x) = h(xg_s) = (\lambda_{x^{-1}}^*(h))(g_s) = ((\rho_g^*)_s\lambda_{x^{-1}}^*(h))(1_{\mathbb{G}}) = (\lambda_{x^{-1}}^*(\rho_g^*)_s(h))(1_{\mathbb{G}}) = ((\rho_g^*)_s(h))(x)$, for all $h \in \mathbb{K}[\mathbb{G}]$ and $x \in \mathbb{G}$, implying $\rho_{g_s}^* = (\rho_g^*)_s$. Similarly we get $\rho_{g_u}^* = (\rho_g^*)_u$. Moreover, we have $\rho_{g_sg_u}^* = \rho_{g_s}^*\rho_{g_u}^* = (\rho_g^*)_s(\rho_g^*)_u = \rho_g^* = (\rho_g^*)_u(\rho_g^*)_s = \rho_{g_u}^*\rho_{g_s}^* = \rho_{g_ug_s}^*$, and since the representation $\mathbb{G} \to \mathbb{K}[\mathbb{G}]\colon g \mapsto \rho_g^*$ is faithful we infer $g = g_sg_u = g_ug_s \in \mathbb{G}$.

**b)** We have $\mathbb{K}[\mathbb{GL}_n] = \mathbb{K}[\mathcal{X}]_{\det_n}$, where $\mathcal{X} = \{X_{11}, \ldots, X_{nn}\}$. Let $\mathcal{B} := \{X_{11}, \ldots, X_{1n}\} \subseteq \mathbb{K}[\mathcal{X}]_{\det_n}$ and $F := \langle \mathcal{B} \rangle_{\mathbb{K}} \leq \mathbb{K}[\mathcal{X}]_{\det_n}$. Then for $A = [a_{ij}] \in \mathbb{GL}_n$ we have $\rho_A^*(X_{1i}) = \sum_{j=1}^n X_{1j}a_{ji}$, for $i \in \{1, \ldots, n\}$. Hence $F$ is $\rho_A^*$-invariant, and its matrix with respect to $\mathcal{B}$ is $(\rho_A^*|_F)_{\mathcal{B}} = A^{\mathrm{tr}}$. Hence we have

$(A_s)^{\mathrm{tr}} = (\rho^*_{A_s}|_F)_{\mathcal{B}} = ((\rho^*_A)_s|_F)_{\mathcal{B}} = ((\rho^*_A|_F)_s)_{\mathcal{B}} = ((\rho^*_A|_F)_{\mathcal{B}})_s = (A^{\mathrm{tr}})_s$, where left and right hand side are abstract and matrix semisimple part, respectively. Similarly, we deduce $(A_u)^{\mathrm{tr}} = (A^{\mathrm{tr}})_u$.

**c)** Since $\varphi(\mathbb{G}) \leq \mathbb{H}$ is closed, it is sufficient to consider the following two cases:

**i)** $\mathbb{G} \leq \mathbb{H}$ is closed and $\varphi$ is the natural embedding. Hence by (5.6) we have $\mathbb{G} = \{h \in \mathbb{H}; \rho^*_h(\mathcal{I}(\mathbb{G})) \subseteq \mathcal{I}(\mathbb{G})\}$, thus $(\rho^*_{\varphi(g)})_s(\mathcal{I}(\mathbb{G})) \subseteq \mathcal{I}(\mathbb{G})$ implies $\varphi(g)_s \in \mathbb{G}$, for all $g \in \mathbb{G}$. Now $\rho^*_{\varphi(g)}$ and $(\rho^*_{\varphi(g)})_s$ induce maps on $\mathbb{K}[\mathbb{H}]/\mathcal{I}(\mathbb{G}) \cong \mathbb{K}[\mathbb{G}]$, indicated by $\bar{\ }$. Thus $\rho^*_{g_s} = (\rho^*_g)_s = (\overline{\rho^*_{\varphi(g)}})_s = \overline{(\rho^*_{\varphi(g)})_s} = \overline{\rho^*_{\varphi(g)_s}} = \rho^*_{\varphi^{-1}(\varphi(g)_s)}$, hence $g_s = \varphi^{-1}(\varphi(g)_s)$, implying $\varphi(g_s) = \varphi(g)_s$. Similarly, we deduce $\varphi(g_u) = \varphi(g)_u$.

**ii)** $\varphi$ is surjective, hence $\mathbb{K}[\mathbb{H}] \subseteq \mathbb{K}[\mathbb{G}]$ and $\varphi^*$ is the natural embedding. For $g \in \mathbb{G}$ we have $(\rho^*_g\varphi^*(h))(x) = (\varphi^*(h))(xg) = h(\varphi(x)\varphi(g)) = (\rho^*_{\varphi(g)}(h))(\varphi(x)) = (\varphi^*\rho^*_{\varphi(g)}(h))(x)$, for all $h \in \mathbb{K}[\mathbb{H}]$ and $x \in \mathbb{G}$, hence $\rho^*_g\varphi^* = \varphi^*\rho^*_{\varphi(g)}$, implying that $\rho^*_g(\mathbb{K}[\mathbb{H}]) \subseteq \mathbb{K}[\mathbb{H}]$ and $\rho^*_{\varphi(g)} = \rho^*_g|_{\mathbb{K}[\mathbb{H}]}$. Thus we obtain $\rho^*_{\varphi(g)_s} = (\rho^*_{\varphi(g)})_s = (\rho^*_g|_{\mathbb{K}[\mathbb{H}]})_s = (\rho^*_g)_s|_{\mathbb{K}[\mathbb{H}]} = \rho^*_{g_s}|_{\mathbb{K}[\mathbb{H}]} = \rho^*_{\varphi(g_s)}$, implying $\varphi(g)_s = \varphi(g_s)$. Similarly, we deduce $\varphi(g)_u = \varphi(g_u)$. ♯

**(6.7) Corollary.** Let $\mathbb{G}$ be an algebraic group.
**a)** For $g \in \mathbb{G}$ the following are equivalent:
**i)** The element $g \in \mathbb{G}$ is semisimple (unipotent).
**ii)** There is an injective homomorphism of algebraic groups $\varphi\colon \mathbb{G} \to \mathbb{GL}_n$, for some $n \in \mathbb{N}$, such that $\varphi(g) \in \mathbb{GL}_n$ is semisimple (unipotent).
**iii)** For any homomorphism of algebraic groups $\varphi\colon \mathbb{G} \to \mathbb{GL}_n$, where $n \in \mathbb{N}$, the image $\varphi(g) \in \mathbb{GL}_n$ is semisimple (unipotent).

**b)** The set $\mathbb{G}_u := \{g \in \mathbb{G}; g \text{ unipotent}\} \subseteq \mathbb{G}$ is closed, called the **unipotent variety** of $\mathbb{G}$.

**Proof. b)** We have $(\mathbb{GL}_n)_u = \{A \in \mathbb{K}^{n\times n}; (A - E_n)^n = 0\} \subseteq \mathbb{K}^{n\times n}$ closed. Thus any injective homomorphism of algebraic groups $\varphi\colon \mathbb{G} \to \mathbb{GL}_n$ shows that $\mathbb{G}_u = \varphi^{-1}((\mathbb{GL}_n)_u) \subseteq \mathbb{G}$ is closed. ♯

For the set $\mathbb{G}_s := \{g \in \mathbb{G}; g \text{ semisimple}\}$ a similar statement does in general not hold, see Exercise (12.20). For examples see Exercises (12.18) and (12.19).

**(6.8) Proposition. a)** Let $\mathcal{S} \subseteq \mathbb{K}^{n\times n}$ be a set of pairwise commuting matrices. Then $\mathcal{S}$ is **trigonalisable**, i. e. there is $B \in \mathbb{GL}_n$ such that $\mathcal{S}^B := B^{-1}\mathcal{S}B \subseteq \mathbb{K}^{n\times n}$ consists of upper triangular matrices.

If moreover $\mathcal{S}$ consists of semisimple matrices, then $\mathcal{S}$ is even **diagonalisable**, i. e. there is $B \in \mathbb{GL}_n$ such that $\mathcal{S}^B \subseteq \mathbb{K}^{n\times n}$ consists of diagonal matrices.

**b)** Let $G \leq \mathbb{GL}_n$ be a (not necessarily closed) subgroup consisting of unipotent matrices. Then $G$ is trigonalisable, i. e. there is $B \in \mathbb{GL}_n$ such that $G^B \leq \mathbb{U}_n := \{[a_{ij}] \in \mathbb{GL}_n; a_{ij} = 0 \text{ for } i > j, a_{ii} = 1\}$, see also Exercise (12.4).

In particular, a **unipotent** algebraic group $\mathbb{H}$, i. e. we have $\mathbb{H} = \mathbb{H}_u$, is isomorphic as an algebraic group to a closed subgroup of $\mathbb{U}_n$, for some $n \in \mathbb{N}$.

**Proof. a)** For general $\mathcal{S}$ we proceed by induction on $n$, the case $n = 1$ being trivial. Now we may assume that there is $A \in \mathcal{S}$ and $\lambda \in \mathbb{K}$ such that $\{0\} < E_\lambda(A) := E_{\lambda,1}(A) = \ker(A - \lambda E_n) < \mathbb{K}^n$. Since $E_\lambda(A)$ is $\mathcal{S}$-invariant, by induction there is $0 \neq x_n \in E_\lambda(A)$ such that $\langle x_n \rangle_\mathbb{K} < \mathbb{K}^n$ is $\mathcal{S}$-invariant. Again by induction there are $x_1, \ldots, x_{n-1} \in \mathbb{K}^n$ such that $\{x_1, \ldots, x_n\} \subseteq \mathbb{K}^n$ is a $\mathbb{K}$-basis and $\langle x_i, \ldots, x_{n-1} \rangle_\mathbb{K} \leq \mathbb{K}^n / \langle x_n \rangle_\mathbb{K}$ is $\mathcal{S}$-invariant, for all $i \in \{1, \ldots, n-1\}$.

If $\mathcal{S}$ consists of semisimple matrices, then we again proceed by induction on $n$, the case $n = 1$ being trivial. Now we may assume that there is $A \in \mathcal{S}$ such that $\mathbb{K}^n \cong \bigoplus_{i=1}^r E_{\lambda_i}(A)$ for some $r > 1$, where $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$ are the pairwise distinct eigenvalues of $A$, and we are done by induction.

**b)** We first show that $G$ acts irreducibly on $\mathbb{K}^n$ if and only if $n = 1$: Let $G$ act irreducibly, and let $\mathcal{A} := \mathbb{K}\langle G \rangle \subseteq \mathbb{K}^{n \times n}$ be the (non-commutative) $\mathbb{K}$-subalgebra of $\mathbb{K}^{n \times n}$ generated by $G$. Hence $\mathcal{A}$ acts faithfully on $\mathbb{K}^n$, thus by Schur's Lemma and the double centraliser theorem we have $\mathcal{A} = \mathbb{K}^{n \times n}$. Since for all $A, B \in G$ we have $\mathrm{Tr}((A - E_n)B) = \mathrm{Tr}(AB) - \mathrm{Tr}(B) = 0$, we conclude $\mathrm{Tr}((A - E_n)C) = 0$ for all $C \in \mathbb{K}^{n \times n}$, implying $\mathrm{Tr}(E_{ij}(A - E_n)E_{kl}) = 0$ for all $i, j, k, l \in \{1, \ldots, n\}$, where $E_{ij} = [\delta_{ik}\delta_{jl}]_{kl} \in \mathbb{K}^{n \times n}$ is the $[i, j]$-th **matrix unit**. Thus we have $A = E_n$, hence $G = \{E_n\}$ and $n = 1$.

We now proceed by induction on $n$, the case $n = 1$ being trivial. Let $\{0\} < W < \mathbb{K}^n$ be a $G$-invariant $\mathbb{K}$-subspace. Then by induction there is $0 \neq x_n \in W$ such that $x_n A = x_n$ for all $A \in G$. Again by induction there are $x_1, \ldots, x_{n-1} \in \mathbb{K}^n$ such that $\{x_1, \ldots, x_n\} \subseteq \mathbb{K}^n$ is a $\mathbb{K}$-basis and $x_i A - x_i \in \langle x_{i+1}, \ldots, x_n \rangle_\mathbb{K}$, for all $i \in \{1, \ldots, n-1\}$ and all $A \in G$. ♯

**(6.9) Theorem.** Let $\mathbb{G}$ be an abelian algebraic group. Then both sets $\mathbb{G}_s$ and $\mathbb{G}_u$ are closed subgroups, and $\mu \colon \mathbb{G}_s \times \mathbb{G}_u \to \mathbb{G} \colon [g_s, g_u] \to g_s g_u$ is an isomorphism of algebraic groups, inducing an isomorphism $(\mathbb{G}_s)^\circ \times (\mathbb{G}_u)^\circ \to \mathbb{G}^\circ$.

**Proof.** We may assume that $\mathbb{G} \leq \mathbb{GL}_n$ closed, for some $n \in \mathbb{N}$. Since $\mathbb{G}$ is abelian, $\mathbb{G}_s, \mathbb{G}_u \leq \mathbb{G}$ are subgroups, and $\mu$ is a bijective homomorphism of algebraic groups. The set $\mathbb{G}_u \subseteq \mathbb{G}$ is closed, and we show that $\mathbb{G}_s \subseteq \mathbb{G}$ also is closed: For any family $\Lambda := \{\lambda_A \in \mathbb{K}; A \in \mathbb{G}_s\}$ let $W_\Lambda := \bigcap_{A \in \mathbb{G}_s} E_\lambda(A) \leq \mathbb{K}^n$. Hence we have $\mathbb{K}^n \cong \bigoplus_{i=1}^r W_{\Lambda_r}$, for some $r \in \mathbb{N}$ and certain families $\Lambda_r$, where the $W_{\Lambda_r}$ are $\mathbb{G}$-invariant. Thus we may assume that $\mathbb{G}_s \subseteq \mathbb{T}_n := \{[a_{ij}] \in \mathbb{GL}_n; a_{ij} = 0 \text{ for } i \neq j\}$ and $\mathbb{G}_u \subseteq \mathbb{U}_n$, thus $\mathbb{G} \subseteq \mathbb{B}_n := \{[a_{ij}] \in \mathbb{GL}_n; a_{ij} = 0 \text{ for } i > j\}$; see also Exercise (12.4). Hence $\mathbb{G}_s = \mathbb{G} \cap \mathbb{T}_n \subseteq \mathbb{G}$ is closed. The morphism $\mathbb{B}_n \to \mathbb{T}_n \colon [a_{ij}] \mapsto \mathrm{diag}[a_{11}, \ldots, a_{nn}]$ restricts to the morphism $\mathbb{G} \to \mathbb{G}_s \colon g \mapsto g_s$, hence $\mu^{-1} \colon \mathbb{G} \to \mathbb{G}_s \times \mathbb{G}_u \colon g \mapsto [g_s, g_s^{-1} g]$ is a morphism. ♯

## 7 Actions on affine varieties

**(7.1) Lemma.** Let $\mathbb{G}$ be an algebraic group acting morphically on $V$, and let $x \in V$. Then we have $\dim(\overline{x\mathbb{G}}) = \dim(\overline{x\mathbb{G}^\circ})$ and $\dim(C_{\mathbb{G}}(x)) = \dim(C_{\mathbb{G}^\circ}(x))$, as well as $\dim(\mathbb{G}) = \dim(C_{\mathbb{G}}(x)) + \dim(\overline{x\mathbb{G}})$.

**Proof.** We have $x\mathbb{G} = \bigcup_{g \in \mathbb{G}^\circ|\mathbb{G}} x\mathbb{G}^\circ g$, implying $\overline{x\mathbb{G}} = \bigcup_{g \in \mathbb{G}^\circ|\mathbb{G}} \overline{x\mathbb{G}^\circ} g$, since $\overline{x\mathbb{G}^\circ} \subseteq V$ is irreducible implying $\dim(\overline{x\mathbb{G}}) = \dim(\overline{x\mathbb{G}^\circ})$; see also Exercise (12.22). Since $C_{\mathbb{G}^\circ}(x) \leq C_{\mathbb{G}}(x)$ is a closed subgroup of finite index, we have $\dim(C_{\mathbb{G}}(x)) = \dim(C_{\mathbb{G}^\circ}(x))$.

Hence to show the last assertion, we may assume that $\mathbb{G}$ is connected. Letting $\varphi$ be the action morphism, the orbit map $\varphi_x \colon \mathbb{G} \to \overline{x\mathbb{G}}$ is a dominant morphism between irreducible varieties. Hence there is $\emptyset \neq U \subseteq \overline{x\mathbb{G}}$ such that $U \subseteq x\mathbb{G}$, and such that $\dim(\varphi_x^{-1}(y)) = \dim(\mathbb{G}) - \dim(\overline{x\mathbb{G}})$ for all $y \in U$. For any $y \in U$ we have $\varphi_x^{-1}(y) = \{h \in \mathbb{G}; xh = y\} = C_{\mathbb{G}}(x)g \subseteq \mathbb{G}$, where $g \in \mathbb{G}$ is fixed such that $y = xg$, implying $\dim(\varphi_x^{-1}(y)) = \dim(C_{\mathbb{G}}(x))$. $\sharp$

**(7.2) Proposition: Closed orbit lemma.**
Let $\mathbb{G}$ be an algebraic group acting morphically on $V$.
**a)** Let $O \subseteq V$ be a $\mathbb{G}$-orbit. Then $\overline{O} \subseteq V$ is $\mathbb{G}$-invariant, $O \subseteq \overline{O}$ is open, and if $O \neq \overline{O}$ then $\dim(\overline{O} \setminus O) < \dim(\overline{O})$.
**b)** For $\mathbb{G}$-orbits $O, O' \subseteq V$ such that $O' \subseteq \overline{O}$ we write $O' \preceq O$. Then the **orbit closure relation** $\preceq$ is a partial order on the set of $\mathbb{G}$-orbits in $V$. Moreover, there are $\preceq$-minimal orbits, all of which are closed.

**Proof. a)** Let $\varphi$ be the action morphism. Since for all $g \in \mathbb{G}$ the morphism $\varphi_g$ is continuous, we from $\varphi_g(O) \subseteq O$ get $\varphi_g(\overline{O}) \subseteq \overline{O}$, hence $\overline{O}$ is $\mathbb{G}$-invariant.

Let $O = \varphi_x(\mathbb{G})$, for some $x \in V$, let $\emptyset \neq U \subseteq \overline{O}$ be open such that $U \subseteq O$, and let $h \in \mathbb{G}$ such that $xh \in U$. Thus $x \in Uh^{-1}$, implying that $O = x\mathbb{G} \subseteq \bigcup_{g \in \mathbb{G}} Ug \subseteq O$, and hence $O = \bigcup_{g \in \mathbb{G}} Ug$, where $Ug \subseteq \overline{O}$ is open for all $g \in \mathbb{G}$.

Let $O \neq \overline{O} = \bigcup_{i=1}^r W_i$, where the $W_i \subseteq \overline{O}$ are the irreducible components; hence $\overline{O} \setminus O = \bigcup_{i=1}^r (W_i \setminus O)$. Since $O \subseteq \overline{O}$ is open and dense, we have $W_i \cap O \neq \emptyset$ for all $i \in \{1, \ldots, r\}$, hence whenever $W_i \not\subseteq O$ we have $\dim(W_i \setminus O) < \dim(W_i)$.

**b)** To show that $\preceq$ is a partial order, we only have to check that $O' \preceq O \preceq O'$ implies $O = O'$: Let $O' \subseteq \overline{O}$ and $O \subseteq \overline{O'}$. Hence $\overline{O'} \subseteq \overline{O} \subseteq \overline{O'}$, and both $O, O' \subseteq \overline{O} = \overline{O'}$ are open and dense, implying that $O \cap O' \neq \emptyset$, thus $O = O'$. $\sharp$

For examples see Exercise (12.23). For the **Kostant-Rosenlicht Theorem**, dealing with orbits of unipotent groups, see Exercise (12.24).

**(7.3) Example: The unipotent variety of $\mathbb{SL}_n$.**
Let $\mathbb{G} := \mathbb{SL}_n$, for some $n \in \mathbb{N}$. Then $\mathbb{GL}_n$ acts morphically on the unipotent variety $\mathbb{G}_u = (\mathbb{GL}_n)_u \subseteq \mathbb{G} \subseteq \mathbb{GL}_n$, which hence is a union of $\mathbb{GL}_n$-conjugacy

classes, and a union of $\mathbb{G}$-conjugacy classes. Since $\mathbb{G} \lhd \mathbb{GL}_n = \mathbb{Z}_n \cdot \mathbb{G}$, where $\mathbb{Z}_n := \mathbb{K}^* \cdot E_n = Z(\mathbb{GL}_n) \leq \mathbb{GL}_n$, elements of $\mathbb{G}$ are $\mathbb{G}$-conjugate if and only if they are $\mathbb{GL}_n$-conjugate.

