Abstract

In this paper the 5-modular decomposition matrix of the principal block of the sporadic simple Conway group Co_3 is determined. The results are obtained by a combination of character theoretic methods and explicit module constructions and analyses, especially condensation techniques, with the assistance of the computer algebra systems GAP, MOC, and MeatAxe.

The 5-modular decomposition matrix of the sporadic simple Conway group Co_3

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1 Introduction and results

1.1 Introduction

In this paper we determine the 5-modular decomposition matrix for the principal block of the sporadic simple Conway group Co_3. The decomposition matrices for the other 5-block of positive defect and for all the 7-, 11- and 23-blocks have been computed in [7]; those for the 2-blocks have been found in [18] up to a few conjectural entries, which have been verified in [16]; and those for the 3-blocks have been computed in [8], although not published yet. Thus, this paper completes the determination of all the decomposition matrices for Co_3.

This paper is part of the larger project to determine the decomposition matrices, or equivalently the Brauer character tables, of the almost simple groups contained in [2]. Up to the sporadic simple McLaughlin group and its derivatives, these have been published in [9]. The results in this paper, see Section 1.2, as well as the other results mentioned above, are intended to be included in a sequel to [9]. They are also included, like all the Brauer character tables in [9] and the many other known decomposition matrices not contained there, in the character table library of the GAP system [17]. The work described here was begun by the author in [13], but not successfully completed then, since at that time the necessary computational devices were not yet available.

In recent years, a whole series of methods has been developed for the practical computation of decomposition numbers. Those which have been applied in this paper will be commented on in Section 1.3. Their effectiveness is best shown by their application to genuine research problems, which is one of the aims of this paper. We would like to remark that, additionally, these methods have been proven very useful for many different other problems in computational representation theory. E. g., on the character theory side, the notion of basic sets of characters, which was developed as a theoretical concept for the MOC system [6], has become important in the modular representation theory of finite groups of Lie type as well. Furthermore, one of the most powerful techniques for the explicit construction and analysis of modules, condensation, which originally was invented in [19] as a workhorse for the determination of decomposition numbers and was only applicable to permutation modules, can now be applied to a much broader range of modules and is central for many other uses, e. g., for the computation of submodule lattices, to establish explicit versions of Morita contexts, the computation of endomorphism rings and the computation of projective resolutions and Gabriel quivers.

1.2 Results

The sporadic simple Conway group Co_3 has two 5-blocks of positive defect and 11 irreducible ordinary characters of defect 0. The non-principal block of positive defect has defect 1, for completeness we state that it contains 5 irreducible ordinary characters and 4 irreducible Brauer characters; its decomposition matrix is already given in [7, page 203]. The principal block has defect 3, it contains 26 irreducible ordinary characters and 18 irreducible Brauer characters, whose degrees are

1, 23, 230, 253, 896, 896^*, $1771a$, $1771b$, 3520, 5290, 20608, 20608^*, 22309, 26335, 52624, 25255, 55705, 53383.

Its decomposition matrix is given in Table 1; the degrees of the irreducible ordinary characters indexing the rows are included in Table 6, their numbering coincides with the one given in [2, 9] and also with that given in GAP. In particular, we note that the decomposition matrix of the principal block is of wedge shape, i. e. , after a suitable reordering of the rows and columns it is lower unitriangular. The rest of the present paper is devoted to outline the strategy and the methods how this decomposition matrix can be determined.

1.3 Methods

The results of this paper are obtained by a combined, interdependent use of character theoretic methods and explicit module constructions and analyses. We emphasize that it is the *interdependent* use of these methods which is successful. The general strategy is to first pursue character theoretic methods to obtain a certain approximation to the decomposition matrix, which then provides hints which modules to construct. Their analysis in turn leads to improvements to the approximation of the decomposition matrix, and so on.

The character tables used have been taken from the character table library of the GAP system [17], which also provides many useful tools to deal with ordinary character tables, or more generally with class functions with values in cyclotomic fields, and subgroup fusions. The computations involving Brauer characters, in particular decomposition into basic sets, filtering out non-positive decompositions and finding improved basic sets, have been performed using the MOC system [6]. The results thus obtained are written down in

Table 1: The decomposition matrix for the principal 5-block of Co_3

χ																		
1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
7	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
8	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
9	.	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
10	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
11	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
13	1	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.
14	1	.	.	.	1	1	1	.	1	.	.	.	.	.	.	.	.	.
15	2	.	.	.	1	1	.	1	.	1	.	.	.	.	.	.	.	.
18	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
19	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
21	.	.	.	1	.	.	.	.	1	.	.	.	1	.	.	.	.	.
25	.	.	.	1	.	.	.	.	.	1	.	.	.	1	.	.	.	.
27	2	.	.	.	1	1	1	1	.	.	.	.	.	.	1	.	.	.
30	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.
32	.	1	1	1	.	.	.	.	.	.	1	1	.	1	.	1	.	.
33	1	1	.	1	1	1	1	.	1	.	1	1	.	.	.	1	1	.
34	2	1	1	1	1	1	.	1	.	1	1	1	.	1	1	.	.	.
36	1	1	.	1	1	1	.	.	1	1	1	1	.	1	1	.	.	1
38	1	.	.	1	1	1	.	.	1	.	1	1	1	1	1	1	.	1
40	2	1	.	2	2	2	.	.	2	.	2	2	1	.	1	1	1	.
42	2	1	.	1	2	2	1	.	2	.	2	2	.	1	1	1	1	.

terms of certain basic sets of projective and Brauer characters. The notion of basic sets is one of the central concepts of MOC, for more details see [6].

Explicit module constructions are performed using the MeatAxe system [14, 15]. For the structural analysis of explicitly given modules, we make use of the notions and techniques developed in [11], which we also assume the reader to be familiar with. One of the main techniques used for the construction and analysis of modules is the condensation method, which in recent years has become one of the most valuable tools in computational representation theory and has been implemented by different authors for different kinds of modules. We will use the implementations [15] for permutation modules and [12] for tensor product modules, where the former is part of the MeatAxe and the latter is available as the TensorCondense package of the MeatAxe. We continue with a few comments on the condensation technique as it is applied here.

1.4 Fixed point condensation

As many interesting modules are too large to be constructed directly, one tries to 'condense' these modules to smaller ones which still reflect enough of the original structure but can be handled on a computer. The following functorial description, following [5], shows which structural information is retained and which is lost under condensation.

Let k be a field, A be a finite-dimensional k-algebra, $e \in A$ be an idempotent and mod–A be the category of finitely generated unital right A-modules. Then the condensation functor with respect to e is given as $C_e := ? \otimes_A Ae : \text{mod–}A \longrightarrow \text{mod–}eAe$. Under this functor, $M \in \text{mod–}A$ is mapped to $M \otimes_A Ae \in \text{mod–}eAe$. The latter can be identified with the subset $Me \subseteq M$. Using this identification, a homomorphism $\alpha \in \mathrm{Hom}_A(M,N)$ is mapped to $\alpha|_{Me} \in \mathrm{Hom}_A(Me,Ne)$. Furthermore, we have $C_e \cong \mathrm{Hom}_A(eA,?)$ as functors, hence C_e is an exact functor. Now let $e = \sum_S \sum_{j=1}^{n_S} e_S^{(j)} \in A$ be an orthogonal decomposition into primitive idempotents, where S runs through the isomorphism types of simple A-modules, for each summand we have $e_S^{(j)}A/\mathrm{Rad}(e_S^{(j)}A) \cong S$ and $n_S \in \mathbf{N}_0$. By $n_S = 0$ we indicate that this type of idempotent does not occur in the above decomposition. Then we have $\dim_k(Se) = n_S \cdot \dim_k(\mathrm{End}_A(S))$.

In the present paper, this is applied in the following special case. Let G be a finite group and $A := k[G]$ be its group algebra over k. Let $K \leq G$ be a subgroup such that $|K| \neq 0 \in k$; in the sequel K is called the condensation subgroup. Then $e = e_K := \frac{1}{|K|} \cdot \sum_{g \in K} g \in k[K] \subseteq k[G]$ is the centrally primitive idempotent of $k[K]$ belonging to the trivial K-module. We have $e \cdot k[G] \cong (1_K)^G$, where $(1_K)^G$ is the permutation representation of G on the cosets of K. Hence by the adjointness of tensor product and homomorphism functors, see [3, Theorem 2.19.], we have $Me \cong \mathrm{Hom}_G((1_K)^G, M) \cong \mathrm{Hom}_K(1_K, M_K) \cong \mathrm{Fix}_M(K)$ as vector spaces, where M is a $k[G]$-module, M_K denotes its restriction to $k[K]$ and $\mathrm{Fix}_M(K) \subseteq M$ consists of the elements of M being fixed by K, which is the name-giving property. As mod–$k[K]$ is a semisimple category the dimension of the condensed module of M can be computed as the ordinary character theoretic scalar product of the trivial K-character and the restriction to K of the Brauer character of M.