By the Jordan normal form theorem we conclude $\mathbb{G}_u = \bigcup_{g \in \mathbb{G}} (\mathbb{U}_n)^g = \mathrm{im}(\kappa)$, where $\kappa \colon \mathbb{U}_n \times \mathbb{G} \to \mathbb{G}_u \colon [u, g] \mapsto u^g$ is the conjugation map. Now $\mathbb{U}_n := \{[a_{ij}] \in \mathbb{GL}_n; a_{ij} = 0 \text{ for } i > j; a_{ii} = 1\} \subseteq \mathbb{G} \subseteq \mathbb{GL}_n \subseteq \mathbb{K}^{n \times n}$ is closed such that $\mathcal{I}(\mathbb{U}_n) = \langle X_{ij}, X_{ii} - 1; i, j \in \{1, \ldots, n\}, i > j \rangle \lhd \mathbb{K}[\mathcal{X}]$, where $\mathcal{X} := \{X_{11}, \ldots, X_{nn}\}$, hence $\mathbb{K}[\mathbb{U}_n] \cong \mathbb{K}[\mathcal{X}]/\mathcal{I}(\mathbb{U}_n) \cong \mathbb{K}[X_{ij}; i, j \in \{1, \ldots, n\}, j > i]$, implying that $\mathbb{U}_n \cong \mathbb{K}^{\frac{n(n-1)}{2}}$ is irreducible. Since $\mathbb{G}$ is irreducible, $\mathbb{G}_u$ is irreducible as well. We proceed to describe the orbit closure relation $\preceq$ on $\mathbb{G}_u$:

Elements of $\mathbb{G}_u$ are conjugate if and only if their Jordan normal forms coincide. The latter up to reordering are uniquely described by the sizes $\lambda_1, \ldots, \lambda_l \in \mathbb{N}$, for some $l \in \mathbb{N}$, of the Jordan blocks $J_{\lambda_i}(1) \in \mathbb{K}^{\lambda_i \times \lambda_i}$ with respect to the eigenvalue $1 \in \mathbb{K}$ occurring. We have $\sum_{i=1}^{l} \lambda_i = n$ and we may assume that $\lambda_1 \geq \cdots \geq \lambda_l \geq 1$; then the conjugacy class associated to $\lambda := [\lambda_1, \ldots, \lambda_l]$ is denoted by $C_\lambda \subseteq \mathbb{G}_u$. Thus the conjugacy classes in $\mathbb{G}_u$ are parametrised by the partitions of $n$:

**(7.4) Definition and Remark. a)** Let $n \in \mathbb{N}_0$. A series $\lambda := [\lambda_1, \lambda_2, \ldots] \subseteq \mathbb{N}_0$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i \geq 1} \lambda_i = n$ is called a **partition** of $n$, the $\lambda_i$ being called its **parts**; we write $\lambda \vdash n$, where we have $\lambda_{n+1} = 0$ and usually omit the zero parts. Let $\mathcal{P}_n$ be the set of partitions of $n$.

Associated to $\lambda = [\lambda_1, \ldots, \lambda_n] \vdash n$ is the **Young diagram** or **Ferrers diagram** $\mathcal{Y}_\lambda := \{[i, j] \in \mathbb{N}^2; i \in \{1, \ldots, n\}, j \in \{1, \ldots, \lambda_i\}\}$, allowing to identify any partition with its Young diagram. Moreover, letting $a_i = a_i(\lambda) := |\{j \in \{1, \ldots, n\}; \lambda_j = i\}| \in \mathbb{N}_0$, for $i \in \{1, \ldots, n\}$, we also write $\lambda = [1^{a_1}, \ldots, n^{a_n}] \vdash n$.

**b)** Let $\lambda = [\lambda_1, \ldots, \lambda_n] \vdash n$, and let $\lambda'_i := |\{j \in \{1, \ldots, n\}; \lambda_j \geq i\}| \in \mathbb{N}_0$ for $i \in \{1, \ldots, n\}$. Hence we have $\lambda'_1 \geq \cdots \geq \lambda'_n \geq 0$ as well as $\sum_{i=1}^{n} \lambda'_i = \sum_{j=1}^{n} |\{i \in \{1, \ldots, n\}; i \leq \lambda_j\}| = \sum_{j=1}^{n} \lambda_j = n$. Thus $\lambda' = [\lambda'_1, \ldots, \lambda'_n] \vdash n$, being called the associated **conjugate partition**.

Hence we have $\lambda'_i = \sum_{j=i}^{n} a_j(\lambda)$, for $i \in \{1, \ldots, n\}$. Moreover, we have $\mathcal{Y}_\lambda = \{[i, j] \in \mathbb{N}^2; i \in \{1, \ldots, n\}, j \in \{1, \ldots, \lambda_i\}\} = \{[i, j] \in \mathbb{N}^2; j \in \{1, \ldots, n\}, i \in \{k \in \{1, \ldots, n\}; \lambda_k \geq j\}\} = \{[i, j] \in \mathbb{N}^2; j \in \{1, \ldots, n\}, i \in \{1, \ldots, \lambda'_j\}\}$, implying that $\mathcal{Y}_{\lambda'} = \{[i, j] \in \mathbb{N}^2; [j, i] \in \mathcal{Y}_\lambda\}$, and thus $(\lambda')' = \lambda$.

**c)** Let $\lambda = [\lambda_1, \ldots, \lambda_n] \vdash n$ and $\mu = [\mu_1, \ldots, \mu_n] \vdash n$. Then $\mu$ is called to **dominate** $\lambda$, if for all $k \in \{1, \ldots, n\}$ we have $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$; we write $\lambda \unlhd \mu$. The **dominance relation** $\unlhd$ is a partial order on $\mathcal{P}_n$; if $\mu \neq \lambda \unlhd \mu$ we write $\lambda \lhd \mu$, and if moreover $\lambda$ is maximal with this property we write $\lambda \lhd_{\max} \mu$; see also Exercise (12.25).

**d)** We have $\lambda \lhd_{\max} \mu$ if and only if $\mu = [\lambda_1, \ldots, \lambda_{r-1}, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{s-1}, \lambda_s - 1, \lambda_{s+1}, \ldots, \lambda_n]$, where $1 \leq r < s \leq n$ as well as $\lambda_{r-1} > \lambda_r$ and $\lambda_s > \lambda_{s+1}$, such that either $s = r + 1$, or $s > r + 1$ and $\lambda_r = \lambda_s$, see Exercise (12.26):

If $\lambda \lhd_{\max} \mu$, let $r := \min\{i \in \{1, \ldots, n\}; \lambda_i \neq \mu_i\}$ and $r < s := \min\{k \in \{r+1, \ldots, n\}; \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \mu_i\} \leq n$. Hence we have $\lambda_r < \mu_r$, and $\mu_r \leq \mu_{r-1} = \lambda_{r-1}$ if $r > 1$, as well as $\lambda_s > \mu_s \geq \mu_{s+1} \geq \lambda_{s+1}$. This yields $\lambda \lhd \nu := [\lambda_1, \ldots, \lambda_{r-1}, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{s-1}, \lambda_s - 1, \lambda_{s+1}, \ldots, \lambda_n] \unlhd \mu$, hence $\nu = \mu$. It remains to show $\lambda_r = \lambda_s$ whenever $s > r+1$: Assume to the contrary that $\lambda_r > \lambda_s$, and let $r < t := 1 + \min\{i \in \{r, \ldots, s-1\}; \lambda_i \neq \lambda_{i+1}\} \leq s$. If $t = s$ then $\lambda \lhd [\lambda_1, \ldots, \lambda_{r-1}, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{s-2}, \lambda_{s-1} - 1, \lambda_s, \ldots, \lambda_n] \lhd \nu = \mu$, while if $t < s$ then $\lambda \lhd [\lambda_1, \ldots, \lambda_{t-1}, \lambda_t + 1, \lambda_{t+1}, \ldots, \lambda_{s-1}, \lambda_s - 1, \lambda_{s+1}, \ldots, \lambda_n] \lhd \nu = \mu$, a contradiction.

Let conversely $\mu$ be as asserted, and let $\nu = [\nu_1, \ldots, \nu_n] \vdash n$ such that $\lambda \lhd_{\max} \nu \unlhd \mu$. Hence for $i \notin \{r, \ldots, s\}$ we have $\nu_i = \lambda_i$. Thus if $s = r+1$ we conclude $\nu_r = \lambda_r + 1$ and $\nu_{r+1} = \lambda_{r+1} - 1$, thus $\nu = \mu$. If $s > r+1$ and hence $\lambda_r = \lambda_s$, then there are $r \leq r' < s' \leq s$ such that $\nu_i = \lambda_i$ for $i \notin \{r', s'\}$ as well as $\nu_{r'} = \lambda_{r'} + 1$ and $\nu_{s'} = \lambda_{s'} - 1$. Since $\lambda_{r'} = \nu_{r'} - 1 \leq \nu_{r'-1} - 1 = \lambda_{r'-1} - 1 < \lambda_{r'-1}$, whenever $r' > 1$, and $\lambda_{s'} = \nu_{s'} + 1 \geq \nu_{s'+1} + 1 = \lambda_{s'+1} + 1 > \lambda_{s'+1}$ this implies $r' = r$ and $s' = s$, hence $\nu = \mu$ in this case as well. ♯

**e)** Finally, $\lambda \unlhd \mu$ implies $\mu' \unlhd \lambda'$: Assume to the contrary that $\mu' \not\!\unlhd \lambda'$. Then for some $k \in \{1, \ldots, n\}$ we have $\sum_{i=1}^{j} \mu'_i \leq \sum_{i=1}^{j} \lambda'_i$ for all $j \in \{1, \ldots, k-1\}$, and $\sum_{i=1}^{k} \mu'_i > \sum_{i=1}^{k} \lambda'_i$. Hence we have $\mu'_k > \lambda'_k$ and $\sum_{i=k+1}^{n} \mu'_i < \sum_{i=k+1}^{n} \lambda'_i$. Considering Young diagrams shows $\sum_{i=k+1}^{n} \mu'_i = \sum_{j=1}^{\mu'_k} (\mu_j - k)$ and similarly $\sum_{i=k+1}^{n} \lambda'_i = \sum_{j=1}^{\lambda'_k} (\lambda_j - k)$. Since $\mu_j \geq k$ for $j \in \{1, \ldots, \mu'_k\}$, this implies $\sum_{j=1}^{\lambda'_k} (\lambda_j - k) > \sum_{j=1}^{\mu'_k} (\mu_j - k) \geq \sum_{j=1}^{\lambda'_k} (\mu_j - k)$, thus $\lambda \not\!\unlhd \mu$, a contradiction.

**(7.5) Proposition.** Let $\mathbb{G} := \mathbb{SL}_n$, for some $n \in \mathbb{N}$, and let $\lambda \vdash n$.
**a)** Let $A \in \mathbb{G}_u$. Then we have $A \in C_\lambda$ if and only if for all $k \in \{1, \ldots, n\}$ we have $\sum_{i=1}^{k} \lambda'_i = n - \mathrm{rk}_{\mathbb{K}}((A - E_n)^k)$.
**b)** The set $C_{\unlhd\lambda} := \bigcup_{\mu \unlhd \lambda} C_\mu \subseteq \mathbb{G}_u$ is closed.

**Proof. a)** For a Jordan block $J_m(1) \in \mathbb{K}^{m \times m}$, for some $m \in \mathbb{N}$, we have $\mathrm{rk}_{\mathbb{K}}((J_m(1) - E_m)^k) = m - k$ for all $k \in \{0, \ldots, m\}$. Thus for $A \in C_\lambda$, where $\lambda = [1^{a_1}, \ldots, n^{a_n}] \vdash n$, we get $\sum_{i=k+1}^{n} (i-k) a_i = \mathrm{rk}_{\mathbb{K}}((A - E_n)^k)$ for all $k \in \{0, \ldots, n-1\}$. Hence the rank vector $[\mathrm{rk}_{\mathbb{K}}((A - E_n)^k); k \in \{0, \ldots, n-1\}] \in \mathbb{Q}^n$ is determined by $\lambda$, and since the above conditions form a unitriangular system of $n$ linear equations for $[a_1, \ldots, a_n] \in \mathbb{Q}^n$, the latter conversely is determined by the rank vector; we anyway have $\mathrm{rk}_{\mathbb{K}}((A - E_n)^0) = n$ and $\mathrm{rk}_{\mathbb{K}}((A - E_n)^n) = 0$. Finally we have $\sum_{i=1}^{k} \lambda'_i = \sum_{i=1}^{n} \lambda'_i - \sum_{i=k+1}^{n} \lambda'_i = n - \sum_{i=k+1}^{n} (\sum_{j=i}^{n} a_j) = n - \sum_{j=k+1}^{n} (j-k) a_j = n - \mathrm{rk}_{\mathbb{K}}((A - E_n)^k)$, for all $k \in \{1, \ldots, n\}$.
**b)** Let $A \in C_\lambda$ and $B \in C_\mu$, where $\mu \vdash n$. Then we have $\mu \unlhd \lambda$ if and only if $\lambda' \unlhd \mu'$, which by the above holds if and only if $\mathrm{rk}_{\mathbb{K}}((A - E_n)^k) \geq \mathrm{rk}_{\mathbb{K}}((B - E_n)^k)$ for all $k \in \{1, \ldots, n\}$. Thus we have

$$C_{\unlhd\lambda} = \{B \in \mathbb{G}_u; \mathrm{rk}_{\mathbb{K}}((B - E_n)^k) \leq \mathrm{rk}_{\mathbb{K}}((A - E_n)^k) \text{ for all } k \in \{1, \ldots, n\}\}.$$

Given $k \in \{1,\ldots,n\}$ and $m \in \{0,\ldots,n\}$, we have $\mathrm{rk}_{\mathbb{K}}((B-E_n)^k) \leq m$, if and only if all $((m+1)\times(m+1))$-minors of $(B-E_n)^k$ vanish. The latter are polynomial conditions in the matrix entries of $B$, hence $C_{\trianglelefteq\lambda} \subseteq \mathbb{G}$ is closed. ♯

**(7.6) Theorem: The unipotent variety of $\mathbb{SL}_n$.**
Let $\mathbb{G} := \mathbb{SL}_n$, for some $n \in \mathbb{N}$, and let $C_\lambda, C_\mu \subseteq \mathbb{G}_u$, where $\lambda, \mu \vdash n$. Then we have $C_\mu \preceq C_\lambda$ if and only if $\mu \trianglelefteq \lambda$.

**Proof.** We have already shown that $C_\mu \preceq C_\lambda$, i. e. $C_\mu \subseteq \overline{C_\lambda} \subseteq \mathbb{G}$, implies $C_\mu \subseteq C_{\trianglelefteq\lambda}$, hence $\mu \trianglelefteq \lambda$. We prove the converse: Let $\lambda = [\lambda_1,\ldots,\lambda_l] \vdash n$, where $\lambda_l > 0$, and let $\mathbb{U}_\lambda \leq \mathbb{U}_n$ the subgroup of all block unitriangular matrices

$$B = \left[\begin{array}{c|c|c|c|c} E_{\lambda_1} & B_{12} & B_{13} & \cdots & B_{1l} \\ \hline 0 & E_{\lambda_2} & B_{23} & \cdots & B_{2l} \\ \hline \vdots & \ddots & \ddots & \ddots & \vdots \\ \hline 0 & \cdots & 0 & E_{\lambda_{l-1}} & B_{l-1,l} \\ \hline 0 & \cdots & 0 & 0 & E_{\lambda_l} \end{array}\right] \in \mathbb{U}_n,$$

where $B_{ij} \in \mathbb{K}^{\lambda_i \times \lambda_j}$ for $i,j \in \{1,\ldots,l\}$. It is immediate that $\mathbb{U}_\lambda \subseteq \mathbb{U}_n$ is closed and that $\mathbb{U}_\lambda \cong \mathbb{K}^N$, for some $N \in \mathbb{N}_0$, hence $\mathbb{U}_\lambda$ is irreducible. Moreover, it is immediate that $\mathrm{rk}_{\mathbb{K}}((B-E_n)^k) \leq n - \sum_{i=1}^k \lambda_i$, for all $k \in \{1,\ldots,n\}$. Let

$$A_\lambda := \left[\begin{array}{c|c|c|c|c} E_{\lambda_1} & A_{12} & 0 & \cdots & 0 \\ \hline 0 & E_{\lambda_2} & A_{23} & \cdots & 0 \\ \hline \vdots & \ddots & \ddots & \ddots & \vdots \\ \hline 0 & \cdots & 0 & E_{\lambda_{l-1}} & A_{l-1,l} \\ \hline 0 & \cdots & 0 & 0 & E_{\lambda_l} \end{array}\right] \in \mathbb{U}_\lambda,$$

where $A_{i,i+1} := \sum_{j=1}^{\lambda_{i+1}} E_{jj} \in \mathbb{K}^{\lambda_i \times \lambda_{i+1}}$ for all $i \in \{1,\ldots,l-1\}$, and where $E_{jj}$ is the $[j,j]$-th matrix unit. It is again immediate that we have equality $\mathrm{rk}_{\mathbb{K}}((A_\lambda - E_n)^k) = n - \sum_{i=1}^k \lambda_i$, for all $k \in \{1,\ldots,n\}$. This implies that $A_\lambda \in C_{\lambda'}$. Moreover, we have $\mathrm{rk}_{\mathbb{K}}((B-E_n)^k) \leq \mathrm{rk}_{\mathbb{K}}((A_\lambda - E_n)^k)$, for all $k \in \{1,\ldots,n\}$ and all $B \in \mathbb{U}_\lambda$, implying that $\mathbb{U}_\lambda \subseteq C_{\trianglelefteq\lambda'}$.

We show that $\mathbb{U}_\lambda \subseteq \overline{C_{\lambda'}} \subseteq C_{\trianglelefteq\lambda'}$: Let still $\kappa\colon \mathbb{U}_n \times \mathbb{G} \to \mathbb{G}_u$ be the conjugation map, and let $V_\lambda := \mathrm{im}(\kappa|_{\mathbb{U}_\lambda\times\mathbb{G}}) \subseteq \mathbb{G}_u$. Hence $V_\lambda$ is irreducible and $\mathbb{G}$-invariant, and thus $\overline{V_\lambda}$ is irreducible and $\mathbb{G}$-invariant. Letting $V_\lambda = \bigcup_{\nu\in\mathcal{N}} C_\nu$, for some $\mathcal{N} \subseteq \mathcal{P}_n$, we get $\overline{V_\lambda} = \bigcup_{\nu\in\mathcal{N}} \overline{C_\nu}$, and hence there is $\widehat{\lambda} \vdash n$ such that $\overline{V_\lambda} = \overline{C_{\widehat{\lambda}}}$. Since $C_{\trianglelefteq\lambda'} \subseteq \mathbb{G}_u$ is closed and $\mathbb{G}$-invariant, we have $\mathbb{U}_\lambda \subseteq V_\lambda \subseteq \overline{V_\lambda} = \overline{C_{\widehat{\lambda}}} \subseteq C_{\trianglelefteq\lambda'}$, implying $\widehat{\lambda} \trianglelefteq \lambda'$. Conversely we have $A_\lambda \in C_{\lambda'} \cap \mathbb{U}_\lambda$, implying $C_{\lambda'} \subseteq V_\lambda \subseteq \overline{C_{\widehat{\lambda}}}$, i. e. $C_{\lambda'} \preceq C_{\widehat{\lambda}}$, and thus $\lambda' \trianglelefteq \widehat{\lambda}$. Hence $\lambda' = \widehat{\lambda}$ and $\mathbb{U}_\lambda \subseteq \overline{C_{\widehat{\lambda}}} = \overline{C_{\lambda'}}$.

Now, to prove that $\mu \trianglelefteq \lambda$ indeed implies $C_\mu \preceq C_\lambda$, we may assume that $\mu := [\lambda_1,\ldots,\lambda_{r-1},\lambda_r - 1,\lambda_{r+1},\ldots,\lambda_{s-1},\lambda_s+1,\lambda_{s+1},\ldots,\lambda_n] \lhd_{\max} \lambda$, where $1 \leq r < s \leq n$. Hence there are representatives $\mathrm{diag}[J_{\lambda_r}(1), J_{\lambda_s}(1), B] \in C_\lambda$ and

$\mathrm{diag}[J_{\lambda_r-1}(1), J_{\lambda_s+1}(1), B] \in C_\mu$, where $m := \lambda_r + \lambda_s \leq n$ and $B \in \mathbb{U}_{n-m}$. Let $\widetilde{\mu} := [\lambda_r - 1, \lambda_s + 1] \vdash m$ and $\widetilde{\lambda} := [\lambda_r, \lambda_s] \vdash m$. Hence we have $\widetilde{\mu} \lhd \widetilde{\lambda}$ as well as $\mathrm{diag}[J_{\lambda_r-1}(1), J_{\lambda_s+1}(1)] \in C_{\widetilde{\mu}} \subseteq (\mathbb{SL}_m)_u$ and $\mathrm{diag}[J_{\lambda_r}(1), J_{\lambda_s}(1)] \in C_{\widetilde{\lambda}} \subseteq (\mathbb{SL}_m)_u$. It is immediate that $\mathbb{SL}_m \to \mathbb{G}\colon A \mapsto \mathrm{diag}[A, E_{m-n}]$ is a closed embedding of algebraic groups, which extends to a closed embedding of affine varieties $\mathbb{SL}_m \times \{B\} \to \mathbb{G}\colon [A, B] \mapsto \mathrm{diag}[A, B]$, where $B \in \mathbb{U}_{n-m}$ is as above. Thus it suffices to show that $C_{\widetilde{\mu}} \preceq C_{\widetilde{\lambda}}$, i. e. $C_{\widetilde{\mu}} \subseteq \overline{C_{\widetilde{\lambda}}}$, since then $C_{\widetilde{\mu}} \times \{B\} \subseteq \overline{C_{\widetilde{\lambda}}} \times \{B\} = \overline{C_{\widetilde{\lambda}} \times \{B\}} \subseteq \overline{C_\lambda}$, implying $C_\mu \subseteq \overline{C_\lambda}$.