1.5 Condensation algebras

In applying the condensation technique we are faced with the following problem. If A is generated as a k-algebra by the subset $\mathcal{A} \subseteq A$, we let $\mathcal{C} := \langle eae; a \in \mathcal{A} \rangle_{k-\text{algebra}} \leq eAe$. The algebra $\mathcal{C}$ is called the condensation algebra, whereas the algebra eAe is called the Hecke algebra; for historical reasons the latter name has become standard. But now the condensation subalgebra $\mathcal{C}$ does not necessarily equal eAe, it may be a proper subalgebra of eAe. As only the action of condensed elements eae on condensed modules can be computed explicitly, we are faced with the task to analyse condensed modules with respect to their structure as $\mathcal{C}$-modules and then to draw conclusions about their eAe-module structure from this analysis.

For a $k[G]$-module M, the following theorem provides one tool to ensure the existence of certain submodules of Me as an $ek[G]e$-module. For the necessary notions and a proof, see [10, Definition I.17.1.,Theorem I.17.3.].

1.6 Theorem. (Zassenhaus and others)

Let (K, R, k) be a modular system for G and let M be an R-free $R[G]$-module of finite R-rank with ordinary character χ, such that $\chi = \chi' + \chi''$ as ordinary characters. Then there exists an R-pure $R[G]$-submodule $N \leq M$ with character χ'.

2 Basic sets of characters

2.1 Subgroup fusions

In the sequel, we will deal with several subgroups of Co_3. We have to find a compatible set of subgroup fusions for the character tables of these subgroups into the character table of Co_3. This is done using the character table library in GAP, its functions dealing with subgroup fusions and a consideration of character table automorphisms. The ordering of conjugacy classes of elements of the occuring groups and of their irreducible ordinary and Brauer characters we use here coincides with the one given in [2, 9], as far as the tables are contained there, and also with that given in GAP.

Table 2: $\mathcal{PS}^1$

χ	Ψ^1																	
1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
7	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
8	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
9	.	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
10	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
13	1	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.
18	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.
19	.	.	.	.	.	.	.	.	1	.	1	1	.	.	.	.	.	.
21	.	1	1	1	.	.	.	.	1	.	1	1	1	.	.	.	.	.
25	.	.	.	1	.	.	.	.	.	1	.	1	.	1	.	.	.	.
27	2	.	.	.	2	2	1	1	1	.	.	.	.	.	1	.	.	.
32	.	1	1	1	.	.	.	.	2	.	2	2	.	1	.	1	.	.
30	.	.	.	.	.	.	.	.	2	.	1	.	.	.	.	1	1	.
36	2	2	.	2	3	3	.	.	5	1	3	5	.	2	1	.	.	1

Table 3: Origin of $\mathcal{PS}^1$

	origin
Ψ_1^1	$(\Phi_{1a})_{3^5:(2\times M_{11})}$
Ψ_2^1	$(5d)_{3_+^{1+4}:4S_6}$
Ψ_3^1	$(10c)_{3^5:(2\times M_{11})}$
Ψ_4^1	$\Phi_{4025}\otimes 23$
Ψ_5^1	$(\Phi_{16a})_{3^5:(2\times M_{11})}$
Ψ_6^1	$(\Phi_{16a^*})_{3^5:(2\times M_{11})}$
Ψ_7^1	$31625b\otimes 23$
Ψ_8^1	$(1750a)_{McL:2}$
Ψ_9^1	$(\Phi_{8f})_{3_+^{1+4}:4S_6}$

	origin
Ψ_{10}^1	$(\Phi_{230a})_{McL:2}$
Ψ_{11}^1	$(\Phi_{6490})_{McL:2}$
Ψ_{12}^1	$(\Phi_{16b^*})_{3^5:(2\times M_{11})}$
Ψ_{13}^1	$(4500b)_{McL:2}$
Ψ_{14}^1	$9625\otimes 275$
Ψ_{15}^1	$9625\otimes 253b$
Ψ_{16}^1	$(10b)_{3^5:(2\times M_{11})}$
Ψ_{17}^1	$(10d)_{3^5:(2\times M_{11})}$
Ψ_{18}^1	$(4500a)_{McL:2}$

2.2 Finding $\mathcal{PS}^1$

We first obtain a lot of projective characters for Co_3 by inducing up the projective indecomposable characters of the largest six maximal subgroups of Co_3. These are listed in [2], and their Brauer character tables are given in [9] or are easily computed using MOC. Additionally, tensoring the defect zero irreducible ordinary characters and the projective indecomposable characters of the block of defect 1 of Co_3 with all irreducible ordinary characters also yields projective characters. These are fed into the MOC system, which finds the first basis $\mathcal{PS}^1$ of projective characters depicted in Table 2. It can be checked, using either MOC or GAP, that the irreducible ordinary characters indexing the rows of Table 2 are a basic set $\mathcal{BS}^\infty$ of Brauer characters; hence it is enough to print only the corresponding rows of the decomposition matrix. The origin of the projective characters in $\mathcal{PS}^1$ is documented in Table 3, e. g., this means that Ψ_1^1 is obtained by inducing up the projective indecomposable character corresponding to the irreducible Brauer character $1a$, i. e., the trivial character, from $3^5:(2\times M_{11})$ and subsequently restricting the induced character to its principal block component. Next, Ψ_2^1 is obtained similarly from the defect zero irreducible ordinary character $5d$ of $3_+^{1+4}:4S_6$, and so on. Note that Φ_{4025} is the projective indecomposable character corresponding to the irreducible Brauer character 4025 belonging to the block of defect 1 of Co_3.

2.3 Wedge shape

It now already follows that the decomposition matrix is of wedge shape, which was not *a priori* clear. This result is achieved by the possibility to sieve easily using MOC through a huge set of projective characters to filter out a suitable basis of projective characters. Knowing wedge shape of the decomposition matrix eases the arguments to follow considerably, as the basis of projective characters can only be changed in subsequent steps by possibly subtracting multiples of columns 'from the right'. In particular, it follows that Ψ_{13}^1, Ψ_{17}^1, Ψ_{18}^1 are projective indecomposable characters.

2.4 Refining projective characters

We now set out to improve the basis of projective characters using the many more projective characters we have constructed above, where it is the task of MOC to filter out the helpful ones. Looking at Ψ_{11}^1 and Ψ_{12}^1, we note that Ψ_{11}^1 contains the sum of a pair of complex conjugate projective indecomposable characters. We try to isolate them as follows. First we find that the projective character originating from $(\Phi_{55})_{HS}$ equals $\mathcal{PS}^1\cdot[0,0,1,0,0,0,0,1,0,4,1,0,-2,2,5,0,-1,1]^{tr}$. Hence $\Psi_{11}^1-\Psi_{13}^1-\Psi_{17}^1$ is a projective character and still contains the sum of the pair of complex conjugate projective characters searched for, which leads to the projective characters denoted by Ψ_{11}^2 and Ψ_{12}^2 in Table 4. Further character theoretic analysis then shows that these indeed are projective indecomposable characters.

2.5 Using defect groups

Let G be a finite group, k be a field, and B be a block of $k[G]$ with defect group $D\le G$. Let $H\le G$ such that $H\cap D^g=\{1\}$ for all $g\in G$. Then by the theory of defect groups, see [1, Section IV.13], it follows that any finitely generated B-module becomes projective under restriction to $k[H]$.

This is applied as follows. Co_3 is a subgroup of Co_1, and does not hit the $5A$ conjugacy class of Co_1, as the subgroup fusion programs of GAP show. Now the irreducible ordinary character $\chi_{47}=25900875$ of Co_1 belongs to a block of defect 1, see [7, page 304ff.], whose defect group is generated by an element in the rational $5A$ conjugacy class, as is easily seen using Brauer's Second Main Theorem on Blocks, see [4, Theorem IV.6.1], and a consideration of ordinary character values. Hence 25900875_{Co_3} is a projective character. Furthermore, we find the following ordinary scalar products $(\chi_{18},25900875_{Co_3})=(20608,25900875_{Co_3})=0$ and $(\chi_{10},25900875_{Co_3})=(3520,25900875_{Co_3})=-1$. From this we conclude that $\Psi_9^2:=\Psi_9^1-\Psi_{11}^2-\Psi_{12}^2$ is a projective character.