Hence we may assume that $\lambda = [n-k, k] \vdash n$ and $\mu = [n-k-1, k+1] \vdash n$, for some $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$. We have $\lambda' = [1^{n-2k}, 2^k] \vdash n$ and $\mu' = [1^{n-2k-2}, 2^{k+1}] \vdash n$, which immediately implies $\mathbb{U}_{\mu'} \subseteq \mathbb{U}_{\lambda'}$. Hence we get $A_{\mu'} \in C_\mu \cap \mathbb{U}_{\mu'} \subseteq C_\mu \cap \mathbb{U}_{\lambda'} \subseteq C_\mu \cap \overline{C_\lambda}$, where $A_{\mu'}$ is as above, implying $C_\mu \subseteq \overline{C_\lambda}$. ♯

**(7.7) Corollary.** For $\mathbb{G} := \mathbb{SL}_n$, where $n \in \mathbb{N}$, we have $\dim(\mathbb{G}_u) = n(n-1)$.

**Proof.** We have $\lambda \trianglelefteq [n]$ for all $\lambda \vdash n$. Hence for $J := J_n(1) \in \mathbb{G}_u$ we conclude that $J^{\mathbb{G}} = C_{[n]} \subseteq \mathbb{G}_u$ is open and dense; the elements of $C_{[n]}$ are called **regular unipotent**. The centraliser $\mathbb{K}$-algebra $C_{\mathbb{K}^{n\times n}}(J) := \{A \in \mathbb{K}^{n\times n}; AJ = JA\} \subseteq \mathbb{K}^{n\times n}$ is closed, and given as $C_{\mathbb{K}^{n\times n}}(J) = \mathbb{K}\langle J\rangle = \mathbb{K}\langle J - E_n\rangle = \{A = [a_{ij}] \in \mathbb{K}^{n\times n}; a_{ij} = 0 \text{ for } i > j, a_{ij} = a_{1,j-i+1} \text{ for } i \leq j\}$. Hence it is immediate that $\dim(C_{\mathbb{K}^{n\times n}}(J)) = n$. Since $C_{\mathbb{G}}(J) = C_{\mathbb{K}^{n\times n}}(J) \cap \mathcal{V}(\det_n) \subset C_{\mathbb{K}^{n\times n}}(J)$ is a hypersurface, we have $\dim(C_{\mathbb{G}}(J)) = n - 1$; see also Exercise (12.27). This yields $\dim(\mathbb{G}_u) = \dim(\mathbb{G}) - \dim(C_{\mathbb{G}}(J)) = (n^2-1) - (n-1) = n(n-1)$. ♯

## 8 Lie algebras

**(8.1) Definition. a)** Let $\mathbb{G}$ be an algebraic group with affine coordinate algebra $\mathbb{K}[\mathbb{G}]$. Since $\mathbb{G} = \coprod_{g \in \mathbb{G}^\circ | \mathbb{G}} \mathbb{G}^\circ g$ is the disjoint union of its irreducible components, we have a notion of regularity for all $g \in \mathbb{G}^\circ g \subseteq \mathbb{G}$. Since $\mathbb{G}$ acts transitively on $\mathbb{G}$ by right multiplication $\rho$, we conclude that $\mathbb{G}$ is smooth. Hence we have $\mathfrak{g} := T_1(\mathbb{G}) = T_1(\mathbb{G}^\circ)$ and $\dim_{\mathbb{K}}(T_1(\mathbb{G})) = \dim(\mathbb{G})$.

**b)** We consider the Lie algebra $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}[\mathbb{G}]) \leq \mathrm{End}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}])$, with Lie product $[\delta, \delta'] = \delta\delta' - \delta'\delta$: The $\mathbb{G}$-action on $\mathbb{G}$ by left multiplication $\lambda$ induces a $\mathbb{K}$-linear $\mathbb{G}$-action $\lambda^*$ on $\mathbb{K}[\mathbb{G}]$. Thus we let

$$L(\mathbb{G}) := \{\delta \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}[\mathbb{G}]); \lambda_x^*\delta = \delta\lambda_x^* \text{ for all } x \in \mathbb{G}\}$$

be the Lie subalgebra of all **left invariant** derivations.

From $\lambda_x^*\delta(fg) = \lambda_x^*(\delta(f)g + f\delta(g)) = \lambda_x^*\delta(f) \cdot \lambda_x^*(g) + \lambda_x^*(f) \cdot \lambda_x^*\delta(g) = \delta\lambda_x^*(f) \cdot \lambda_x^*(g) + \lambda_x^*(f) \cdot \delta\lambda_x^*(g) = \delta(\lambda_x^*(f) \cdot \lambda_x^*(g)) = \delta\lambda_x^*(fg)$, for $\delta \in L(\mathbb{G})$ and $f, g \in \mathbb{K}[\mathbb{G}]$, we conclude that the condition of being left invariant can be checked on $\mathbb{K}$-algebra generators of $\mathbb{K}[\mathbb{G}]$.

**(8.2) Theorem.** Let $\mathbb{G}$ be an algebraic group.
**a)** Then the map $L(\mathbb{G}) \to \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1)\colon \delta \mapsto \delta^\bullet(1)$ is an isomorphism of $\mathbb{K}$-vector spaces, where $\delta^\bullet(1)\colon f \mapsto \delta(f)(1)$, for $f \in \mathbb{K}[\mathbb{G}]$.

Its inverse is given by $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1) \to L(\mathbb{G})\colon \gamma \mapsto \widehat{\gamma}$, where the **right convolution** $\widehat{\gamma}$ is defined by $\widehat{\gamma}(f)\colon \mathbb{G} \to \mathbb{K}\colon x \mapsto \gamma\lambda^*_{x^{-1}}(f)$, for $f \in \mathbb{K}[\mathbb{G}]$.

**b)** By transport of structure $T_1(\mathbb{G}) \cong \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1) =: \mathfrak{g}$ becomes a non-commutative associative algebra, by letting $\gamma\cdot\gamma' := (\gamma\otimes\gamma')\mu^* \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1)$, for all $\gamma, \gamma' \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1)$, where $\mu^*\colon \mathbb{K}[\mathbb{G}] \to \mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}]$ is the comorphism associated to the multiplication $\mu\colon \mathbb{G} \times \mathbb{G} \to \mathbb{G}$.

Thus $\mathfrak{g}$ becomes a Lie algebra, called the **Lie algebra** of the algebraic group $\mathbb{G}$.

**c)** If $\mathbb{H}$ is an algebraic group with Lie algebra $\mathfrak{h}$, and $\varphi\colon \mathbb{G} \to \mathbb{H}$ is a homomorphism of algebraic groups, then the differential $d_1(\varphi)\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of non-commutative associative algebras, and thus of Lie algebras.

**Proof. a)** For $\gamma \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1)$ and $f \in \mathbb{K}[\mathbb{G}]$ we show that $\widehat{\gamma}(f) \in \mathbb{K}[\mathbb{G}]$, implying that $\widehat{\gamma} \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}[\mathbb{G}])$: For $x \in \mathbb{G}$ we have $\lambda_{x^{-1}}\colon \mathbb{G} \times \{x^{-1}\} \xrightarrow{\mathrm{id}\otimes\epsilon_{x^{-1}}} \mathbb{G} \times \mathbb{G} \xrightarrow{\lambda} \mathbb{G}$, implying $\lambda^*_{x^{-1}}\colon \mathbb{K}[\mathbb{G}] \xrightarrow{\lambda^*} \mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}] \xrightarrow{\mathrm{id}\otimes\epsilon^*_{x^{-1}}} \mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K} \cong \mathbb{K}[\mathbb{G}]$, and thus $\gamma\lambda^*_{x^{-1}} = (\gamma \otimes \epsilon^*_{x^{-1}})\lambda^*$, implying that $x \mapsto \gamma\lambda^*_{x^{-1}}(f)$ is polynomial.

We have $\widehat{\gamma}(fg)(x) = \gamma\lambda^*_{x^{-1}}(fg) = \gamma(\lambda^*_{x^{-1}}(f)\lambda^*_{x^{-1}}(g)) = \gamma\lambda^*_{x^{-1}}(f) \cdot \lambda^*_{x^{-1}}(g)(1) + \lambda^*_{x^{-1}}(f)(1) \cdot \gamma\lambda^*_{x^{-1}}(g) = (\widehat{\gamma}(f)g + f\widehat{\gamma}(g))(x)$, for all $f, g \in \mathbb{K}[\mathbb{G}]$ and $x \in \mathbb{G}$, hence $\widehat{\gamma} \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}[\mathbb{G}])$. For all $f \in \mathbb{K}[\mathbb{G}]$ and $x, y \in \mathbb{G}$ we have $(\lambda^*_y\widehat{\gamma}(f))(x) = \widehat{\gamma}(f)(y^{-1}x) = \gamma\lambda^*_{x^{-1}y}(f) = \gamma\lambda^*_{x^{-1}}(\lambda^*_y(f)) = (\widehat{\gamma}\lambda^*_y(f))(x)$, thus $\widehat{\gamma} \in L(\mathbb{G})$.

For all $\delta \in L(\mathbb{G})$ and $f \in \mathbb{K}[\mathbb{G}]$ and $x \in \mathbb{G}$ we have $\widehat{\delta^\bullet(1)}(f)(x) = \delta^\bullet(1)\lambda^*_{x^{-1}}(f) = (\delta\lambda^*_{x^{-1}}(f))(1) = (\lambda^*_{x^{-1}}\delta(f))(1) = \delta(f)(x)$, thus $\widehat{\delta^\bullet(1)} = \delta$. Conversely, for all $\gamma \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1)$ and $f \in \mathbb{K}[\mathbb{G}]$ we have $\widehat{\gamma}^\bullet(1)(f) = \widehat{\gamma}(f)(1) = \gamma\lambda^*_1(f) = \gamma(f)$.

**b)** We show how multiplication, i. e. concatenation of maps, in $L(\mathbb{G})$ transports to $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1)$: For $f \in \mathbb{K}[\mathbb{G}]$ let $\mu^*(f) = \sum_{i=1}^r g_i \otimes h_i \in \mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}]$, for some $r \in \mathbb{N}$ and suitable $f_i, g_i \in \mathbb{K}[\mathbb{G}]$. Hence we deduce $(\gamma \cdot \gamma')(f) = (\gamma \otimes \gamma')(\sum_{i=1}^r g_i \otimes h_i) = \sum_{i=1}^r \gamma(g_i)\gamma'(h_i)$.

For all $x \in \mathbb{G}$ we have $\lambda^*_{x^{-1}}(f)(y) = f(xy) = \mu^*(f)([x, y]) = \sum_{i=1}^r g_i(x)h_i(y)$, for all $y \in \mathbb{G}$, and hence $\lambda^*_{x^{-1}}(f) = \sum_{i=1}^r g_i(x)h_i$. Thus $\widehat{\gamma}(f)(x) = \gamma\lambda^*_{x^{-1}}(f) = \sum_{i=1}^r g_i(x)\gamma(h_i)$, and hence $\widehat{\gamma}(f) = \sum_{i=1}^r g_i \cdot \gamma(h_i)$. This yields $(\widehat{\gamma}\widehat{\gamma'})^\bullet(1)(f) = \widehat{\gamma}\widehat{\gamma'}(f)(1) = \widehat{\gamma}(\sum_{i=1}^r g_i \cdot \gamma'(h_i))(1) = \sum_{i=1}^r \gamma(g_i)\gamma'(h_i) = (\gamma \cdot \gamma')(f)$.

**c)** Let $\gamma \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1)$ and $\delta := d_1(\varphi)(\gamma) = \gamma\varphi^* \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{H}], \mathbb{K}_1)$. We first show that $\widehat{\gamma}\varphi^* = \varphi^*\widehat{\delta}$: For all $h \in \mathbb{K}[\mathbb{H}]$ and $x \in \mathbb{G}$ we have $\widehat{\gamma}\varphi^*(h)(x) = \gamma\lambda^*_{x^{-1}}\varphi^*(h)$ and $\varphi^*\widehat{\delta}(h)(x) = \widehat{\delta}(h)(\varphi(x)) = \delta\lambda^*_{\varphi(x)^{-1}}(h) = \gamma\varphi^*\lambda^*_{\varphi(x)^{-1}}(h)$, where indeed for all $y \in \mathbb{G}$ we have $(\lambda^*_{x^{-1}}\varphi^*(h))(y) = \varphi^*(h)(xy) = h(\varphi(xy)) = h(\varphi(x)\varphi(y)) = \lambda^*_{\varphi(x)^{-1}}(h)(\varphi(y)) = \varphi^*\lambda^*_{\varphi(x)^{-1}}(h)(y)$.

Let $\gamma' \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}], \mathbb{K}_1)$ and $\delta' := d_1(\varphi)(\gamma') \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{H}], \mathbb{K}_1)$. Then for all

$h \in \mathbb{K}[\mathbb{H}]$ we have $(\delta \cdot \delta')(h) = (\widehat{\delta\widehat{\delta'}}(h))(1) = \delta\widehat{\delta'}(h) = \gamma\varphi^*\widehat{\delta'}(h) = \gamma\widehat{\gamma'}\varphi^*(h) = (\widehat{\gamma}\widehat{\gamma'}\varphi^*(h))(1) = (\gamma \cdot \gamma')\varphi^*(h)$. ♯

Given a closed subgroup $\mathbb{H} \leq \mathbb{G}$, the embedding $\mathfrak{h} \leq \mathfrak{g}$ of their Lie algebras can be described using the right convolution, see Exercise (12.29).

**(8.3) Example: The additive and the multiplicative group.**
**a)** Let $\mathbb{G} := \mathbb{G}_a = \mathbb{K}$ be the additive group, hence $\mathbb{K}[\mathbb{G}_a] \cong \mathbb{K}[X]$. Thus we have $\dim_{\mathbb{K}}(T_0(\mathbb{G}_a)) = 1$, hence $T_0(\mathbb{G}_a)$ is a commutative Lie algebra. From $\partial\lambda^*_{-x}(X) = \partial(X + x) = 1 = \lambda^*_{-x}\partial(X)$, for all $x \in \mathbb{G}$, we deduce that $\partial$ is left invariant, hence $L(\mathbb{G}_a) = \langle\partial\rangle_{\mathbb{K}}$.
**b)** Let $\mathbb{G} := \mathbb{G}_m = \mathbb{K}_X$ be the multiplicative group, hence $\mathbb{K}[\mathbb{G}_m] \cong \mathbb{K}[X]_X$. Thus we have $\dim_{\mathbb{K}}(T_1(\mathbb{G}_m)) = 1$, hence $T_1(\mathbb{G}_m)$ is a commutative Lie algebra. We have $L(\mathbb{G}_m) \cong T_1(\mathbb{G}_m) \cong \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[X]_X, \mathbb{K}_1) \cong \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[X], \mathbb{K}_1) \cong \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[X]_{\langle X-1\rangle}, \mathbb{K}_1) \cong \mathrm{Hom}_{\mathbb{K}}(\langle X-1\rangle/\langle X-1\rangle^2, \mathbb{K}_1)$. Letting $\gamma := \gamma(X)\partial \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[X], \mathbb{K}_1)$, where $\gamma(X) = \gamma(X-1) := 1$, for the right convolution $\widehat{\gamma}$ associated to $\gamma$ we have $\widehat{\gamma}(X)(x) = \gamma\lambda^*_{x^{-1}}(X) = \gamma(xX) = x$, for all $x \in \mathbb{G}$, and thus $\widehat{\gamma}(X) = X$. Hence we have $\widehat{\gamma}(X)\partial = X\partial \in L(\mathbb{G}_m)$ and thus $L(\mathbb{G}_m) = \langle X\partial\rangle_{\mathbb{K}}$.

**(8.4) Example: General and special linear groups.**
**a)** Let $\mathbb{G} := \mathbb{GL}_n$ be the general linear group, hence $\mathbb{K}[\mathbb{GL}_n] \cong \mathbb{K}[\mathcal{X}]_{\det_n}$, where $\mathcal{X} = \{X_{11}, \ldots, X_{nn}\}$ and $\det_n = \sum_{\sigma\in\mathcal{S}_n}(\mathrm{sgn}(\sigma) \cdot \prod_{i=1}^n X_{i,i^\sigma}) \in \mathbb{K}[\mathcal{X}]$ is the $n$-th determinant polynomial. Thus $\dim_{\mathbb{K}}(T_{E_n}(\mathbb{GL}_n)) = n^2$, and $T_{E_n}(\mathbb{GL}_n) \cong \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}]_{\det_n}, \mathbb{K}_{E_n}) \cong \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}], \mathbb{K}_{E_n}) \cong T_{E_n}(\mathbb{K}^{n\times n}) \cong \mathbb{K}^{n\times n}$, where $\delta = \sum_{i=1}^n\sum_{j=1}^n \delta(X_{ij})\partial_{ij} \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}], \mathbb{K}_{E_n})$ is mapped to $[\delta(X_{ij})]_{ij} \in \mathbb{K}^{n\times n}$.

For all $i, j \in \{1, \ldots, n\}$ we have $\mu^*(X_{ij}) = \sum_{k=1}^n X_{ik} \otimes X_{kj} \in \mathbb{K}[\mathbb{GL}_n] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{GL}_n]$, hence for $\delta, \delta' \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}], \mathbb{K}_{E_n})$ we get $(\delta\cdot\delta')(X_{ij}) = \sum_{k=1}^n \delta(X_{ik}) \otimes \delta'(X_{kj})$, which hence transported to $\mathbb{K}^{n\times n}$ yields the usual matrix product. Hence we have $T_{E_n}(\mathbb{GL}_n) \cong \mathfrak{gl}_n := \mathbb{K}^{n\times n}$ as Lie algebras, where the latter is endowed with the usual Lie product.

**b)** Let $\varphi_{\det}\colon \mathbb{GL}_n \to \mathbb{G}_m\colon A \mapsto \det(A)$ be the determinant homomorphism. Considering $\mathbb{GL}_n \subseteq \mathbb{K}^{n\times n}$ and $\mathbb{G}_m \subseteq \mathbb{K}$ as principal open subsets, we extend to the morphism $\varphi_{\det}\colon \mathbb{K}^{n\times n} \to \mathbb{K}\colon A \mapsto \det(A)$, and identify $T_{E_n}(\mathbb{GL}_n) \cong \mathcal{T}_{E_n}(\mathbb{K}^{n\times n}) \cong \mathbb{K}^{n\times n}$ and $T_1(\mathbb{G}_m) \cong \mathcal{T}_1(\mathbb{K}) \cong \mathbb{K}$. Hence for the differential we get $d_{E_n}(\varphi_{\det})\colon \mathbb{K}^{n\times n} \to \mathbb{K}\colon [t_{ij}] \mapsto \sum_{i=1}^n\sum_{j=1}^n t_{ij} \cdot \partial_{ij}(\det_n)(E_n)$. From $\partial_{ij}(\det_n) = (-1)^{i+j} \cdot \det_{n-1}(\{X_{kl}; k \neq i, l \neq j\})$ we get $\partial_{ij}(\det_n)(E_n) = \delta_{ij}$, implying that $d_{E_n}(\varphi_{\det})\colon [t_{ij}] \mapsto \sum_{i=1}^n t_{ii} = \mathrm{Tr}([t_{ij}])$ is the usual matrix trace.

**c)** Let $\mathbb{G} := \mathbb{SL}_n \leq \mathbb{GL}_n \subseteq \mathbb{K}^{n\times n}$ be the special linear group, hence we have $\mathbb{K}[\mathbb{SL}_n] \cong \mathbb{K}[\mathcal{X}]/\langle\det_n - 1\rangle \cong \mathbb{K}[\mathcal{X}]_{\det_n}/\langle\det_n - 1\rangle_{\det_n}$. Thus $\dim_{\mathbb{K}}(T_{E_n}(\mathbb{SL}_n)) = n^2 - 1$, and $\mathcal{T}_{E_n}(\mathbb{SL}_n) = \mathcal{V}(\partial_{E_n}(\det_n - 1)) = \{[t_{ij}] \in \mathbb{K}^{n\times n}; \sum_{i=1}^n\sum_{j=1}^n t_{ij} \cdot \partial_{ij}(\det_n - 1)(E_n) = 0\} = \{[t_{ij}] \in \mathbb{K}^{n\times n}; \mathrm{Tr}([t_{ij}]) = 0\} \leq \mathcal{T}_{E_n}(\mathbb{K}^{n\times n})$, hence $T_{E_n}(\mathbb{SL}_n) \cong \mathfrak{sl}_n := \{A \in \mathfrak{gl}_n; \mathrm{Tr}(A) = 0\}$ is the Lie subalgebra of $\mathfrak{gl}_n = \mathbb{K}^{n\times n}$ consisting of the matrices of trace zero.

For Lie algebras of the examples in Exercise (12.4) see Exercise (12.30). For a differential of a homomorphism of algebraic groups see Exercise (12.32).

**(8.5) Example: Classical groups.**
Let $J = [b_{ij}] \in \mathbb{K}^{n\times n}$ be the matrix of a non-degenerate $\mathbb{K}$-bilinear form on $\mathbb{K}^n$, and let $\mathbb{G} = \{A \in \mathbb{K}^{n\times n}; AJA^{\mathrm{tr}} = J\} \leq \mathbb{GL}_n \subseteq \mathbb{K}^{n\times n}$ be the associated classical group. Letting $\mathcal{X} = \{X_{11}, \ldots, X_{nn}\}$ and $f_{rs} := (\sum_{i=1}^n \sum_{j=1}^n X_{ri}b_{ij}X_{sj}) - b_{rs} \in \mathbb{K}[\mathcal{X}]$, where $r, s \in \{1, \ldots, n\}$, we have $\mathbb{G} = \mathcal{V}(f_{rs}; r, s \in \{1, \ldots, n\}) \subseteq \mathbb{K}^{n\times n}$.

For $k, l \in \{1, \ldots, n\}$ we have $\partial_{kl}(f_{rs}) = \delta_{kr} \cdot \sum_{j=1}^n b_{lj}X_{sj} + \delta_{ks} \cdot \sum_{i=1}^n X_{ri}b_{il} \in \mathbb{K}[\mathcal{X}]$, implying $\partial_{kl}(f_{rs})(E_n) = \delta_{kr}b_{ls} + \delta_{ks}b_{rl}$. Thus for the total differentials we get $\partial_{E_n}(f_{rs}) = \sum_{k=1}^n \sum_{l=1}^n \partial_{kl}(f_{rs})(E_n) \cdot X_{kl} = \sum_{k=1}^n \sum_{l=1}^n (\delta_{kr}b_{ls} + \delta_{ks}b_{rl}) \cdot X_{kl} = \sum_{l=1}^n b_{ls}X_{rl} + \sum_{l=1}^n b_{rl}X_{sl}$, for all $r, s \in \{1, \ldots, n\}$, implying $[\partial_{E_n}(f_{rs})]_{rs} = [X_{ij}] \cdot J + J \cdot [X_{ij}]^{\mathrm{tr}} \in \mathbb{K}[\mathcal{X}]^{n\times n}$. Hence we have $T_{E_n}(\mathbb{G}) \cong \bigcap_{r,s\in\{1,\ldots,n\}} \ker(\partial_{E_n}(f_{rs})) = \{A \in \mathfrak{gl}_n; AJ + JA^{\mathrm{tr}} = 0\} =: \mathfrak{g}_J$, where the Lie algebra structure is inherited from $\mathfrak{gl}_n$.