2.6 Finding $\mathcal{PS}^2$

Continuing this kind of analysis, we finally arrive at the basis $\mathcal{PS}^2$ shown in Table 4. Here an underlined entry means that it cannot be changed any more by further subtraction 'from the right'. An underlined irreducible ordinary character means that its decomposition into irreducible Brauer characters is completely known, and an underlined column number indicates that its entries form a projective indecomposable character. So far, we have determined 13 irreducible Brauer characters, they are $\varphi_1=1$, $\varphi_2=23$, $\varphi_3=230$, $\varphi_4=253$, $\varphi_5=896$, $\varphi_6=896^*$, $\varphi_7=1771a$, $\varphi_8=1771b$, $\varphi_9=3520$, $\varphi_{10}=5290$, $\varphi_{11}=20608$, $\varphi_{12}=20608^*$, $\varphi_{14}=26335$.

Table 4: $\mathcal{PS}^2$

χ	Ψ^2																	
$\underline{1}$	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{2}$	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{4}$	.	$\underline{1}$	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{3}$	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{6}$	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{7}$	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{8}$	.	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.
$\underline{9}$	.	$\underline{1}$	$\underline{1}$	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.
$\underline{10}$	.	.	.	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.
$\underline{13}$	$\underline{1}$	.	.	$\underline{1}$	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.
$\underline{18}$	.	.	.	.	.	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.
$\underline{19}$	.	.	.	.	.	.	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.
21	.	.	.	1	.	.	.	.	1	.	.	.	$\underline{1}$	.	.	.	.	.
$\underline{25}$	.	.	.	$\underline{1}$	.	.	.	.	.	$\underline{1}$	.	.	.	$\underline{1}$	.	.	.	.
27	2	.	.	.	2	2	$\underline{1}$	$\underline{1}$	1	.	.	.	.	.	$\underline{1}$	.	.	.
32	.	$\underline{1}$	$\underline{1}$	1	.	.	.	.	.	.	$\underline{1}$	$\underline{1}$	.	1	.	$\underline{1}$	.	.
30	.	.	.	.	.	.	.	.	2	.	.	.	.	.	.	1	$\underline{1}$	.
36	2	$\underline{1}$	.	2	3	3	.	.	3	$\underline{1}$	$\underline{1}$	$\underline{1}$	.	1	1	.	.	$\underline{1}$

Table 5: A few permutation characters of Co_3

H	$[Co_3 : H]$	χ_i in $1_H \uparrow^{Co_3}$
$McL:2$	276	$1, 5$
McL	552	$1, 2, 4, 5$
HS	11178	$1, 2, 5, 9, 15$
$U_4(3):2^2$	37950	$1, 2 \cdot 5, 13, 15, 20$
M_{23}	48600	$1, 2, 4, 5, 9, 13, 15, 22$
$2 \cdot S_6(2)$	170775	$1, 5, 14, 15, 20, 27, 29$
$U_3(5):3$	1311552	$1, 3, 4, 5, 2 \cdot 9, 15, 16, 17, 2 \cdot 20, 21, 22,$
		$24, 25, 27, 28, 29, 31, 32, 2 \cdot 34, 38, 39$
U	1416800	$1, 4, 5, 14, 15, 2 \cdot 20, 21, 22, 27,$
		$29, 30, 31, 32, 35, 38, 39, 40$

Table 6: Dimensions of the condensed irreducible ordinary characters of Co_3 with respect to $SL_2(7)$

i	χ_i	d_i	i	χ_i	d_i	i	χ_i	d_i
1	1	1	15	8855	35	29	73600	222
2	23	2	16	9625	28	30	80960	248
3	253	2	17	9625	28	31	91125	272
4	253	4	18	20608	58	32	93312	278
5	275	5	19	20608	58	33	129536	392
6	896	2	20	23000	71	34	129536	394
7	896	2	21	26082	78	35	177100	524
8	1771	7	22	31625	101	36	184437	550
9	2024	12	23	31625	99	37	221375	656
10	3520	12	24	31625	103	38	226688	668
11	3520	12	25	31878	102	39	246400	724
12	4025	17	26	40250	116	40	249480	742
13	5544	24	27	57960	179	41	253000	754
14	7084	24	28	63250	186	42	255024	762