**a)** For the symplectic group $\mathbb{S}_{2m} \leq \mathbb{GL}_{2m}$ we have $J = \begin{bmatrix} 0 & J_m \\ -J_m & 0 \end{bmatrix} \in \mathbb{K}^{2m\times 2m}$. Indexing rows and columns by $\mathcal{I} := \{-m, \ldots, -1, 1, \ldots, m\}$, we have $J = [\delta_{i,-j} \cdot \frac{j}{|j|}]_{ij}$. Thus for $A = [a_{ij}] \in \mathbb{K}^{2m\times 2m}$ we have $AJ = [\sum_{k\in\mathcal{I}} a_{ik}b_{kj}]_{ij} = [a_{i,-j} \cdot \frac{j}{|j|}]_{ij}$ and $JA^{\mathrm{tr}} = [\sum_{k\in\mathcal{I}} b_{ik}a_{jk}]_{ij} = [a_{j,-i} \cdot \frac{-i}{|i|}]_{ij}$. Hence $A \in \mathfrak{sp}_{2m} := \{A \in \mathfrak{gl}_{2m}; AJ + JA^{\mathrm{tr}} = 0\}$ if and only if $a_{i,-j} \cdot \frac{j}{|j|} = a_{j,-i} \cdot \frac{i}{|i|}$, or equivalently $a_{-i,-j} = a_{ji} \cdot \frac{-i}{|i|} \cdot \frac{j}{|j|}$, for all $i, j \in \mathcal{I}$.

Hence there is no condition for $a_{i,-i}$, and we obtain $\binom{2m}{2} = m(2m-1)$ $\mathbb{K}$-linearly independent equations. Thus we have $\dim(\mathbb{S}_{2m}) = \dim_{\mathbb{K}}(\mathfrak{sp}_{2m}) = (2m)^2 - m(2m-1) = m(2m+1) = \frac{n(n+1)}{2}$, where $n = 2m$. Moreover, we have $\mathbb{S}_{2m} \leq \mathbb{SL}_{2m}$ and thus $\mathfrak{sp}_{2m} \leq \mathfrak{sl}_{2m}$: Indeed, from $a_{-i,-i} = a_{ii} \cdot \frac{-i}{|i|} \cdot \frac{i}{|i|} = -a_{ii}$, for all $i \in \mathcal{I}$, we for $A \in \mathfrak{sp}_{2m}$ get $\mathrm{Tr}(A) = 0$.

**b)** For the orthogonal group $\mathbb{O}_n \leq \mathbb{GL}_n$, where $\mathrm{char}(\mathbb{K}) \neq 2$, we have $J = J_n = [\delta_{i,n+1-j}]_{ij} \in \mathbb{K}^{n\times n}$. Thus for $A = [a_{ij}] \in \mathbb{K}^{n\times n}$ we have $AJ = [\sum_{k=1}^n a_{ik}b_{kj}]_{ij} = [a_{i,n+1-j}]_{ij}$ and $JA^{\mathrm{tr}} = [\sum_{k=1}^n b_{ik}a_{jk}]_{ij} = [a_{j,n+1-i}]_{ij}$. Hence $A \in \mathfrak{o}_n := \{A \in \mathfrak{gl}_n; AJ_n + J_nA^{\mathrm{tr}} = 0\}$ if and only if $a_{i,n+1-j} + a_{j,n+1-i} = 0$, for all $i, j \in \{1, \ldots, n\}$. Hence there is the equation $2a_{i,n+1-i} = 0$, implying $a_{i,n+1-i} = 0$, and we obtain $\binom{n}{2} + n = \frac{n(n+1)}{2}$ $\mathbb{K}$-linearly independent equations. Thus we have $\dim(\mathbb{O}_n) = \dim_{\mathbb{K}}(\mathfrak{so}_n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

For the special orthogonal group $\mathbb{SO}_n = \ker((\varphi_{\det})|_{\mathbb{O}_n}) \leq \mathbb{O}_n \leq \mathbb{GL}_n$ we from $[\mathbb{O}_n \colon \mathbb{SO}_n] = 2$ deduce $\mathbb{O}_n^\circ = \mathbb{SO}_n^\circ \leq \mathbb{SO}_n \leq \mathbb{O}_n$, hence $T_{E_n}(\mathbb{SO}_n) = T_{E_n}(\mathbb{O}_n)$, and we let $\mathfrak{o}_n =: \mathfrak{so}_n \leq \mathfrak{sl}_n$: Indeed, from $a_{ii} + a_{n+1-i,n+1-i} = 0$, for all $i \in \{1, \ldots, n\}$, we for $A \in \mathfrak{so}_n$ get $\mathrm{Tr}(A) = 0$; see also Exercise (12.31).

**(8.6) Example: Orthogonal groups in characteristic 2.**
**a)** Let $\mathrm{char}(\mathbb{K}) = 2$ and let $\mathbb{O}_{2m} \leq \mathbb{S}_{2m} \leq \mathbb{GL}_{2m}$ be the even-dimensional

orthogonal group. Indexing rows and columns by $\mathcal{I} := \{-m, \ldots, -1, 1, \ldots, m\}$, the underlying quadratic form is given by $q(x) = \sum_{i=1}^{m} x_i x_{-i}$, for all $x \in \mathbb{K}^{2m}$. Hence we have $q(x) = xQx^{\text{tr}}$, where $Q = \begin{bmatrix} 0 & J_m \\ 0 & 0 \end{bmatrix} \in \mathbb{K}^{2m\times 2m}$. For $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{K}^{2m\times 2m}$, where $A_{11}, A_{12}, A_{21}, A_{22} \in \mathbb{K}^{m\times m}$, we have $q(xA) = xAQA^{\text{tr}}x^{\text{tr}}$, where $AQA^{\text{tr}} = \begin{bmatrix} A_{11}J_mA_{12}^{\text{tr}} & A_{11}J_mA_{22}^{\text{tr}} \\ A_{21}J_mA_{12}^{\text{tr}} & A_{21}J_mA_{22}^{\text{tr}} \end{bmatrix} \in \mathbb{K}^{2m\times 2m}$. Thus we have $A \in \mathbb{O}_{2m}$, i. e. $q(xA) = q(x)$ for all $x \in \mathbb{K}^{2m}$, if and only if $A_{11}J_mA_{22}^{\text{tr}} + A_{12}J_mA_{21}^{\text{tr}} = J_m \in \mathbb{K}^{m\times m}$ and both $A_{11}J_mA_{12}^{\text{tr}} \in \mathbb{K}^{m\times m}$ and $A_{21}J_mA_{22}^{\text{tr}} \in \mathbb{K}^{m\times m}$ are symmetric matrices with zero diagonal. Since the $\mathbb{K}$-bilinear form associated to $q$ is given by $J = J_{2m} = \begin{bmatrix} 0 & J_m \\ J_m & 0 \end{bmatrix} \in \mathbb{K}^{2m\times 2m}$, we have $A \in \mathbb{S}_{2m}$ if and only if $\begin{bmatrix} A_{12}J_mA_{11}^{\text{tr}} + A_{11}J_mA_{12}^{\text{tr}} & A_{12}J_mA_{21}^{\text{tr}} + A_{11}J_mA_{22}^{\text{tr}} \\ A_{22}J_mA_{11}^{\text{tr}} + A_{21}J_mA_{12}^{\text{tr}} & A_{22}J_mA_{21}^{\text{tr}} + A_{21}J_mA_{22}^{\text{tr}} \end{bmatrix} = AJ_{2m}A^{\text{tr}} = J_{2m} = \begin{bmatrix} 0 & J_m \\ J_m & 0 \end{bmatrix} \in \mathbb{K}^{2m\times 2m}$, which hence holds if and only if $A_{11}J_mA_{22}^{\text{tr}} + A_{12}J_mA_{21}^{\text{tr}} = J_m \in \mathbb{K}^{m\times m}$ and both $A_{11}J_mA_{12}^{\text{tr}} \in \mathbb{K}^{m\times m}$ and $A_{21}J_mA_{22}^{\text{tr}} \in \mathbb{K}^{m\times m}$ are symmetric matrices.

Thus comparing the equations collected for membership in $\mathbb{O}_{2m} \leq \mathbb{S}_{2m}$ and $\mathbb{S}_{2m}$, we deduce that the additional equations are $[A_{11}J_mA_{12}^{\text{tr}}]_{rr} = 0$ for all $r \in \{-m, \ldots, -1\}$, and $[A_{21}J_mA_{22}^{\text{tr}}]_{rr} = 0$ for all $r \in \{1, \ldots, m\}$. These in turn are given as $\sum_{s=1}^{m} a_{r,-s}a_{rs} = 0$ for all $r \in \mathcal{I}$. Letting $\mathcal{X} = \{X_{kl}; k, l \in \mathcal{I}\}$ and $f_{rr} := \sum_{s=1}^{m} X_{r,-s}X_{rs} \in \mathbb{K}[\mathcal{X}]$, we for $k, l \in \mathcal{I}$ have $\partial_{kl}(f_{rr}) = \delta_{kr} \cdot X_{k,-l} \in \mathbb{K}[\mathcal{X}]$, implying $\partial_{kl}(f_{rr})(E_{2m}) = \delta_{kr}\delta_{k,-l}$. For the total differentials we get $\partial_{E_{2m}}(f_{rr}) = \sum_{k\in\mathcal{I}} \sum_{l\in\mathcal{I}} \partial_{kl}(f_{rr})(E_{2m}) \cdot X_{kl} = X_{r,-r}$, for all $r \in \mathcal{I}$.

Hence we get $T_{E_{2m}}(\mathbb{O}_{2m}) \cong \mathfrak{o}_{2m} := \{A = [a_{ij}] \in \mathfrak{sp}_{2m}; a_{i,-i} = 0 \text{ for all } i \in \mathcal{I}\}$. Comparing with the equations collected for $\mathbb{S}_{2m}$ where we had no condition for $a_{i,-i}$, we deduce that there are $\binom{2m}{2} + 2m = m(2m+1)$ $\mathbb{K}$-linearly independent equations, and thus we have $\dim(\mathbb{O}_{2m}) = \dim_{\mathbb{K}}(\mathfrak{o}_{2m}) = (2m)^2 - m(2m+1) = m(2m-1) = \frac{n(n-1)}{2}$, where $n = 2m$. Again, for the special orthogonal group $\mathbb{SO}_{2m} \leq \mathbb{O}_{2m}$ we from $[\mathbb{O}_{2m} \colon \mathbb{SO}_{2m}] = 2$ get $T_{E_{2m}}(\mathbb{SO}_{2m}) = T_{E_{2m}}(\mathbb{O}_{2m})$, and we let $\mathfrak{o}_{2m} := \mathfrak{so}_{2m} \leq \mathfrak{sp}_{2m}$.

**b)** Let $\text{char}(\mathbb{K}) = 2$ and let $\mathbb{O}_n \leq \mathbb{GL}_n$ be the odd-dimensional orthogonal group, where $n = 2m+1 \geq 3$. Indexing rows and columns by $\mathcal{I} := \{-m, \ldots, -1, 1, \ldots, m, 0\}$, the underlying quadratic form is given by $q(x) = x_0^2 + \sum_{i=1}^{m} x_i x_{-i}$, for all $x \in \mathbb{K}^n$, and the $\mathbb{K}$-bilinear form associated to $q$ is given by $J = \begin{bmatrix} J_{2m} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{K}^{n\times n}$. Let $\varphi \colon \mathbb{O}_n \to \mathbb{S}_{2m} \colon A = \left[\begin{array}{c|c} A' & a^{\text{tr}} \\ \hline 0 & 1 \end{array}\right] \mapsto A'$ be the bijective homomorphism of algebraic groups from (4.5), where the vector $a \in \mathbb{K}^{2m}$ is given by $a_i^2 = q(e_iA)$, for all $i \in \mathcal{I}' := \{-m, \ldots, -1, 1, \ldots, m\}$. Thus for $A = [a_{ij}] \in \mathbb{K}^{n\times n}$ we have $A \in \mathbb{O}_n$ if and only if $A' \in \mathbb{S}_{2m}$ and $a_i^2 = q([a_{i,-m}, \ldots, a_{i,m}]) = \sum_{j=1}^{m} a_{ij}a_{i,-j}$ for all $i \in \mathcal{I}'$, as well as $a_{00} = 1$ and $a_{0j} = 0$ for all $j \in \mathcal{I}'$.

Letting $\mathcal{X}' := \{X_{ij} \in \mathcal{X}; i,j \in \mathcal{I}'\}$, in addition to the polynomials in $\mathbb{K}[\mathcal{X}'] \subseteq \mathbb{K}[\mathcal{X}]$ describing membership of $A'$ in $\mathbb{S}_{2m}$ we get the polynomials $f_{i0} := X_{i0}^2 + \sum_{j=1}^m X_{ij}X_{i,-j} \in \mathbb{K}[\mathcal{X}]$ for all $i \in \mathcal{I}'$, as well as $f_{00} := X_{00}+1 \in \mathbb{K}[\mathcal{X}]$ and $f_{0j} := X_{0j} \in \mathbb{K}[\mathcal{X}]$ for all $j \in \mathcal{I}'$. For the total differentials we get $\partial_{E_n}(f_{0j}) = X_{0j}$ and $\partial_{E_n}(f_{i0}) = \partial_{i0}(X_{i0}^2)(E_n) \cdot X_{i0} + \sum_{j\in\mathcal{I}'} X_{i,-j}(E_n) \cdot X_{ij} = X_{i,-i}$ for all $i,j \in \mathcal{I}'$. Thus $T_{E_n}(\mathbb{O}_n) \cong \mathfrak{o}_{2m+1} := \{A = \left[\begin{array}{c|c} A' & * \\ \hline 0 & 0 \end{array}\right] \in \mathfrak{gl}_{2m+1}; A' \in \mathfrak{o}_{2m}\}$; indeed a comparison of dimensions shows $\dim(\mathbb{O}_{2m+1}) = \dim_{\mathbb{K}}(\mathfrak{o}_{2m+1}) = \dim_{\mathbb{K}}(\mathfrak{o}_{2m}) + 2m = m(2m+1) = \frac{n(n-1)}{2} = \dim(\mathbb{S}_{2m})$.

The comorphism $\varphi^* \colon \mathbb{K}[\mathbb{S}_{2m}] \to \mathbb{K}[\mathbb{O}_n]$ is induced by the natural embedding $\mathbb{K}[\mathcal{X}'] \subseteq \mathbb{K}[\mathcal{X}]$. Hence on the associated Zariski tangential spaces we have $d_{E_n}(\varphi) \colon \mathfrak{o}_{2m+1} \to \mathfrak{sp}_{2m} \colon A = \left[\begin{array}{c|c} A' & * \\ \hline 0 & 0 \end{array}\right] \mapsto A'$, implying $d_{E_n}(\varphi)(\mathfrak{o}_{2m+1}) = \mathfrak{o}_{2m} < \mathfrak{sp}_{2m}$, thus $d_{E_n}(\varphi)$ is not an isomorphism of $\mathbb{K}$-vector spaces.

## 9 The Lang-Steinberg Theorem

**(9.1) Lemma.** Let $\mathbb{G}$ be an algebraic group with Lie algebra $\mathfrak{g}$.
**a)** Identifying $T_1(\mathbb{G}\times\mathbb{G}) := T_{[1,1]}(\mathbb{G}\times\mathbb{G}) \cong T_1(\mathbb{G}) \oplus T_1(\mathbb{G}) \cong \mathfrak{g}\oplus\mathfrak{g}$, for the differential of the multiplication map we have $d_1(\mu) \colon \mathfrak{g}\oplus\mathfrak{g} \to \mathfrak{g} \colon [t,t'] \mapsto t+t'$.
**b)** For the differential of the inversion map we have $d_1(\iota) = -\mathrm{id}_{\mathfrak{g}} \colon \mathfrak{g} \to \mathfrak{g}$.

**Proof. a)** The identification $T_1(\mathbb{G}\times\mathbb{G}) \cong T_1(\mathbb{G})\oplus T_1(\mathbb{G})$ is given by restricting $\gamma \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}], \mathbb{K}_1)$ to $\mathbb{K}[\mathbb{G}] \otimes_{\mathbb{K}} \{1\}$ and $\{1\} \otimes_{\mathbb{K}} \mathbb{K}[\mathbb{G}]$, respectively, and conversely for $\delta,\delta' \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}],\mathbb{K}_1)$ we have $(\delta \bullet \delta')(g\otimes h) = \delta(g)h(1) + g(1)\delta'(h)$, for all $g,h \in \mathbb{K}[\mathbb{G}]$; see also Exercise (12.33).

For $f \in \mathbb{K}[\mathbb{G}]$ let $\mu^*(f) = \sum_{i=1}^r g_i \otimes h_i$, for some $r \in \mathbb{N}$ and $g_i, h_i \in \mathbb{K}[\mathbb{G}]$. Hence $f(x) = f(1\cdot x) = f(x\cdot 1) = \sum_{i=1}^r g_i(1)h_i(x) = \sum_{i=1}^r g_i(x)h_i(1)$, for all $x \in \mathbb{G}$, and thus $f = \sum_{i=1}^r g_i(1)h_i = \sum_{i=1}^r g_ih_i(1)$. Hence for $\delta,\delta' \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathbb{G}],\mathbb{K}_1)$ we have $(\delta\bullet\delta')\mu^*(f) = \sum_{i=1}^r \delta(g_i)h_i(1) + g_i(1)\delta'(h_i) = \delta(f) + \delta'(f)$.

**b)** We have $\mu(\mathrm{id}_{\mathbb{G}} \times \iota) = \nu_1 \colon \mathbb{G} \to \mathbb{G}\times\mathbb{G} \to \mathbb{G} \colon x \mapsto 1$. The identification $T_1(\mathbb{G}\times\mathbb{G}) \cong \mathfrak{g}\oplus\mathfrak{g}$ yields $d_1(\mathrm{id}_{\mathbb{G}}\times\iota) = d_1(\mathrm{id}_{\mathbb{G}}) \oplus d_1(\iota)$. Since $\nu_1^* \colon \mathbb{K} \mapsto \mathbb{K}[\mathbb{G}]$ is the natural map, this implies $0 = d_1(\nu_1) = d_1(\mu(\mathrm{id}_{\mathbb{G}}\times\iota)) = d_1(\mu)d_1(\mathrm{id}_{\mathbb{G}}\times\iota) = d_1(\mu)(d_1(\mathrm{id}_{\mathbb{G}}) \oplus d_1(\iota)) = d_1(\mathrm{id}_{\mathbb{G}}) + d_1(\iota)$, hence $d_1(\iota) = -d_1(\mathrm{id}_{\mathbb{G}}) = -\mathrm{id}_{\mathfrak{g}}$. ♯

**(9.2) Theorem.** Let $\mathbb{G}$ be an algebraic group with Lie algebra $\mathfrak{g}$.
**a)** For $x \in \mathbb{G}$ let $\kappa_x \colon \mathbb{G} \to \mathbb{G} \colon y \mapsto x^{-1}yx$ and $\mathrm{Ad}(x) := d_1(\kappa_{x^{-1}}) \colon \mathfrak{g} \to \mathfrak{g}$. Then $\mathrm{Ad} \colon \mathbb{G} \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}) \subseteq \mathbb{GL}(\mathfrak{g}) \cong \mathbb{GL}_{\dim(\mathbb{G})}$ is a rational representation, called the **adjoint representation**; we have $Z(\mathbb{G}) \leq \ker(\mathrm{Ad})$.
**b)** We have $d_1(\mathrm{Ad}) \colon \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}(\mathfrak{g}) \colon x \mapsto \mathrm{ad}(x)$, where $\mathrm{ad}(x) \colon \mathfrak{g} \to \mathfrak{g} \colon y \mapsto [x,y]$ is the **left adjoint action**.

**Proof. a)** Since $\kappa_x$ is an isomorphism of algebraic groups, $\mathrm{Ad}(x) \in \mathbb{GL}(\mathfrak{g})$ is a Lie algebra automorphism. For $x, y \in \mathbb{G}$ we have $\mathrm{Ad}(xy) = d_1(\kappa_{y^{-1}x^{-1}}) = d_1(\kappa_{x^{-1}}\kappa_{y^{-1}}) = d_1(\kappa_{x^{-1}})d_1(\kappa_{y^{-1}}) = \mathrm{Ad}(x)\mathrm{Ad}(y)$, implying that $\mathrm{Ad}\colon \mathbb{G} \to \mathbb{GL}(\mathfrak{g})$ is a group homomorphism. We show that Ad is a morphism:

Let $\mathbb{G} \leq \mathbb{GL}_n$ closed, hence we have $\mathfrak{g} \leq \mathfrak{gl}_n$. Letting $\mathbb{H} := \{\widehat{A} = \left[\begin{array}{c|c} A & 0 \\ \hline * & * \end{array}\right] \in \mathbb{GL}(\mathfrak{gl}_n); A \in \mathbb{GL}(\mathfrak{g})\} \leq \mathbb{GL}(\mathfrak{gl}_n)$ closed, we have the morphism of algebraic groups $\pi\colon \mathbb{H} \to \mathbb{GL}(\mathfrak{g})\colon \widehat{A} \to A$. For $x \in \mathbb{G}$ we have an extension $\kappa_{x^{-1}}\colon \mathbb{GL}_n \to \mathbb{GL}_n$, inducing $\widehat{\mathrm{Ad}}\colon \mathbb{G} \to \mathbb{H} \leq \mathbb{GL}(\mathfrak{gl}_n)\colon x \mapsto \left[\begin{array}{c|c} \mathrm{Ad}(x) & 0 \\ \hline * & * \end{array}\right]$, hence $\mathrm{Ad} = \pi\widehat{\mathrm{Ad}}$. Thus extending $\widehat{\mathrm{Ad}}$ to $\mathbb{GL}_n$ it suffices to consider $\mathrm{Ad}\colon \mathbb{GL}_n \to \mathbb{GL}(\mathfrak{gl}_n) \cong \mathbb{GL}_{n^2}$:

Letting $\mathcal{X} := \{X_{11}, \ldots, X_{nn}\}$ we for $i, j \in \{1, \ldots, n\}$ have $\kappa^*_{x^{-1}}(X_{ij})(y) = X_{ij}(xyx^{-1}) = [xyx^{-1}]_{ij} = \sum_{k=1}^n \sum_{l=1}^n x_{ik}y_{kl}x'_{lj} = \sum_{k=1}^n \sum_{l=1}^n x_{ik}X_{kl}(y)x'_{lj} = (x \cdot [X_{kl}] \cdot x^{-1})_{ij}(y)$, for all $y = [y_{ij}] \in \mathbb{GL}_n$, where $x = [x_{ij}] \in \mathbb{GL}_n$ and $x^{-1} = [x'_{ij}] \in \mathbb{GL}_n$. Hence $\kappa^*_{x^{-1}}([X_{ij}]) = x \cdot [X_{ij}] \cdot x^{-1} \in \mathbb{K}[\mathcal{X}]^{n\times n}$. Hence for $\delta \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}], \mathbb{K}_{E_n})$ we have $\mathrm{Ad}(x)(\delta)(X_{ij}) = \delta\kappa^*_{x^{-1}}(X_{ij}) = \delta(x \cdot [X_{kl}] \cdot x^{-1})_{ij} = (x \cdot [\delta(X_{kl})] \cdot x^{-1})_{ij}$. By the identification $T_{E_n}(\mathbb{GL}_n) \cong T_{E_n}(\mathbb{K}^{n\times n}) \cong \mathfrak{gl}_n$, where $\delta$ is mapped to $[\delta(X_{ij})]_{ij} \in \mathfrak{gl}_n$, we get $\mathrm{Ad}(x)\colon \mathfrak{gl}_n \to \mathfrak{gl}_n\colon A \mapsto xAx^{-1}$. Hence with respect to the $\mathbb{K}$-basis $\{E_{ij;kl} \in \mathbb{K}^{n^2\times n^2}; i, j, k, l \in \{1, \ldots, n\}\} \subseteq \mathrm{End}_{\mathbb{K}}(\mathfrak{gl}_n)$, where $E_{rs} \cdot E_{ij;kl} = \delta_{ir}\delta_{js}E_{kl} \in \mathfrak{gl}_n$, the matrix of $\mathrm{Ad}(x)$ is given as $[x_{ki}x'_{jl}]_{ij;kl} = x^{\mathrm{tr}} \otimes x^{-1} \in \mathbb{K}^{n^2\times n^2}$, the latter being a matrix Kronecker product.

**b)** We have $d_{E_n}(\mathrm{Ad}) = d_{E_{n^2}}(\pi)d_{E_n}(\widehat{\mathrm{Ad}})$, where $d_{E_n}(\widehat{\mathrm{Ad}})\colon \mathfrak{g} \to T_{E_{n^2}}(\mathbb{H}) \leq T_{E_{n^2}}(\mathbb{GL}(\mathfrak{gl}_n)) \cong \mathrm{End}_{\mathbb{K}}(\mathfrak{gl}_n) \cong \mathbb{K}^{n^2\times n^2}$, as well as $d_{E_{n^2}}(\pi)\colon T_{E_{n^2}}(\mathbb{H}) \cong \{\widehat{A} = \left[\begin{array}{c|c} A & 0 \\ \hline * & * \end{array}\right] \in \mathrm{End}_{\mathbb{K}}(\mathfrak{gl}_n); A \in \mathrm{End}_{\mathbb{K}}(\mathfrak{g})\} \to \mathrm{End}_{\mathbb{K}}(\mathfrak{g})\colon \widehat{A} \mapsto A$. Thus it suffices to consider $\mathrm{Ad}\colon \mathbb{GL}_n \to \mathbb{GL}(\mathfrak{gl}_n)$ and its differential $d_{E_n}(\mathrm{Ad})\colon \mathfrak{gl}_n \to \mathrm{End}_{\mathbb{K}}(\mathfrak{gl}_n)$:

Let $\sigma\colon \mathfrak{gl}_n \to \mathrm{End}_{\mathbb{K}}(\mathfrak{gl}_n)\colon x \mapsto \sigma(x)$, where $\sigma(x)\colon \mathfrak{gl}_n \to \mathfrak{gl}_n\colon A \mapsto xA$. Hence $[\sigma(x)(E_{ij})]_{kl} = [xE_{ij}]_{kl} = \delta_{jl}x_{ki}$, for $i, j, k, l \in \{1, \ldots, n\}$, and for $f_{ij;kl} := \delta_{jl}X_{ki} \in \mathbb{K}[\mathcal{X}]$ we get $\partial_{rs}(f_{ij;kl})(E_n) = \delta_{jl}\delta_{kr}\delta_{is}$, for $r, s \in \{1, \ldots, n\}$. Thus on $\mathcal{T}_{E_n}(\mathfrak{gl}_n) \cong \mathfrak{gl}_n$ and $\mathcal{T}_{E_n}(\mathrm{End}_{\mathbb{K}}(\mathfrak{gl}_n)) \cong \mathrm{End}_{\mathbb{K}}(\mathfrak{gl}_n)$ we have $d_{E_n}(\sigma)(E_{rs})\colon E_{ij} \mapsto \sum_{k=1}^n \sum_{l=1}^n \partial_{rs}(f_{ij;kl})(E_n) \cdot E_{kl} = \delta_{is}E_{rj} = E_{rs}E_{ij}$, hence $d_{E_n}(\sigma)(x)$ is left multiplication with $x \in \mathfrak{gl}_n$.

Similarly, $\tau\colon \mathfrak{gl}_n \to \mathrm{End}_{\mathbb{K}}(\mathfrak{gl}_n)\colon x \mapsto \tau(x)$, where $\tau(x)\colon \mathfrak{gl}_n \to \mathfrak{gl}_n\colon A \mapsto Ax$ yields $[E_{ij}x]_{kl} = \delta_{ik}x_{jl}$, and for $g_{ij;kl} := \delta_{ik}X_{jl} \in \mathbb{K}[\mathcal{X}]$ we get $\partial_{rs}(g_{ij;kl})(E_n) = \delta_{ik}\delta_{jr}\delta_{ls}$. Thus we have $d_{E_n}(\tau_x)(E_{rs})\colon E_{ij} \mapsto \sum_{k=1}^n \sum_{l=1}^n \partial_{rs}(g_{ij;kl})(E_n) \cdot E_{kl} = \delta_{jr}E_{is} = E_{ij}E_{rs}$, hence $d_{E_n}(\tau)(x)$ is right multiplication with $x \in \mathfrak{gl}_n$.

Now we have $\mathrm{Ad} = \mu(\sigma \times \tau)(\mathrm{id}_{\mathbb{G}} \times \iota)\colon \mathbb{G} \to \mathbb{G} \times \mathbb{G} \to \mathbb{GL}(\mathfrak{g}) \times \mathbb{GL}(\mathfrak{g}) \to \mathbb{GL}(\mathfrak{g})$. This yields $d_{E_n}(\mathrm{Ad}) = d_{E_{n^2}}(\mu)(d_{E_n}(\sigma) \oplus (-d_{E_n}(\tau)) = d_{E_n}(\sigma) - d_{E_n}(\tau)$, and thus for $x \in \mathfrak{gl}_n$ we get $d_{E_n}(\mathrm{Ad})(x)\colon \mathfrak{gl}_n \to \mathfrak{gl}_n\colon y \mapsto xy - yx = [x, y]$. ♯

Further differentiation formulae are given in Exercises (12.34) and (12.35).

**(9.3) Definition and Remark. a)** Let $\mathbb{G}$ be an algebraic group. A homomorphism of algebraic groups $\Phi\colon \mathbb{G} \to \mathbb{G}$, such that $d_1(\Phi)\colon T_1(\mathbb{G}) \to T_1(\mathbb{G})$ is nilpotent, is called a **Frobenius endomorphism** on $\mathbb{G}$.

If $\mathbb{H} \leq \mathbb{G}$ is a closed subgroup which is **$\Phi$-invariant**, i. e. we have $\Phi(V) \subseteq V$, then the restriction of $\Phi|_{\mathbb{H}}$ is a Frobenius endomorphism on $\mathbb{H}$.

**b)** Let $\mathrm{char}(\mathbb{K}) = p > 0$ and $q := p^f$ for some $f \in \mathbb{N}$. Then $\Phi_q\colon \mathbb{K}^n \to \mathbb{K}^n\colon [x_1, \ldots, x_n] \mapsto [x_1^q, \ldots, x_n^q]$ is called the associated **geometric Frobenius morphism** on $\mathbb{K}^n$. Hence the set of **fixed points** $(\mathbb{K}^n)^{\Phi_q} := \{x \in \mathbb{K}^n; \Phi_q(x) = x\} = \mathbb{F}_q^n$ coincides with the finite set of **$\mathbb{F}_q$-rational points** of $\mathbb{K}^n$.

From $\Phi_q^*\colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}]\colon X_i \mapsto X_i^q$ we for $x \in \mathbb{K}^n$ and $\delta \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[\mathcal{X}], \mathbb{K}_x)$ as well as $i \in \{1, \ldots, n\}$ get $d_x(\Phi_q)(\delta)(X_i) = \delta\Phi_q^*(X_i) = \delta(X_i^q) = \delta(X_i) \cdot (qX_i^{q-1})(x) = 0$, implying $d_x(\Phi_q) = 0$.

Considering $\mathbb{GL}_n = (\mathbb{K}^{n\times n})_{\det_n} \subseteq \mathbb{K}^{n\times n}$ as a principal open subset, the geometric Frobenius morphism on $\mathbb{K}^{n\times n}$ restricts to the **standard** Frobenius endomorphism $\Phi_q\colon \mathbb{GL}_n \to \mathbb{GL}_n\colon [a_{ij}] \mapsto [a_{ij}^q]$ on $\mathbb{GL}_n$, where we still have $d_1(\Phi_q) = 0$. Moreover, $\widetilde{\Phi}_q\colon \mathbb{GL}_n \to \mathbb{GL}_n\colon A \mapsto \Phi_q(A^{-\mathrm{tr}}) = \Phi_q(A)^{-\mathrm{tr}}$ is a homomorphism of algebraic groups and we have $\widetilde{\Phi}_q^2 = \Phi_{q^2}$, hence $d_1(\widetilde{\Phi}_q)^2 = 0$; the morphism $\widetilde{\Phi}$ is called the **non-standard** Frobenius endomorphism on $\mathbb{GL}_n$.

**c)** Given a Frobenius endomorphism $\Phi$ on $\mathbb{G}$, let $\mathbb{G}^\Phi := \{g \in \mathbb{G}; \Phi(g) = g\} \leq \mathbb{G}$ be the subgroup of fixed points of $\Phi$ on $\mathbb{G}$. Iterating yields a chain of closed subgroups $\mathbb{G} \geq \Phi(\mathbb{G}) \geq \Phi^2(\mathbb{G}) \geq \ldots$, which hence eventually becomes stable. Thus there is $n \in \mathbb{N}$ such that $\mathbb{H} := \Phi^n(\mathbb{G}) = \Phi^{n+1}(\mathbb{G})$, implying that $\Phi|_{\mathbb{H}}\colon \mathbb{H} \to \mathbb{H}$ is surjective. Since we have $\mathbb{G}^\Phi \leq \mathbb{H}$, as far as fixed points are concerned we might restrict ourselves to surjective Frobenius endomorphisms. Moreover if $\mathbb{G} \leq \mathbb{GL}_n$ is a $\Phi_q$-invariant closed subgroup, and we have $\Phi_q|_{\mathbb{G}} = \Phi^d$, for some $d \in \mathbb{N}$, then we have $\mathbb{G}^\Phi \leq \mathbb{G}^{\Phi^d} = \mathbb{G}^{\Phi_q} \leq \mathbb{GL}_n^{\Phi_q}$, hence $\mathbb{G}^\Phi$ is finite.

For the standard Frobenius endomorphism $\Phi_q$ on $\mathbb{GL}_n$ we get the **general linear group** $\mathrm{GL}_n(\mathbb{F}_q) = \mathbb{GL}_n^{\Phi_q}$, and since it is immediate that $\mathbb{SL}_n \leq \mathbb{GL}_n$ as well as $\mathbb{S}_{2m} \leq \mathrm{GL}_n$ and $\mathbb{O}_n \leq \mathrm{GL}_n$ are $\Phi_q$-invariant, we get the **special linear group** $\mathrm{SL}_n(\mathbb{F}_q) = \mathbb{SL}_n^{\Phi_q}$, the **symplectic group** $\mathrm{Sp}_{2m}(\mathbb{F}_q) = \mathbb{S}_{2m}^{\Phi_q}$, for $\mathrm{char}(\mathbb{K}) \neq 2$ the **general orthogonal groups** $\mathrm{GO}_{2m+1}(\mathbb{F}_q) = \mathbb{O}_{2m+1}^{\Phi_q}$ and $\mathrm{GO}_{2m}^+(\mathbb{F}_q) = \mathbb{O}_{2m}^{\Phi_q}$ as well as the **special orthogonal groups** $\mathrm{SO}_{2m+1}(\mathbb{F}_q) = \mathrm{GO}_{2m+1}(\mathbb{F}_q) \cap \mathrm{SL}_{2m+1}(\mathbb{F}_q) = \mathbb{SO}_{2m+1}^{\Phi_q}$ and $\mathrm{SO}_{2m}^+(\mathbb{F}_q) = \mathrm{GO}_{2m}^+(\mathbb{F}_q) \cap \mathrm{SL}_{2m}(\mathbb{F}_q) = \mathbb{SO}_{2m}^{\Phi_q}$, and for $\mathrm{char}(\mathbb{K}) = 2$ the **general orthogonal group** $\mathrm{GO}_{2m}^+(\mathbb{F}_q) = \mathbb{O}_{2m}^{\Phi_q}$; since in the latter case $\mathbb{SO}_{2m} = \mathbb{O}_{2m}^\circ$ is $\Phi_q$-invariant we also get the **special orthogonal group** $\mathrm{SO}_{2m}^+(\mathbb{F}_q) = \mathbb{SO}_{2m}^{\Phi_q}$.

For the non-standard Frobenius endomorphism $\widetilde{\Phi}_q$ on $\mathbb{GL}_n$ we get the **general unitary group** $\mathrm{GU}_n(\mathbb{F}_{q^2}) = \mathbb{GL}_n^{\widetilde{\Phi}_q}$, and since it is immediate that $\mathbb{SL}_n \leq \mathbb{GL}_n$ and is $\widetilde{\Phi}_q$-invariant, we get the **special unitary group** $\mathrm{SU}_n(\mathbb{F}_{q^2}) := \mathrm{GU}_n(\mathbb{F}_{q^2}) \cap \mathrm{SL}_n(\mathbb{F}_{q^2}) = \mathbb{SL}_n^{\widetilde{\Phi}_q}$. By a non-standard Frobenius endomorphism on $\mathbb{SO}_{2m}$ we get the **non-split special orthogonal group** $\mathrm{SO}^-_{2m}(\mathbb{F}_q)$, see Exercise (12.36).

**(9.4) Theorem: Lang (1965), Steinberg (1968).**
Let $\mathbb{G}$ be a connected algebraic group, and let $\Phi\colon \mathbb{G} \to \mathbb{G}$ be a Frobenius homomorphism. Then the **Lang map** $\mathcal{L}\colon \mathbb{G} \to \mathbb{G}\colon x \mapsto x^{-1}\Phi(x)$ is surjective.

**Proof.** We consider the **$\Phi$-conjugation** action $\mathbb{G}\times\mathbb{G} \to \mathbb{G}\colon [z,x] \mapsto x^{-1}z\Phi(x)$. For $z \in \mathbb{G}$ we have the orbit map $\mathcal{L}_z\colon \mathbb{G} \to \mathbb{G}\colon x \mapsto x^{-1}z\Phi(x) = x^{-1}z\Phi(x)z^{-1}z$, where in particular we have $\mathcal{L}_1 = \mathcal{L}$:

We have $\mathcal{L}_z = \rho_z\mu(\iota \times (\kappa_{z^{-1}}\Phi))$, where $\kappa_z\colon \mathbb{G} \to \mathbb{G}\colon x \mapsto z^{-1}xz$, implying $d_1(\mathcal{L}_z) = d_1(\rho_z)(d_1(\kappa_{z^{-1}}\Phi) - \mathrm{id}_{T_1(\mathbb{G})})$. Letting $d \in \mathbb{N}$ such that $d_1(\Phi)^d = 0$, we have $(\kappa_{z^{-1}}\Phi)^d = \kappa_{\widehat{z}^{-1}}\Phi^d$, where $\widehat{z} := z \cdot \Phi(z) \cdot \Phi^2(z) \cdot \cdots \cdot \Phi^{d-1}(z) \in \mathbb{G}$. Hence $d_1(\kappa_{z^{-1}}\Phi)^d = d_1(\kappa_{\widehat{z}^{-1}})d_1(\Phi)^d = 0$, implying that $d_1(\kappa_{z^{-1}}\Phi)$ is nilpotent as well, and thus $d_1(\kappa_{z^{-1}}\Phi) - \mathrm{id}_{T_1(\mathbb{G})}\colon T_1(\mathbb{G}) \to T_1(\mathbb{G})$ is a $\mathbb{K}$-isomorphism. Since $d_1(\rho_z)\colon T_1(\mathbb{G}) \to T_z(\mathbb{G})$ is a $\mathbb{K}$-isomorphism, this implies that $d_1(\mathcal{L}_z)\colon T_1(\mathbb{G}) \to T_z(\mathbb{G})$ is a $\mathbb{K}$-isomorphism as well. For $y \in \mathbb{G}$ we have $\mathcal{L}_z(x) = x^{-1}z\Phi(x) = y^{-1}(xy^{-1})^{-1}z\Phi(xy^{-1})\Phi(y) = y^{-1}\mathcal{L}_z(xy^{-1})\Phi(y)$, for all $x \in \mathbb{G}$, thus $\mathcal{L}_z = \rho_{\Phi(y)}\lambda_y\mathcal{L}_z\rho_{y^{-1}}$. Hence we have $d_y(\mathcal{L}_z) = d_z(\rho_{\Phi(y)}\lambda_y)d_1(\mathcal{L}_z)d_y(\rho_{y^{-1}})$, implying that $d_y(\mathcal{L}_z)\colon T_y(\mathbb{G}) \to T_{\mathcal{L}_z(y)}(\mathbb{G})$ is a $\mathbb{K}$-isomorphism, for any $y \in \mathbb{G}$.

Let $V_z := \overline{\mathcal{L}_z(\mathbb{G})} \subseteq \mathbb{G}$, hence $V_z$ is irreducible. Since the regular points of $V_z$ form a nonempty open subset, and $\mathcal{L}_z(\mathbb{G})$ contains a nonempty open subset of $V_z$, these sets intersect non-trivially, and hence there is $y \in \mathbb{G}$ such that $\mathcal{L}_z(y) \in V_z$ is regular. Since $T_{\mathcal{L}_z(y)}(\mathbb{G}) = \mathrm{im}(d_y(\mathcal{L}_z)) \leq T_{\mathcal{L}_z(y)}(V_z) \leq T_{\mathcal{L}_z(y)}(\mathbb{G})$ we deduce $\dim(V_z) = \dim_{\mathbb{K}}(T_{\mathcal{L}_z(y)}(V_z)) = T_{\mathcal{L}_z(y)}(\mathbb{G}) = \dim(\mathbb{G})$, and thus $V_z = \mathbb{G}$, see also Exercise (11.32). Being a $\mathbb{G}$-orbit, $\mathcal{L}_z(\mathbb{G}) \subseteq \overline{\mathcal{L}_z(\mathbb{G})} = V_z = \mathbb{G}$ is open. Hence any two such $\mathbb{G}$-orbits intersect non-trivially, thus $\mathcal{L}_z(\mathbb{G}) = \mathbb{G}$ for all $z \in \mathbb{G}$. ♯

**(9.5) Corollary.** Let $\mathbb{G}$ be a connected algebraic group with Frobenius homomorphism $\Phi\colon \mathbb{G} \to \mathbb{G}$. Then $\mathbb{G}^\Phi$ is finite.

**Proof.** Given $x, y \in \mathbb{G}$ we have $\mathcal{L}(x) = \mathcal{L}(y)$ if and only if $yx^{-1} = \Phi(yx^{-1})$, which holds if and only if $\mathbb{G}^\Phi x = \mathbb{G}^\Phi y$. Hence the fibres of $\mathcal{L}$ are the right cosets of $\mathbb{G}^\Phi$ in $\mathbb{G}$. Since $\mathcal{L}$ is dominant there is $z \in \mathbb{G}$ such that $\dim(\mathcal{L}^{-1}(z)) = \dim(\mathbb{G}) - \dim(\mathbb{G}) = 0$, hence $\mathcal{L}^{-1}(z)$ is finite. ♯

A different proof, for a $\Phi_q$-invariant connected closed subgroup $\mathbb{G} \leq \mathbb{GL}_n$ whose Frobenius endomorphism $\Phi$ fulfils $\Phi^d = \Phi_q|_{\mathbb{G}}$ for some $d \in \mathbb{N}$, showing that $\mathcal{L}$ is a finite dominant morphism, is given in Exercise (12.38). For not necessarily connected algebraic groups see Exercise (12.37).

**(9.6) Proposition.** Let $\mathbb{G}$ be an algebraic group with Frobenius endomorphism $\Phi$. Let $\Omega$ be a $\mathbb{G}$-set, and let $\varphi\colon \Omega \to \Omega$ be **$\Phi$-equivariant**, i. e. we have $\varphi(\omega g) = \varphi(\omega)\Phi(g)$, for all $\omega \in \Omega$ and $g \in \mathbb{G}$.

**a)** If $\mathbb{G}$ is connected and $\Omega$ is a transitive $\mathbb{G}$-set, then the set of fixed points $\Omega^\varphi := \{\omega \in \Omega; \varphi(\omega) = \omega\}$ is non-empty.
**b)** If moreover the stabiliser $\mathrm{Stab}_\mathbb{G}(\omega) \leq \mathbb{G}$ is a connected closed subgroup, for some and hence all $\omega \in \Omega$, then $\Omega^\varphi$ is a transitive $\mathbb{G}^\Phi$-set.

**Proof. a)** For any $\omega \in \Omega$ we have $\varphi(\omega) = \omega g$ for some $g \in \mathbb{G}$. Letting $h \in \mathbb{G}$ such that $g = h^{-1}\Phi(h)$, we get $\varphi(\omega h^{-1}) = \varphi(\omega)\Phi(h^{-1}) = \varphi(\omega)g^{-1}h^{-1} = \omega h^{-1}$, hence $\omega h^{-1} \in \Omega^\varphi$.
**b)** Let $\omega \in \Omega^\varphi$ be fixed such that $\mathbb{H} := \mathrm{Stab}_\mathbb{G}(\omega) \leq \mathbb{G}$ is closed and connected. Then for $g \in \mathbb{G}^\Phi$ we have $\varphi(\omega g) = \varphi(\omega)\Phi(g) = \omega g$, hence $\omega g \in \Omega^\varphi$. Moreover, for $h \in \mathbb{H}$ we have $\omega\Phi(h) = \varphi(\omega)\Phi(h) = \varphi(\omega h) = \varphi(\omega) = \omega$, hence $\mathbb{H}$ is $\Phi$-invariant, and thus $\Phi|_\mathbb{H}$ is a Frobenius endomorphism. For $\omega' \in \Omega^\varphi$ arbitrary let now $g \in \mathbb{G}$ such that $\omega g = \omega'$. Then we have $\omega g = \omega' = \varphi(\omega') = \varphi(\omega g) = \varphi(\omega)\Phi(g) = \omega\Phi(g)$, hence $g\Phi(g^{-1}) \in \mathbb{H}$. Thus there is $h \in \mathbb{H}$ such that $g\Phi(g^{-1}) = h^{-1}\Phi(h)$, implying $hg = \Phi(hg) \in \mathbb{G}^\Phi$ and $\omega' = \omega hg$. ♯

A generalisation to the case of unconnected stabilisers is given in Exercise (12.40). An application to conjugacy classes in $\mathbb{G}^\Phi$ is given in Exercises (12.39) and (12.41).

## 10 Generation and connectedness

**(10.1) Proposition.** Let $\mathbb{G}$ be an algebraic group, let $V_\lambda$ irreducible affine varieties together with morphisms $\varphi_\lambda : V_\lambda \to \mathbb{G}$ such that $1_\mathbb{G} \in W_\lambda := \varphi_\lambda(V_\lambda)$, for all $\lambda \in \Lambda$, where $\Lambda$ is an index set. Then $H := \langle W_\lambda; \lambda \in \Lambda\rangle \leq \mathbb{G}$ is closed and irreducible. Moreover, there is $[\lambda_1, \ldots, \lambda_r] \subseteq \Lambda$, for some $r \in \mathbb{N}_0$, and signs $\epsilon_i \in \{\pm 1\}$, for $i \in \{1, \ldots, r\}$, such that $H = W_{\lambda_1}^{\epsilon_1} \cdot \cdots \cdot W_{\lambda_r}^{\epsilon_r}$.

**Proof.** We may assume that for any $\lambda \in \Lambda$ there is $\lambda' \in \Lambda$ such that $\varphi_{\lambda'} = \iota_\mathbb{G}\varphi_\lambda$. For any $r \in \mathbb{N}_0$ and $\alpha := [\alpha_1, \ldots, \alpha_r] \subseteq \Lambda$ let $\varphi_\alpha : \prod_{i=1}^r V_{\alpha_i} \to \mathbb{G} : [x_1, \ldots, x_r] \mapsto x_1 \cdot \cdots \cdot x_r$. Letting $W_\alpha := W_{\alpha_1} \cdot \cdots \cdot W_{\alpha_r} = \mathrm{im}(\varphi_\alpha)$ we conclude that $\overline{W_\alpha} \subseteq \mathbb{G}$ is closed and irreducible. For $\beta := [\beta_1, \ldots, \beta_s] \subseteq \Lambda$, where $s \in \mathbb{N}_0$, let $\alpha\beta := [\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s] \subseteq \Lambda$ be the concatenation of $\alpha$ and $\beta$. Then we have $W_\alpha W_\beta = W_{\alpha\beta} \subseteq \mathbb{G}$, and moreover even $\overline{W_\alpha}\,\overline{W_\beta} \subseteq \overline{W_{\alpha\beta}}$: For $h \in W_\beta$ the map $W_\alpha \to W_{\alpha\beta} : g \mapsto gh$ is continuous, hence we have $\overline{W_\alpha}h = \overline{W_\alpha h} \subseteq \overline{W_{\alpha\beta}}$, and thus $\overline{W_\alpha}W_\beta \subseteq \overline{W_{\alpha\beta}}$. Similarly, from $gW_\beta \subseteq \overline{W_{\alpha\beta}}$ for $g \in \overline{W_\alpha}$ we obtain $g\overline{W_\beta} \subseteq \overline{W_{\alpha\beta}}$, and thus $\overline{W_\alpha}\,\overline{W_\beta} \subseteq \overline{W_{\alpha\beta}}$.

Choose $\alpha \subseteq \Lambda$ such that $\dim(\overline{W_\alpha}) \in \mathbb{N}_0$ is maximal. For any $\beta \subseteq \Lambda$ we since $1_\mathbb{G} \in \overline{W_\beta}$ have $\overline{W_\alpha} = \overline{W_\alpha} \cdot 1_\mathbb{G} \subseteq \overline{W_\alpha}\,\overline{W_\beta} \subseteq \overline{W_{\alpha\beta}}$, hence by the maximality of $\dim(\overline{W_\alpha})$ we conclude $\overline{W_\alpha} = \overline{W_\alpha}\,\overline{W_\beta} = \overline{W_{\alpha\beta}}$, and $\overline{W_\beta} = 1_\mathbb{G} \cdot \overline{W_\beta} \subseteq \overline{W_\alpha}\,\overline{W_\beta} = \overline{W_\alpha}$. In particular, we have $\overline{W_\alpha}\,\overline{W_\alpha} = \overline{W_\alpha}$ and $\overline{W_\alpha}^{-1} \subseteq \overline{W_\alpha}$, implying that $\overline{W_\alpha} \leq \mathbb{G}$ is a closed subgroup, such that $W_\beta \subseteq \overline{W_\alpha}$ for all $\beta \subseteq \Lambda$. Finally, there is $\emptyset \neq U \subseteq \overline{W_\alpha}$ open, and hence dense, such that $U \subseteq W_\alpha = \mathrm{im}(\varphi_\alpha)$. By (5.2) we have $UU = \overline{W_\alpha}$, implying $H = W_{\alpha\alpha} = W_\alpha W_\alpha = \overline{W_\alpha}$. ♯

**(10.2) Theorem.** **a)** The symplectic group $\mathbb{S}_{2m}$ is connected, and $\mathbb{S}_{2m} \leq \mathbb{SL}_{2m}$.
**b)** The special orthogonal group $\mathbb{SO}_n$ for $\mathrm{char}(\mathbb{K}) \neq 2$ is connected.
**c)** The special orthogonal group $\mathbb{SO}_{2m}$ for $\mathrm{char}(\mathbb{K}) = 2$ is connected.

**Proof.** **a)** Indexing rows and column by $\mathcal{I} := \{-m, \ldots, -1, 1, \ldots, m\}$, for $i, j \in \mathcal{I}$ such that $j \neq \{\pm i\}$ and $t \in \mathbb{K}$ let $x_{ij}(t) := E_n + t(E_{ij} - \frac{i}{|i|} \cdot \frac{j}{|j|} \cdot E_{-j,-i}) \in \mathbb{SL}_{2m}$ and $x_{i,-i}(t) := E_n + tE_{i,-i} \in \mathbb{SL}_{2m}$ be **symplectic transvections**. Hence for the unipotent **root subgroups** $U_{ij} := \{x_{ij}(t); t \in \mathbb{K}\} \leq \mathbb{SL}_{2m}$ the map $\mathbb{G}_a \cong \mathbb{K} \to U_{ij} \colon t \mapsto x_{ij}(t)$ is an isomorphism of algebraic groups. It is immediate that $U_{ij} \leq \mathbb{S}_{2m}$, and by [12, p.186] we have $\langle U_{ij}; i, j \in \mathcal{I}, i \neq j \rangle = \mathbb{S}_{2m}$.

**b)** Since $\mathbb{SO}_1 = \{1\}$ and $\mathbb{SO}_2 \cong \mathbb{G}_m$, see Exercise (12.7), we may assume $n \geq 3$. Hence letting $\Omega_n := [\mathbb{O}_n, \mathbb{O}_n] \trianglelefteq \mathbb{O}_n$, by [15, Thm.11.45, 11.51] we have $\Omega_n = [\mathbb{SO}_n, \mathbb{SO}_n] = \mathbb{SO}_n$.

Let $n = 2m$. Indexing rows and columns by $\mathcal{I} := \{-m, \ldots, -1, 1, \ldots, m\}$, for $i, j \in \mathcal{I}$ such that $j \neq \{\pm i\}$ and $t \in \mathbb{K}$ let $x_{ij}(t) := E_n + t(E_{ij} - E_{-j,-i}) \in \mathbb{SL}_{2m}$ and $U_{ij} := \{x_{ij}(t); t \in \mathbb{K}\} \leq \mathbb{SL}_{2m}$. It is immediate that $U_{ij} \leq \mathbb{SO}_{2m}$, and by [12, p.185] we have $\langle U_{ij}; i, j \in \mathcal{I}, j \neq \{\pm i\} \rangle = \Omega_{2m}$.

Let $n = 2m+1$. Indexing rows and columns by $\mathcal{I} := \{-m, \ldots, -1, 0, 1, \ldots, m\}$, for $i, j \in \mathcal{I} \setminus \{0\}$ such that $j \neq \{\pm i\}$ and $t \in \mathbb{K}$ let $x_{ij}(t) := E_n + t(E_{ij} - E_{-j,-i}) \in \mathbb{SL}_{2m+1}$ and $x_{i0}(t) := E_n + t(E_{i0} - E_{0,-i}) - \frac{t^2}{2} \cdot E_{i,-i} \in \mathbb{SL}_{2m+1}$, as well as $U_{ij} := \{x_{ij}(t); t \in \mathbb{K}\} \leq \mathbb{SL}_{2m+1}$. It is immediate that $U_{ij} \leq \mathbb{SO}_{2m+1}$, and by [12, p.187] we have $\langle U_{ij}; i, j \in \mathcal{I}, i \neq 0, j \neq \{\pm i\} \rangle = \Omega_{2m+1}$.

**c)** Since $\mathbb{G}_m \cong \mathbb{T} := \{\mathrm{diag}[t, t^{-1}]; t \in \mathbb{K} \setminus \{0\}\} \lhd \mathbb{O}_2 \leq \mathbb{SL}_2$ is connected such that $[\mathbb{O}_2 \colon \mathbb{T}] = 2$, see Exercise (12.7), we deduce that $\mathbb{T} \leq \mathbb{O}_2$ is the only closed subgroup of index 2, and thus $\mathbb{SO}_2 = \mathbb{T} \cong \mathbb{G}_m$. Hence we may assume $2m \geq 4$, and letting $\Omega_{2m} := [\mathbb{O}_{2m}, \mathbb{O}_{2m}] \trianglelefteq \mathbb{O}_{2m}$ by [15, Thm.11.45, 11.51] we have $\Omega_{2m} = [\mathbb{SO}_{2m}, \mathbb{SO}_{2m}] = \mathbb{SO}_{2m}$.

Indexing rows and columns by $\mathcal{I} := \{-m, \ldots, -1, 1, \ldots, m\}$, for $i, j \in \mathcal{I}$ such that $j \neq \{\pm i\}$ and $t \in \mathbb{K}$ let $x_{ij}(t) := E_n + t(E_{ij} + E_{-j,-i}) \in \mathbb{SL}_{2m}$ and $U_{ij} := \{x_{ij}(t); t \in \mathbb{K}\} \leq \mathbb{SL}_{2m}$; these are the same generators as for the case $n = 2m$ in (b), and a subset of the generators of $\mathbb{S}_{2m}$ in (a). It is immediate that $U_{ij} \leq \mathbb{O}_{2m}$, and by [12, p.185] we have $\langle U_{ij}; i, j \in \mathcal{I}, j \neq \{\pm i\} \rangle = \Omega_{2m}$. ♯

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# III Exercises and references

## 11 Exercises to Part I

**(11.1) Exercise: Polynomial functions.**
Let $K$ be a field and $\mathcal{X} := \{X_1, \ldots, X_n\}$. For any $x = [x_1, \ldots, x_n] \in K^n$ let $\epsilon_x^*\colon K[\mathcal{X}] \to K\colon f \mapsto f(x)$ be the associated **evaluation map**, and for any $f \in K[\mathcal{X}]$ let $f^\bullet\colon K^n \to K\colon x \mapsto f(x)$ be the **polynomial function** afforded by $f$.
**a)** Show that in general $f$ is not necessarily uniquely determined by $f^\bullet$.
**b)** Show that if $K$ is infinite then $f$ indeed is uniquely determined by $f^\bullet$.
**c)** Show that an algebraically closed field is infinite.

**(11.2) Exercise: Unions of algebraic sets.**
Let $\mathbb{K}$ be an algebraically closed field, let $\mathcal{X} := \{X_1, \ldots, X_n\}$, and let $I, I' \trianglelefteq \mathbb{K}[\mathcal{X}]$. Show that $\mathcal{V}(I) \cup \mathcal{V}(I') = \mathcal{V}(I \cdot I') = \mathcal{V}(I \cap I')$.

**Proof.** See [9, La.1.1.5]. $\sharp$

**(11.3) Exercise: Hilbert's Nullstellensatz.**
Let $\mathbb{K}$ be an algebraically closed field, let $\mathcal{X} := \{X_1, \ldots, X_n\}$, let $I \lhd \mathbb{K}[\mathcal{X}]$, and let $f \in \mathcal{I}(\mathcal{V}(I))$. Assuming the weak form of Hilbert's Nullstellensatz, show that $f \in \sqrt{I}$.

**Hint (Rabinowitch, 1929).** Consider $J := \langle I, 1 - fY \rangle \trianglelefteq \mathbb{K}[\mathcal{X}][Y]$, where $Y$ is an indeterminate over $\mathbb{K}[\mathcal{X}]$, and show that $\mathcal{V}(J) = \emptyset$.

**Proof.** See [4, Thm.5.4]. $\sharp$

**(11.4) Exercise: Topological spaces.**
Let $V$ be a topological space. Show the following:
**a)** A subset $U \subseteq V$ is irreducible if and only if its closure $\overline{U} \subseteq V$ is.
**b)** If $V$ is irreducible, then any open subset $\emptyset \neq U \subseteq V$ is dense and irreducible.
**c)** If $V$ is irreducible and $\varphi\colon V \to W$ is a continuous map, where $W$ is a topological space, then $\varphi(V) \subseteq W$ is irreducible.
**d)** $V$ is Hausdorff if and only if the **diagonal** $\{[v, v] \in V \times V; v \in V\} \subseteq V \times V$ is closed, where $V \times V$ is endowed with the product topology.
**e)** If $V$ is Noetherian, then $V$ is Hausdorff if and only if $V$ discrete.
**f)** If $V \neq \emptyset$ is Noetherian, then it is a finite union $V = V_1 \cup \cdots \cup V_r$, where the $V_i \subseteq V$ are closed and irreducible. If moreover $V_i \not\subseteq V_j$ for all $i \neq j$, then the $V_i \subseteq V$ are precisely the maximal irreducible closed subsets, being called the **irreducible components** of $V$.
**g)** If $V \neq \emptyset$ is Noetherian and $U \subseteq V$ is open and dense, then $U$ has a non-empty intersection with any irreducible component of $V$.

**Proof.** See [11, Exc.1.2.2, La.1.2.3] or [6, Exc.I.1.6, I.1.7]. $\sharp$

**(11.5) Exercise: Regular maps.**
Let $\mathbb{K}$ be an algebraically closed field, let $V \subseteq \mathbb{K}^n$ and $W \subseteq \mathbb{K}^m$ be algebraic, and let $\varphi\colon V \to W$ be regular. Show that $\varphi$ is continuous with respect to the Zariski topologies on $V$ and $W$.

**Proof.** See [6, La.I.3.1]. $\sharp$

**(11.6) Exercise: Affine varieties.**
Let $\mathbb{K}$ be an algebraically closed field and let $V \subseteq \mathbb{K}^n$ be algebraic. Show that both the set $V$ and its Zariski topology can be recovered from the affine coordinate algebra $\mathbb{K}[V]$:
**a)** Show that $\mathcal{V}$ induces a bijection between $\{I \trianglelefteq \mathbb{K}[V]\}$ and $\{U \subseteq V \text{ closed}\}$.
**b)** Show that this restricts to a bijection between $\{I \lhd \mathbb{K}[V] \text{ maximal}\}$ and $V$.

**Proof.** See [11, Prop.1.3.3]. $\sharp$

**(11.7) Exercise: Morphisms of affine varieties.**
Let $\mathbb{K}$ be an algebraically closed field, let $V, W$ be affine varieties, let $\varphi\colon V \to W$ be a morphism and let $\varphi^*\colon \mathbb{K}[W] \to \mathbb{K}[V]$ be the associated comorphism.
**a)** Show that $\varphi^*$ is injective if and only if $\varphi$ is **dominant**, i. e. $\varphi(V) \subseteq W$ is dense.
**b)** Show that $\varphi^*$ is surjective if and only if $\varphi$ is a **closed embedding**, i. e. $\varphi(V) \subseteq W$ is closed and $\varphi\colon V \to \varphi(V)$ is an isomorphism of affine varieties.

**Proof.** See [11, La.1.9.1] or [9, Prop.2.2.1]. $\sharp$

**(11.8) Exercise: Morphisms.**
Let $\mathbb{K}$ be an algebraically closed field.
**a)** Let $\varphi\colon \mathbb{K}^2 \to \mathbb{K}^2\colon [x, y] \mapsto [xy, y]$. Show that $\varphi(\mathbb{K}^2) \subseteq \mathbb{K}^2$ is neither open nor closed.
**b)** Let $\psi\colon \mathbb{K} \to \mathbb{K}^2\colon x \mapsto [x^2, x^3]$. Show that $\psi(\mathbb{K}) \subseteq \mathbb{K}^2$ is closed, and that $\psi\colon \mathbb{K} \to \psi(\mathbb{K})$ is bijective, but not an isomorphism of affine varieties.
**c)** Give an example of a continuous map between affine varieties which is not a morphism.

**Proof. a) b)** See [10, Exc.I.4.5] or [7, Ex.I.4.N, I.4.O] or [6, Exc.I.3.2].
**c)** See [10, Exc.I.1.6]. $\sharp$

**(11.9) Exercise: Products of affine varieties.**
Let $\mathbb{K}$ be an algebraically closed field and let $V \subseteq \mathbb{K}^n$ and $W \subseteq \mathbb{K}^m$ be algebraic.
**a)** Show that $V \times W \subseteq \mathbb{K}^n \times \mathbb{K}^m$ is algebraic.
**b)** Let $\mathbb{K}[\mathcal{X}]$ and $\mathbb{K}[\mathcal{Y}]$ be the affine coordinate algebras of $\mathbb{K}^n$ and $\mathbb{K}^m$, respectively. Show that the affine coordinate algebra of $\mathbb{K}^n \times \mathbb{K}^m$ can be identified with $\mathbb{K}[\mathcal{X} \,\dot\cup\, \mathcal{Y}]$, and that the natural map $\mathbb{K}[\mathcal{X}] \otimes_{\mathbb{K}} \mathbb{K}[\mathcal{Y}] \to \mathbb{K}[\mathcal{X} \,\dot\cup\, \mathcal{Y}]$ induces an isomorphism $\mathbb{K}[V] \otimes_{\mathbb{K}} [W] \to \mathbb{K}[V \times W]$ of $\mathbb{K}$-algebras.
**c)** Show that $V \times W$ is irreducible if and only if both $V$ and $W$ are irreducible.
**d)** Show that the Zariski topology on $V \times W$ is finer than the product topology induced by the Zariski topologies on $V$ and $W$, and give an example where the former is strictly finer.

**Proof. a) b) c)** See [9, Sect.1.3.7, Prop.1.3.8]. **d)** See [11, Exc.1.5.5]. ♯

**(11.10) Exercise: Principal open subsets.**
Let $V$ be an affine variety over $\mathbb{K}$.
**a)** For $0 \neq f, g \in \mathbb{K}[V]$ show that $V_{fg} = V_f \cap V_g$ and $V_{f^r} = V_f$, for all $r \in \mathbb{N}$. Moreover, show that $V_f \subseteq V_g$ if and only if $\sqrt{\langle f \rangle} \subseteq \sqrt{\langle g \rangle} \trianglelefteq \mathbb{K}[V]$.
**b)** Show that $\{V_f \subseteq V; 0 \neq f \in \mathbb{K}[V]\}$ is a basis of the Zariski topology.

**Proof.** See [11, Sect.1.3.5, La.1.3.6]. ♯

**(11.11) Exercise: Open subsets of affine varieties.**
Let $\mathbb{K}$ be an algebraically closed field and $U := \mathbb{K}^2 \setminus \{[0,0]\}$. Show that $U$ cannot be endowed with the structure of an affine variety, such that the inclusion maps $U \subseteq \mathbb{K}^2$ and $(\mathbb{K}^2)_f \subseteq U$, for all $f \in \mathbb{K}[X_1, X_2] \setminus \mathcal{I}([0,0])$, are morphisms.

**Proof.** See [7, Ch.I.4, p.35]. ♯

**(11.12) Exercise: Localisation.**
Let $R$ be a ring and let $U \subseteq R$ be multiplicatively closed such that $1 \in U$.
**a)** Show that the localisation $R_U$ is a ring, that $\nu\colon R \to R_U\colon r \mapsto \frac{r}{1}$ is a ring homomorphism, and $R_U$ has the following universal property: If $\varphi\colon R \to S$ is a ring homomorphism such that $\varphi(U) \subseteq S^*$, then there is unique ring homomorphism $\widehat{\varphi}\colon R_U \to S$ such that $\widehat{\varphi}\nu = \varphi$.
**b)** Show that for $J \trianglelefteq R_U$ we have $(\nu^{-1}(J))_U = J$, and conclude that the map $\iota\colon \{J \trianglelefteq R_U\} \to \{I \trianglelefteq R\}\colon J \mapsto \nu^{-1}(J)$ is an inclusion-preserving and intersection-preserving injection, mapping prime ideals to prime ideals.
**c)** Show that for an ideal $I \trianglelefteq R$ we have $I \subseteq \nu^{-1}(I_U) = \{f \in R; fu \in I \text{ for some } u \in U\} \trianglelefteq R$, conclude that if $U \cap I = \emptyset$ then we have $I_U \neq R_U$, and that a prime ideal $P \triangleleft R$ is in $\operatorname{im}(\iota)$ if and only if $U \cap P = \emptyset$.
**d)** Let $U \subseteq S$ be multiplicatively closed. Show that $R_S \cong (R_U)_{S_U}$.

**Proof.** See [1, Ch.2.1] or [4, Thm.4.1, 4.3]. ♯

**(11.13) Exercise: Integral ring extensions.**
Let $R \subseteq S$ be a ring extension.
**a)** Show that an element $s \in S$ is integral over $R$, if and only if there is an $R$-subalgebra of $S$ containing $s$, which is a finitely generated $R$-module. Conclude that $R \subseteq S$ is a **finite** ring extension, i. e. $S$ is a finitely generated $R$-algebra and integral over $R$, if and only if $S$ is a finitely generated $R$-module.
**b)** Show that the **integral closure** $\overline{R} := \{s \in S; s \text{ integral over } R\} \subseteq S$ of $R$ in $S$ is a subring of $S$, and that $\overline{\overline{R}} = \overline{R}$ holds. Show that a factorial domain $R$ is **integrally closed**, i. e. we have $R = \overline{R} \subseteq S := \mathrm{Q}(R)$.
**c)** Let $R \subseteq S$ be an integral ring extension, and let $S$ be a domain. Show that $R$ is a field if and only $S$ is a field.
**d)** Let $R \subseteq S$ be an integral ring extension, and let $J \lhd S$ and $I := J \cap R \lhd R$. Show that $\dim(I) = \dim(J) \in \mathbb{N}_0 \,\dot\cup\, \{\infty\}$ and $\mathrm{ht}(I) = \mathrm{ht}(J) \in \mathbb{N}_0 \,\dot\cup\, \{\infty\}$.

**Proof.** **a) b)** See [3, Ch.9].
**c)** See [4, La.9.1] or [1, La.4.16]. **d)** See [5, Cor.6.10]. ♯

**(11.14) Exercise: Infinite dimension (Nagata, 1962).**
Let $K$ be a field and $R := K[X_i; i \in \mathbb{N}]$, let $d_0 := 0$, and for $i \in \mathbb{N}$ let $d_i \in \mathbb{N}$ such that $d_i < d_{i+1}$, and $P_i := \langle X_{d_{i-1}+1}, \ldots, X_{d_i} \rangle \lhd R$, and let $U := R \setminus \bigcup_{i \geq 1} P_i \subseteq R$.
**a)** Show that $R$ is not Noetherian.
**b)** Show that the localisation $R_U$ is Noetherian.
**c)** Show that $\dim(R_U) = \sup\{d_i - d_{i-1}; i \in \mathbb{N}\} \in \mathbb{N} \,\dot\cup\, \{\infty\}$.

**Proof.** See [1, Exc.9.6]. ♯

**(11.15) Exercise: Dimension and height.**
Give an example of a finitely generated $K$-algebra, where $K$ is a field, which is not a domain, possessing an ideal $I \lhd R$ such that $\dim(I) + \mathrm{ht}(I) \neq \dim(R)$.

**(11.16) Exercise: Catenary rings.**
A finite dimensional Noetherian ring $R$ is called **catenary**, if for any prime ideals $P \subseteq Q \lhd R$ all maximal chains $P = P_0 \subset \cdots \subset P_r = Q$ of prime ideals have length $r = \mathrm{ht}(Q) - \mathrm{ht}(P)$.

Let $K$ be a field, and let $R$ be a finitely generated $K$-algebra which is a domain. Show that $R$ is catenary.

**(11.17) Exercise: Dimension of varieties.**
Let $V$ be an affine variety over $\mathbb{K}$
**a)** Show that $\dim(V) = 0$ if and only if $V$ is a finite set. Which are the irreducible varieties amongst them?
**b)** Let $V$ be irreducible. Show that $\dim(V)$ is the maximum of the lengths $d \in \mathbb{N}_0$ of chains $\emptyset \neq V_0 \subset \cdots \subset V_d = V$ of closed irreducible subsets.

**c)** Let $V$ be irreducible such that $\mathbb{K}[V]$ is a factorial domain. Show that any closed subset $W \subset V$ having equidimension $\dim(W) = \dim(V) - 1$ is of the form $W = \mathcal{V}(f)$ for some $f \in \mathbb{K}[V]$.
**d)** Let $V, W$ be irreducible. Show that $\dim(V \times W) = \dim(V) \cdot \dim(W)$.

**Proof.** **a)** See [10, Exc.I.3.1]. **b)** See [10, Exc.I.3.4].
**c)** See [10, Exc.I.3.6]. **d)** See [10, Prop.I.3.1]. $\sharp$

**(11.18) Exercise: Finite morphisms.**
Let $V, W$ be affine varieties and let $\varphi\colon V \to W$ be a finite morphism. Show that there is $c \in \mathbb{N}$ such that $|\varphi^{-1}(y)| \leq c$, for all $y \in W$.

**Proof.** See [9, La.2.2.3]. $\sharp$

**(11.19) Exercise: Finite morphisms.**
Let $\mathbb{K}$ be an algebraically closed field, and let $\varphi\colon \mathbb{K}^2 \to \mathbb{K}^2 \colon [x, y] \mapsto [xy, y]$, see Exercise (11.8).
**a)** Determine the dimensions of the irreducible components of the fibres of $\varphi$.
**b)** Is $\varphi$ a dominant morphism? Is $\varphi$ a finite morphism?

**(11.20) Exercise: Constructible sets.**
Let $V$ be a topological space. A subset $U \cap W \subseteq V$, where $U \subseteq V$ is open and $W \subseteq V$ is closed, is called **locally closed**. A finite union of locally closed subsets is called **constructible**.
**a)** Show that the set of constructible subsets is the smallest set of subsets containing all open subsets and being closed under taking finite unions and complements.
**b)** Let $V$ be Noetherian, and let $W \subseteq V$ be constructible. Show that there is $U \subseteq \overline{W}$ open and dense such that $U \subseteq W$.

**Proof.** **a)** See [6, Exc.II.3.18] or [10, Exc.I.4.3]. **b)** See [9, Exc.2.7.7]. $\sharp$

**(11.21) Exercise: Chevalley's Theorem (1955).**
Let $V, W$ be affine varieties and let $\varphi\colon V \to W$ be a morphism. Show that $\varphi$ maps constructible subsets to constructible subsets, see Exercise (11.20).

**Hint.** It suffices to consider $\varphi(V)$, reduce to the case $V, W$ irreducible, and proceed by induction on $\dim(W)$.

**Proof.** See [6, Exc.II.3.19] or [10, Thm.I.4.4]. $\sharp$

**(11.22) Exercise: Open morphisms.**
Let $V, W$ be irreducible affine varieties and let $\varphi\colon V \to W$ be a dominant morphism, such that for all $Z \subseteq W$ closed and irreducible the preimage $\varphi^{-1}(Z) \subseteq V$ is equidimensional of dimension $\dim(Z) + \dim(V) - \dim(W)$. Show that $\varphi$ is an **open map**, i. e. maps open sets to open sets.

**Hint.** Use Chevalley's Theorem.

**Proof.** See [10, Thm.I.4.5]. $\sharp$

**(11.23) Exercise: Upper semicontinuity of dimension.**
Let $V, W$ be irreducible affine varieties and let $\varphi\colon V \to W$ be a dominant morphism. For any $x \in V$ let $\epsilon_\varphi(x) \in \mathbb{N}_0$ be the maximum of the dimensions of the irreducible components of $\varphi^{-1}(\varphi(x)) \subseteq V$ containing $x$. Show that for any $n \in \mathbb{N}_0$ the set $\{x \in V; \epsilon_\varphi(x) \geq n\} \subseteq V$ is closed.

**Proof.** See [10, Prop.I.4.4]. $\sharp$

**(11.24) Exercise: Diagonalisable matrices.**
Let $\mathbb{K}$ be an algebraically closed field, and let $n \in \mathbb{N}$. Show that the set of diagonalisable matrices is dense in $\mathbb{K}^{n \times n}$.

**Hint.** Let $\mathbb{D}_n := \{[a_{ij}] \in \mathbb{K}^{n\times n}; a_{ij} = 0 \text{ for } i \neq j\}$ be the set of diagonal matrices, and consider fibres of $\varphi\colon \mathbb{D}_n \times \mathbb{GL}_n \to \mathbb{K}^{n\times n}\colon [x, g] \mapsto x^g := g^{-1}xg$.

**Proof.** See [9, Exc.2.7.9]. $\sharp$

**(11.25) Exercise: Derivations.**
Let $A$ be an $R$-algebra, let $U \subseteq A$ be multiplicatively closed such that $1 \in U$, and let $\nu_U\colon A \to A_U$ be the natural map. Let $M$ be an $A$-module such that the elements of $U$ act invertibly on $M$. Show that $\nu_U^*\colon \mathrm{Der}_R(A_U, M) \to \mathrm{Der}_R(A, M)$ is an isomorphism of abelian groups.

**Proof.** See [4, Exc.25.3]. $\sharp$

**(11.26) Exercise: Partial derivatives.**
Let $\mathcal{X} := \{X_1, \ldots, X_n\}$ be indeterminates over the perfect field $K$.
**a)** Let $\mathrm{char}(K) = 0$. Show that $f \in K[\mathcal{X}]$ is constant if and only if $\partial_i(f) = 0$ for all $i \in \{1, \ldots, n\}$.
**b)** Let $\mathrm{char}(K) = p > 0$, and let $f \in K[\mathcal{X}]$ such that $\partial_i(f) = 0$ for all $i \in \{1, \ldots, n\}$. Show that there is $g \in K[\mathcal{X}]$ such that $f = g^p$.
**c)** Let $f \in K[\mathcal{X}]$ be irreducible. Show that $\partial_i(f) \neq 0$ for some $i \in \{1, \ldots, n\}$.

**Proof.** See [9, Exc.1.8.14]. $\sharp$

**(11.27) Exercise: Tangent spaces.**
Let $V$ and $W$ be affine varieties over $\mathbb{K}$, and let $x \in V$ and $y \in W$. Show that $T_{[x,y]}(V \times W) \cong T_x(V) \oplus_{\mathbb{K}} T_y(W)$ as $\mathbb{K}$-vector spaces.

**Proof.** See [10, Prop.I.5.1]. $\sharp$

**(11.28) Exercise: Linear spaces.**
Let $\mathbb{K}$ be an algebraically closed field, let $V \leq \mathbb{K}^n$ and $W \leq \mathbb{K}^m$ be $\mathbb{K}$-subspaces, and let $\varphi \colon V \to W$ be a $\mathbb{K}$-linear map.
**a)** Show that $V$ is an irreducible affine variety such that $\dim(V) = \dim_{\mathbb{K}}(V)$. Show that for any $x \in V$ there is a natural identification of $T_x(V)$ with $V$.
**b)** Show that $\varphi$ is a morphism of affine varieties. Show that for any $x \in V$ using the above identifications the differential $d_x(\varphi)$ can be identified with $\varphi$.

**Proof.** See [10, Ch.I.5.4]. $\sharp$

**(11.29) Exercise: Zariski tangent spaces.**
Let $V \subseteq \mathbb{K}^n$ be closed, let $0 \neq f \in \mathbb{K}[V]$ and let $x \in V_f$. Using the closed embedding $V_f \to \mathbb{K}^{n+1} \colon y \mapsto [y, f(y)^{-1}]$ give a definition of a Zariski tangent space $\mathcal{T}_x(V_f)$, and show that it can be naturally identified with $\mathcal{T}_x(V)$.

**(11.30) Exercise: Regular points.**
Let $V$ be an irreducible affine variety over $\mathbb{K}$.
**a)** Show that for any $x \in V$ we have $\dim_{\mathbb{K}}(T_x(V)) \geq \dim(V)$.
**b)** Show that the set of regular points is an open subset of $V$.

**Hint for (a).** Consider the local ring $\mathcal{O}_x$ associated to $x$, and by using the Nakayama Lemma show that any subset $\mathcal{S} \subseteq \mathcal{P}_x$ generates the maximal ideal $\mathcal{P}_x$ as an $\mathcal{O}_x$-module if and only if it generates $\mathcal{P}_x/\mathcal{P}_x^2$ as a $\mathbb{K}$-vector space.

**Hint for (b).** Use the Jacobian matrix.

**Proof. a)** See [10, Ch.I.5.3]. **b)** See [10, Thm.I.5.2]. $\sharp$

**(11.31) Exercise: Singular points.**
Let $\mathrm{char}(\mathbb{K}) \neq 2$. Show that the following hypersurfaces $\mathcal{V}(f_i) \subseteq \mathbb{K}^2$ and $\mathcal{V}(g_j) \subseteq \mathbb{K}^3$ are irreducible, and determine their singular points. For the case $\mathbb{K} := \mathbb{C}$ draw pictures of the $\mathbb{R}$-**rational** points $\mathcal{V}(f_i) \cap \mathbb{R}^2$ and $\mathcal{V}(g_j) \cap \mathbb{R}^3$.

**a)** **i)** $f_1 = Y^4 + X^4 - X^2$ **ii)** $f_2 = Y^6 - XY + X^6$
**iii)** $f_3 = Y^4 + Y^2 + X^4 - X^3$ **iv)** $f_4 = Y^4 - XY^2 - X^2Y + X^4$

**b)** **i)** $g_1 = Z^2 - XY^2$ **ii)** $g_2 = Z^2 - Y^2 - X^2$ **iii)** $g_3 = Y^3 + XY + X^3$

**Proof. a)** See [6, Exc.I.5.1]. **b)** See [6, Exc.I.5.2]. $\sharp$

**(11.32) Exercise: Dominance criterion.**
Let $V, W$ be irreducible affine varieties, and let $\varphi\colon V \to W$ be a morphism such that for all $x \in V$ the differential $d_x(\varphi)\colon T_x(V) \to T_{\varphi(x)}(W)$ is a $\mathbb{K}$-isomorphism. Show that $\varphi$ is dominant.

## 12 Exercises to Part II

**(12.1) Exercise: Algebraic groups.**
**a)** Show that the direct product $\mathbb{G} \times \mathbb{G}'$ of algebraic groups $\mathbb{G}$ and $\mathbb{G}'$ again is an algebraic group.
**b)** Let $\mathbb{G}$ be an algebraic group, let $\mathbb{H} \subseteq \mathbb{G}$ be a closed subgroup, and let $\varphi\colon \mathbb{H} \to \mathbb{G}$ be the inclusion map. Show that $\mathbb{H}$ again is an algebraic group, such that $\varphi$ is a homomorphism of algebraic groups.

**Proof.** See [11, Exc.2.1.2]. $\sharp$

**(12.2) Exercise: Automorphisms of algebraic groups.**
Let $\mathbb{K}$ be an algebraically closed field.
**a)** Show that the maps $\mathbb{G}_a \to \mathbb{G}_a\colon x \mapsto ax$, for $a \in \mathbb{K} \setminus \{0\}$, are the only automorphisms of $\mathbb{G}_a$ as an algebraic group.
**b)** Show that $\mathrm{id}\colon \mathbb{G}_m \to \mathbb{G}_m\colon x \mapsto x$ and $\iota\colon \mathbb{G}_m \to \mathbb{G}_m\colon x \mapsto x^{-1}$ are the only automorphisms of $\mathbb{G}_m$ as an algebraic group.
**c)** Show that $\mathbb{G}_a$ and $\mathbb{G}_m$ are not isomorphic.

**Proof.** See [10, Exc.II.7.1, II.7.4]. $\sharp$

**(12.3) Exercise: Determinant polynomials.**
Let $K$ be a field and let $\det_n \in K[X_{11}, X_{12}, \ldots, X_{nn}]$ be the $n$-th determinant polynomial, for some $n \in \mathbb{N}$.
**a)** Show that $\det_n \in K[X_{11}, X_{12}, \ldots, X_{nn}]$ is irreducible.
**b)** Show that $\det_n - a \in K[X_{11}, X_{12}, \ldots, X_{nn}]$ is irreducible, for any $a \in K$.

**(12.4) Exercise: Examples of algebraic groups.**
**a)** Show that the following are algebraic groups, where $n \in \mathbb{N}$:
**i)** The **scalar group** $\mathbb{Z}_n := \{\alpha \cdot E_n \in \mathbb{GL}_n; 0 \neq \alpha \in \mathbb{K}\}$,
**ii)** the **torus** $\mathbb{T}_n := \{[a_{ij}] \in \mathbb{GL}_n; a_{ij} = 0 \text{ for } i \neq j\}$,
**iii)** the **unipotent group** $\mathbb{U}_n := \{[a_{ij}] \in \mathbb{GL}_n; a_{ij} = 0 \text{ for } i > j, a_{ii} = 1\}$,
**iv)** the **Borel group** $\mathbb{B}_n := \{[a_{ij}] \in \mathbb{GL}_n; a_{ij} = 0 \text{ for } i > j\}$,
**v)** the **monomial group** $\mathbb{N}_n := \{A \in \mathbb{GL}_n \text{ monomial}\}$.
**b)** Show that any finite group is an algebraic group.

**Proof.** See [11, Exc.2.1.3] or [10, Exc.II.7.7]. $\sharp$

**(12.5) Exercise: Projective special linear groups.**
Let $\mathbb{K}$ be an algebraically closed field, let $A := \mathbb{K}[X_{11}, X_{12}, X_{21}, X_{22}]/\langle \det_2 - 1\rangle$ be the affine coordinate algebra of $\mathbb{SL}_2$, let $B := \mathbb{K}\langle X_{ij}X_{kl}; i, j, k, l \in \{1, 2\}\rangle \subseteq A$, and let $\mathbb{PSL}_2$ be the affine variety having $B$ as its coordinate algebra.
**a)** If $\mathrm{char}(\mathbb{K}) \neq 2$, show that $B = \{f \in A; f(x) = f(-x) \text{ for all } x \in \mathbb{SL}_2\}$.
**b)** For $\mathrm{char}(\mathbb{K})$ arbitrary, show that $\mathbb{PSL}_2$ is endowed with the structure of an algebraic group, such that there is a surjective homomorphism $\varphi\colon \mathbb{SL}_2 \to \mathbb{PSL}_2$ of algebraic groups with $\ker(\varphi) = \{\pm E_2\}$.
**c)** If $\mathrm{char}(\mathbb{K}) = 2$, show that $\varphi$ is an isomorphism of groups, but not an isomorphism of algebraic groups.

**Proof.** See [11, Exc.2.1.4.(3)]. ♯

**(12.6) Exercise: Symplectic groups.**
**a)** Show that $\mathbb{S}_2 = \mathbb{SL}_2 \leq \mathbb{GL}_2$.
**b)** Let $m \in \mathbb{N}$. Show that $\mathbb{S}_{2m}$ has a closed subgroup isomorphic to $\mathbb{GL}_m$.

**Proof.** See [14, Prop.3.1]. ♯

**(12.7) Exercise: Orthogonal groups.**
Let $\mathbb{K}$ be an algebraically closed field.
**a)** Let $\mathrm{char}(\mathbb{K}) \neq 2$. Show that $\mathbb{SO}_2 = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in \mathbb{K}^{2\times 2}; 0 \neq a \in \mathbb{K} \right\}$.
**b)** For $\mathrm{char}(\mathbb{K})$ arbitrary, give a similar description of $\mathbb{O}_2$.
**c)** Let $\mathrm{char}(\mathbb{K}) \neq 2$ and let $\omega \in \mathbb{K}$ such that $\omega^3 = -2$. Show that

$$\varphi\colon \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & \omega t & t^2 \\ 0 & 1 & -\omega t \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ \omega t & 1 & 0 \\ t^2 & -\omega t & 1 \end{bmatrix},$$

for all $t \in \mathbb{K}$, defines a surjective homomorphism of algebraic groups $\varphi\colon \mathbb{SL}_2 \to \mathbb{SO}_3$, and determine $\ker(\varphi) \lhd \mathbb{SL}_2$.

**Hint for (c).** Consider the natural action of $\mathbb{SL}_2$ on the homogeneous polynomials of degree 2 in two variables.

**Proof.** See [9, 1.3.15, 1.3.16. Exc.1.8.19]. ♯

**(12.8) Exercise: Connectedness.**
Determine the identity component and the dimension of the algebraic groups in Exercises (12.4) and (12.5).

**(12.9) Exercise: Closed subgroups.**
Show that a closed subset of an algebraic group $\mathbb{G}$, which contains $1_{\mathbb{G}}$ and is closed under taking products, is a subgroup.

**Proof.** See [10, Exc.II.7.5]. $\sharp$

**(12.10) Exercise: Abelian subgroups.**
Let $\mathbb{G}$ be an algebraic group, and let $H \leq \mathbb{G}$ be an abelian subgroup. Show that $\overline{H} \leq \mathbb{G}$ is an abelian subgroup.

**Proof.** See [9, Exc.3.6.2]. $\sharp$

**(12.11) Exercise: Finite normal subgroups.**
Let $\mathbb{G}$ be a connected algebraic group.
**a)** Show that a finite normal subgroup of $\mathbb{G}$ is central.
**b)** Let $\varphi \colon \mathbb{G} \to \mathbb{G}$ be a surjective homomorphism of algebraic groups. Show that $\ker(\varphi) \trianglelefteq \mathbb{G}$ is finite.

**Proof.** **a)** See [11, Exc.2.2.2.(3)]. **b)** See [11, Exc.4.3.6.(6)b)]. $\sharp$

**(12.12) Exercise: Normalisers.**
Let $\mathbb{G}$ be an algebraic group, and let $\mathbb{H} \leq \mathbb{G}$ be closed. Show that the **normaliser** $N_{\mathbb{G}}(\mathbb{H}) := \{g \in \mathbb{G}; \mathbb{H}^g = \mathbb{H}\} \leq \mathbb{G}$ is a closed subgroup.

**Proof.** See [10, Cor.II.8.2]. $\sharp$

**(12.13) Exercise: Translation of functions.**
Let $\mathbb{G}$ be an algebraic group over $\mathbb{K}$, and let $\rho$ and $\lambda$ be its regular right and left translation actions, respectively.
**a)** Show that $\mathbb{K}[\mathbb{G}]$ is the union of finite dimensional $\mathbb{K}$-subspaces which are $\rho_g^*$-invariant for all $g \in \mathbb{G}$.
**b)** Let $F \leq \mathbb{K}[\mathbb{G}]$ such that $\dim_{\mathbb{K}}(F) < \infty$. Show that there is $F \leq E \leq \mathbb{K}[\mathbb{G}]$ such that $\dim_{\mathbb{K}}(E) < \infty$, which is $\rho_g^*$-invariant and $\lambda_h^*$-invariant for all $g, h \in \mathbb{G}$.
**c)** Let $\mathbb{H} \leq \mathbb{G}$ be a closed subgroup, and let $\mathcal{I}(\mathbb{H}) \lhd \mathbb{K}[\mathbb{G}]$ be the associated vanishing ideal. Show that $\mathbb{H} = \{g \in \mathbb{G}; \rho_g^*(\mathcal{I}(\mathbb{H})) \subseteq \mathcal{I}(\mathbb{H})\}$ and $\mathbb{H} = \{g \in \mathbb{G}; \lambda_g^*(\mathcal{I}(\mathbb{H})) \subseteq \mathcal{I}(\mathbb{H})\}$.

**Proof.** **a)** See [10, Exc.II.8.3]. **a)** See [10, Exc.II.8.4].
**c)** See [11, La.2.3.6] or [10, La.II.8.5]. $\sharp$

**(12.14) Exercise: linearisation.**
Let $\mathbb{G}$ be an algebraic group, and let $\lambda$ be its regular left translation action. Show that for all $g \in \mathbb{G}$ and the associated semisimple and unipotent parts $g_s \in \mathbb{G}$ and $g_u \in \mathbb{G}$, respectively, we have $(\lambda_g^*)_s = \lambda_{g_s}^*$ and $(\lambda_g^*)_s = \lambda_{g_s}^*$.

**Proof.** See [11, Exc.2.4.10.(1)]. $\sharp$

**(12.15) Exercise: Linearisation of actions.**
Let $\mathbb{G}$ be an algebraic group over $\mathbb{K}$ acting morphically on $V$. Show that there is a closed embedding $\psi\colon V \to \mathbb{K}^n$ and an algebraic representation $\varphi\colon \mathbb{G} \to \mathbb{GL}_n$, for some $n \in \mathbb{N}$, such that $\psi(xg) = \psi(x)\varphi(g)$, for all $x \in V$ and $g \in \mathbb{G}$.

**Proof.** See [11, Exc.2.3.7]. $\sharp$

**(12.16) Exercise: Additive Jordan decomposition.**
Let $A \in \mathbb{K}^{n\times n}$ and let $A_s, A_n \in \mathbb{K}^{n\times n}$ be its semisimple and its nilpotent part, respectively. Show that there are $f, g \in \mathbb{K}[T]$ such that $f(0) = 0 = g(0)$ and such that $A_s = f(A)$ and $A_n = g(A)$.

**Proof.** See [11, Prop.2.4.4(ii)] or [10, La.VI.15.1.A]. $\sharp$

**(12.17) Exercise: Jordan decomposition.**
Let $\mathbb{K}$ be an algebraically closed field.
**a)** Let $\mathrm{char}(\mathbb{K}) = p > 0$. Show that $A \in \mathbb{GL}_n$ is unipotent if and only if $A^{p^k} = E_n$, for some $k \in \mathbb{N}_0$.
**b)** Let $\mathrm{char}(\mathbb{K}) = 0$. Show that any semisimple element of $\mathbb{GL}_n$ has finite order.
**c)** Let $\mathbb{K} := \overline{\mathbb{F}}_q$ be the algebraic closure of the finite field $\mathbb{F}_q$. Give a description of the Jordan decomposition in $\mathbb{GL}_n$ in terms of element orders.

**Proof.** **a)** See [10, Ch.VI.15.1]. **b)** See [10, Exc.VI.15.5]. **c)** See [9, Ch.3.5]. $\sharp$

**(12.18) Exercise: Semisimple and unipotent elements.**
Determine the subsets $\mathbb{G}_s$ and $\mathbb{G}_u$ of the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$.

**(12.19) Exercise: Special linear group $\mathbb{SL}_2$.**
**a)** Show that any element of $\mathbb{SL}_2(\mathbb{K})$ is conjugate to precisely one of the following elements, where $0 \neq x \in \mathbb{K}$:

$$\pm \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

**b)** Determine the Jordan decomposition of the elements of $\mathbb{SL}_2(\mathbb{K})$. Is $\mathbb{SL}_2(\mathbb{K})_s$ a subgroup? Is $\mathbb{SL}_2(\mathbb{K})_u$ a subgroup?
**c)** Determine the centralisers of the elements of $\mathbb{SL}_2(\mathbb{K})$. Which are connected?

**Proof.** See [9, Exc.2.7.12]. $\sharp$

**(12.20) Exercise: Semisimple elements.**
Give examples showing that the set $\mathbb{G}_s$ of semisimple elements of an algebraic group $\mathbb{G}$ is neither necessarily closed nor necessarily open.

**(12.21) Exercise: One-dimensional groups.**
Let $\mathbb{G}$ be a connected algebraic group such that $\dim(\mathbb{G}) = 1$.
**a)** Show that $\mathbb{G}$ is commutative.
**b)** Show that if $\mathbb{G}$ is not unipotent then $\mathbb{G} \cong \mathbb{G}_m$ as algebraic groups.

**Hint for (a).** For $g \in \mathbb{G}$ consider the morphism $\varphi_g \colon \mathbb{G} \to \mathbb{G} \colon x \mapsto g^x = x^{-1}gx$.

**Proof.** See [11, La.2.6.2]. $\sharp$

**(12.22) Exercise: Orbits.**
Let $\mathbb{G}$ be an algebraic group acting morphically on $V$ and let $x \in V$. Show that $x(\mathbb{G}^\circ) \subseteq x\mathbb{G}$ is open and closed.

**Proof.** See [9, Exc.2.7.10]. $\sharp$

**(12.23) Exercise: Orbit closure relation.**
**a)** Determine the orbits of the natural action of $\mathbb{GL}_n(\mathbb{K})$ on $\mathbb{K}^n$, where $n \in \mathbb{N}$. How are they related with respect to the partial order $\preceq$?
**b)** How are the conjugacy classes of $\mathbb{SL}_2(\mathbb{K})$ related with respect to the partial order $\preceq$?

**Proof. a)** See [10, Exc.II.8.1]. $\sharp$

**(12.24) Exercise: Kostant-Rosenlicht Theorem.**
Let $\mathbb{G}$ be an algebraic group acting morphically on $V$ and let $x \in V$.
**a)** Let $W := \overline{x\mathbb{G}} \subseteq V$. Show that $\mathbb{K}[W]^{\mathbb{G}} = \mathbb{K} \cdot 1_{\mathbb{K}[W]}$.
**b)** Let $\mathbb{G}$ be unipotent. Show that $x\mathbb{G} \subseteq V$ is closed.

**Hint for (b).** For $U := x\mathbb{G} \subseteq \overline{x\mathbb{G}} =: W$ consider $\mathcal{I}(W \setminus U) \trianglelefteq \mathbb{K}[W]$, and use local finiteness and trigonalisability.

**Proof. b)** See [8, Prop.I.4.10] or [11, Exc.4.3.6.(3)]. $\sharp$

**(12.25) Exercise: Hasse diagrams.**
Let $n \in \mathbb{N}_0$. The **Hasse diagram** of the dominance partial order $\trianglelefteq$ on $\mathcal{P}_n$ is defined as the directed graph on the vertex set $\mathcal{P}_n$, having a directed edge $\lambda \to \mu$ if and only if $\lambda \vartriangleleft_{\max} \mu$.
**a)** Draw the Hasse diagrams for $n \leq 8$. For which $n \in \mathbb{N}_0$ is $\trianglelefteq$ a total order?
**b)** Show that the **lexicographical order** on $\mathcal{P}_n$ is a total order refining the dominance partial order.

**(12.26) Exercise: Dominance partial order.**
Let $\lambda = [\lambda_1, \ldots, \lambda_n] \vdash n$. Show that for $\mu \vdash n$ we have $\lambda \vartriangleleft_{\max} \mu$ if and only if $\mu = [\lambda_1, \ldots, \lambda_{r-1}, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{s-1}, \lambda_s - 1, \lambda_{s+1}, \ldots, \lambda_n]$, for some $1 \leq r < s \leq n$ such that $\lambda_{r-1} > \lambda_r$ and $\lambda_s > \lambda_{s+1}$, and such that either $s = r + 1$ or $s > r + 1$ and $\lambda_r = \lambda_s$.

**Proof.** See [13, Thm.1.4.10]. ♯

**(12.27) Exercise: Centralisers of unipotent elements.**
Let $\mathbb{G} := \mathbb{SL}_n$ for some $n \in \mathbb{N}$, and let $\lambda = [\lambda_1, \ldots, \lambda_l] \vdash n$, where $\lambda_l > 0$. For $B \in C_\lambda$ show that $\dim(C_{\mathbb{G}}(B)) = n - 1 + 2 \cdot \sum_{i=1}^{l}(i-1)\lambda_i$.

**Proof.** See [9, Prop.2.6.1]. ♯

**(12.28) Exercise: Restricted Lie algebras.**
Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{K}) = p > 0$, let $A$ be a Lie $\mathbb{K}$-algebra, and for all $x \in A$ let $\mathrm{ad}(x)\colon A \to A\colon y \mapsto [y, x]$ be the associated adjoint action. Then $A$ is called a **restricted** Lie algebra, if there is a **$p$-power operation** $[p]\colon A \to A\colon x \mapsto x^{[p]}$ having the following properties: We have $\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^p$ and $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ as well as $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{1}{i} \cdot \alpha_i(x, y)$, for all $x, y \in A$ and $\lambda \in \mathbb{K}$, where $\alpha_i(x, y) \in A$ is defined by the $\mathbb{K}$-linear expansion $\mathrm{ad}(\lambda x + y)^{p-1}(x) = \sum_{i=1}^{p-1} \lambda^i \alpha_i(x, y)$.

**a)** If $A$ is a restricted Lie algebra, determine $(x + y)^{[p]} \in A$ for commuting elements $x, y \in A$. Show that any non-commutative $\mathbb{K}$-algebra $A$ becomes a restricted Lie algebra with respect to $[p]\colon A \to A\colon x \mapsto x^p$.
**b)** Given any $\mathbb{K}$-algebra $A$, show that $\mathrm{Der}_{\mathbb{K}}(A, A)$ becomes a restricted Lie algebra with respect to $[p]\colon A \to A\colon \delta \mapsto \delta^p$. Show that the Lie algebra associated to an algebraic group is restricted. Determine the $p$-power operation on the Lie algebras associated to the groups $\mathbb{G}_a$ and $\mathbb{G}_m$.
**c)** Given a homomorphism $\varphi\colon \mathbb{G} \to \mathbb{H}$ of algebraic groups, show that $d_1(\varphi)\colon \mathfrak{g} \to \mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the associated Lie algebras, is a homomorphism of restricted Lie algebras, i. e. we have $d_1(\varphi)(x^{[p]}) = d_1(\varphi)(x)^{[p]}$ for all $x \in \mathfrak{g}$.

**Proof.** See [8, Ch.I.3.1] or [11, Ch.3.3] or [10, Exc.III.9.3]. ♯

**(12.29) Exercise: Right convolution.**
Let $\mathbb{G} \leq \mathbb{GL}_n$ be a closed subgroup with Lie algebra $\mathfrak{g} \leq \mathfrak{gl}_n$.
**a)** Let $\mathbb{H} \leq \mathbb{G}$ be a closed subgroup having vanishing ideal $\mathcal{I}(\mathbb{H}) \lhd \mathbb{K}[\mathbb{G}]$. Show that $L(\mathbb{H}) = \{\delta \in L(\mathbb{G}); \delta(\mathcal{I}(\mathbb{H})) \subseteq \mathcal{I}(\mathbb{H})\}$.
**b)** Let $\mathbb{K}[\mathcal{X}]_{\det_n}$ denote the affine coordinate algebra of $\mathbb{GL}_n$, let $\mathcal{I}(\mathbb{G}) \lhd \mathbb{K}[\mathcal{X}]_{\det_n}$ be the vanishing ideal of $\mathbb{G}$, and let $J := \mathcal{I}(\mathbb{G}) \cap \mathbb{K}[\mathcal{X}] \lhd \mathbb{K}[\mathcal{X}]$. Show that $\mathbb{G} = \{x \in \mathbb{GL}_n; \rho_x^*(J) \subseteq J\}$ and $\mathfrak{g} = \{x \in \mathfrak{gl}_n; \widehat{x}(J) \subseteq J\}$.

**Proof. a)** See [10, La.III.9.4] or [8, Prop.I.3.8].
**b)** See [10, Exc.III.9.1] or [8, Cor.I.3.8]. ♯

**(12.30) Exercise: Lie algebras of algebraic groups.**
Determine the Lie algebras associated to the algebraic groups in Exercise (12.4).

**Proof.** See [11, Exc.3.3.10.(2)]. ♯

**(12.31) Exercise: Skew-symmetric matrices.**
Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{K}) \neq 2$, and let $S := \{A \in \mathfrak{gl}_n; A = -A^{\mathrm{tr}}\}$ be the set of all **skew-symmetric** matrices. Show that $S$ is a Lie algebra, which is isomorphic to the Lie algebra $\mathfrak{o}_n$.

**Proof.** See [11, Exc.3.3.10.(4)]. ♯

**(12.32) Exercise: Differential of homomorphisms.**
Determine the differential $d_{E_2}(\varphi)$ of the homomorphism $\varphi \colon \mathbb{SL}_2 \to \mathbb{PSL}_2$ of algebraic groups in Exercise (12.5). Is $d_{E_2}(\varphi)$ an isomorphism?

**Proof.** See [11, Exc.3.3.10.(3)]. ♯

**(12.33) Exercise: Differential of multiplication.**
Let $\mathbb{G}$ be an algebraic group with multiplication map $\mu \colon \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, and let $x, y \in \mathbb{G}$. Identifying $T_{[x,y]}(\mathbb{G}\times\mathbb{G})$ with $T_x(\mathbb{G}) \oplus T_y(\mathbb{G})$, determine the differential $d_{[x,y]}(\mu) \colon T_x(\mathbb{G}) \oplus T_y(\mathbb{G}) \to T_{xy}(\mathbb{G})$.

**(12.34) Exercise: Differential of the commutator map.**
Let $\mathbb{G}$ be an algebraic group with Lie algebra $\mathfrak{g}$.
**a)** For $x \in \mathbb{G}$ let $\gamma_x \colon \mathbb{G} \to \mathbb{G} \colon y \mapsto y^{-1}x^{-1}yx$ be the **commutator map**. Show that $d_1(\gamma_x) = -\mathrm{Ad}(x) - \mathrm{id}_{\mathfrak{g}} \colon \mathfrak{g} \to \mathfrak{g}$.
**b)** For $y \in \mathbb{G}$ let $\kappa_y \colon \mathbb{G} \to \mathbb{G} \colon x \mapsto x^{-1}yx$ be the associated orbit map of the conjugation action. Derive a formula for the differential $d_1(\kappa_y) \colon \mathfrak{g} \to T_y(\mathbb{G})$.

**Proof.** See [10, Prop.III.10.1.(c)] or [8, I.3.16]. ♯

**(12.35) Exercise: Differential of right translation.**
Let $\mathbb{G}$ be an algebraic group with Lie algebra $\mathfrak{g}$, let $E \leq \mathbb{K}[\mathbb{G}]$ be a $\mathbb{K}$-subspace such that $n := \dim_{\mathbb{K}}(E) < \infty$, which is $\rho_g^*$-invariant for all $g \in \mathbb{G}$, and let $\rho \colon \mathbb{G} \to \mathbb{GL}(E)$ be the rational representation induced by right multiplication. Identifying $T_1(\mathbb{GL}(E)) \cong \mathfrak{gl}_n$ with $\mathrm{End}_{\mathbb{K}}(E)$, show that $d_1(\rho) \colon \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}(E) \colon \gamma \mapsto \widehat{\gamma}$, where $\widehat{\gamma}$ is the right convolution associated to $\gamma$.

**Proof.** See [10, III.10.2] or [8, Prop.I.3.11]. ♯

**(12.36) Exercise: Non-split orthogonal groups.**
Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{K}) = p > 0$, let $q := p^f$ for some $f \in \mathbb{N}$, and let $\Phi_q$ be the standard Frobenius endomorphism on $\mathbb{GL}_{2m}$. Moreover, let

$$T := \begin{bmatrix} E_{m-1} & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & E_{m-1} \end{bmatrix} \in \mathbb{K}^{(2m)\times(2m)}.$$

Show that $T \in \mathbb{O}_{2m} \leq \mathbb{GL}_{2m}$, and that $\Phi'_q\colon \mathbb{SO}_{2m} \to \mathbb{SO}_{2m}\colon A \mapsto \Phi_q(T^{-1}AT)$ is a Frobenius endomorphism on $\mathbb{SO}_{2m}$.

The fixed point set $\mathrm{SO}^-_{2m}(\mathbb{F}_q) = \mathbb{SO}_{2m}^{\Phi'_q}$ is called the associated **non-split special orthogonal group**.

**Hint.** Distinguish the cases $p = 2$ and $p > 2$.

**Proof.** See [9, Ex.4.1.10.(d)]. $\sharp$

**(12.37) Exercise: Lang map.**
Let $\mathbb{G}$ be a not necessarily connected algebraic group with Frobenius endomorphism $\Phi$, and let $z \in \mathbb{G}$. Describe the image of $\mathcal{L}_z\colon \mathbb{G} \to \mathbb{G}\colon x \mapsto x^{-1}z\Phi(x)$.

**Proof.** See [8, Thm.V.16.3]. $\sharp$

**(12.38) Exercise: Lang-Steinberg Theorem.**
Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{K}) = p > 0$, let $q := p^f$ for some $f \in \mathbb{N}$, let $\mathbb{G} \leq \mathbb{GL}_n$ be a $\Phi_q$-invariant connected closed subgroup, let $\Phi$ be a Frobenius endomorphism on $\mathbb{G}$ such that $\Phi^d = \Phi_q|_{\mathbb{G}}$ for some $d \in \mathbb{N}$, and let $\mathcal{L}\colon \mathbb{G} \to \mathbb{G}\colon x \mapsto x^{-1}\Phi(x)$ be the Lang map.
**a)** From $\mathbb{G}^\Phi := \{g \in \mathbb{G}; \Phi(g) = g\}$ being finite deduce that $\mathcal{L}$ is dominant.
**b)** Show that $\mathbb{K}[\mathbb{G}]$ is a finitely generated $\mathcal{L}^*(\mathbb{K}[\mathbb{G}])$-module, i. e. $\mathcal{L}$ is finite.
**c)** Deduce that $\mathcal{L}$ is surjective.

**Proof.** See [9, Thm.4.1.12]. $\sharp$

**(12.39) Exercise: Conjugacy classes.**
**a)** Let $\mathbb{G}$ be a connected algebraic group with Frobenius endomorphism $\Phi$, let $C \subseteq \mathbb{G}$ be a $\Phi$-invariant conjugacy class of $\mathbb{G}$, and let $g \in C$ such that $C_{\mathbb{G}}(g)$ is connected. Show that $C^\Phi \subseteq \mathbb{G}^\Phi$ is non-empty conjugacy class of $\mathbb{G}^\Phi$.
**b)** Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{K}) = p > 0$, let $q := p^f$ for some $f \in \mathbb{N}$, let $\Phi_q$ be the standard Frobenius endomorphism on $\mathbb{GL}_n$, and let $A \in \mathrm{GL}_n(\mathbb{F}_q)$. Show that $C_{\mathbb{GL}_n}(A)$ is connected. In the light of (a), which well-known fact from linear algebra is recovered?

**Hint for (b).** Consider $\{C \in \mathbb{K}^{n\times n}; AC = CA\}$.

**Proof. a)** See [11, Exc.3.3.17.(1)]. **b)** See [9, Ex.4.3.6]. $\sharp$

**(12.40) Exercise: Component groups.**
Let $\mathbb{G}$ be a connected algebraic group with Frobenius endomorphism $\Phi$. Let $\Omega$ be a transitive $\mathbb{G}$-set, and let $\varphi\colon \Omega \to \Omega$ be $\Phi$-equivariant. Moreover, let $\omega \in \Omega$ and let $\mathbb{H} := \mathrm{Stab}_{\mathbb{G}}(\omega) \leq \mathbb{G}$ be closed.
**a)** Show that $\Phi$ induces a group homomorphism on the finite **component group** $C(\omega) := \mathbb{H}/\mathbb{H}^\circ$.

**b)** Show that the **$\Phi$-conjugacy classes** of $C(\omega)$, i. e. the orbits of the $\Phi$-conjugation action $C(\omega) \times C(\omega) \to C(\omega)\colon [g,h] \mapsto h^{-1}g\Phi(h)$, are in natural bijection with the $\mathbb{G}^\Phi$-orbits in $\Omega^\varphi$.

**Proof.** See [9, 4.3.4, Thm.4.3.5]. $\sharp$

**(12.41) Exercise: Conjugacy classes of $\mathbb{SL}_2$.**
Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{K}) = p > 0$, let $q := p^f$ for some $f \in \mathbb{N}$, let $\Phi_q$ be the standard Frobenius endomorphism on $\mathbb{SL}_2$.

Determine the $\Phi$-invariant conjugacy classes of $\mathbb{SL}_2$, see also Exercise (12.19). How do their $\mathbb{F}_q$-rational points split into conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_q)$, and into $\mathrm{GL}_2(\mathbb{F}_q)$-orbits? Determine the associated component groups and their $\Phi$-conjugacy classes, see Exercise (12.40).

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