3 Applying condensation

3.1 Matrix generators

We are now going to construct and analyse several matrix and permutation representations of Co_3 explicitly. We start our constructions by accessing the defining integral representation of $2.Co_1$ in its action as the full automorphism group of the rank 24 Leech lattice. Explicit matrices for a set of standard generators, see [21], are available via the library [20]. We then use the information on generators for the maximal subgroups of Co_1 also given there to obtain matrix generators $M_{24,\mathbf{Z}} \in GL_{24}(\mathbf{Z})$ for Co_3. These then can be reduced modulo any prime to yield modular representations.

3.2 Permutation characters

As we are also going to use several permutation representations of Co_3, we collect the degrees and the decompositions into irreducible ordinary characters of a few of them in Table 5, where we give the ordinal numbers and multiplicities of the occuring irreducible ordinary characters. The subgroup U mentioned in Table 5 is defined in Section 3.8. It is a standard task of the MeatAxe to construct these permutation representations, except possibly the last one, whose construction is described in Section 3.8.

E. g., the permutation representation P_{170775} on the cosets of the subgroup $2 \cdot S_6(2)$ of index 170775 is found as follows. We reduce $M_{24,\mathbf{Z}}$ modulo the prime 3, find its 3-modular constituent $M_{22,3}$ of degree 22, form the symmetric tensor square $M_{22,3}^{2+}$, and find the constituents $M_{126,3}$ and $M_{126,3}^*$ of $M_{22,3}^{2+}$, where '*' means taking the contragradient dual. These are now restricted to a maximal subgroup $2 \cdot S_6(2)$, where again generators can be found in [20]. Exactly one of the restrictions has a trivial socle constituent. Now the vector permutation technique yields the desired permutation representation.

3.3 The condensation subgroup

As a condensation subgroup we choose a subgroup $K := SL_2(7) < McL:2 < Co_3$. It turns out, using GAP, that the subgroup fusion from $SL_2(7)$ to Co_3 is uniquely determined by this condition. Using the remarks in Section 1.4, we then compute the dimensions of the condensed modules of the irreducible ordinary representations of Co_3, they are given in Table 6. By expressing Brauer characters as $\mathbf{Z}$-linear combinations of the irreducible ordinary characters, it is easy to find the dimensions of the condensed modules of the irreducible modular representations known so far. We condense sufficiently many elements of Co_3 and end up with condensed modules, whose structural analysis with respect to the corresponding condensation algebra is indicated in the sections to come. In all of the analysis of submodule structures of condensed modules to follow, we make use of the MeatAxe and the techniques and ideas described in [11] without further notice.

3.4 Analysis of P_{170775}

We are now going to analyse the condensed module M of P_{170775} to find the decomposition of the irreducible ordinary character $\chi_{27} = 57960$ into irreducible Brauer characters. The condensed module has dimension 537 and, by the MeatAxe, is found to have the following constituents with multiplicities:

$$6 \cdot 1a, 3 \cdot 2a, 3 \cdot 2b, 2 \cdot 5a, 2 \cdot 7a, 2 \cdot 8a, 12a, 21a, 71a, 158a, 217a.$$

We successively use the smaller permutation representations mentioned in Section 3.2, whose modular constituents are already known, to establish a correspondence between as many of the constituents found by the MeatAxe and the condensed modules of the known irreducible modular representations as possible. Furthermore, using the decomposition of $\chi_{15} = 8855$ into the basic set $\mathcal{BS}^1$ given in Section 2.2, we find that we already know all of its modular constituents, it decomposes into irreducible Brauer characters as $8855 = 2 \cdot 1 + 896 + 896^* + 1771b + 5290$. Putting this information together, we find that a 5-modular reduction of 8855 condenses to a module with constituents $2 \cdot 1a$, $2a$, $2b$, $8a$, $21a$. By the Zassenhaus Theorem 1.6, there is a submodule U_{35} of M of dimension 35 which is the condensed module of a 5-modular reduction of 8855. Hence U_{35} contains the uniquely determined $21a$-local submodule of M. It turns out that this local submodule already has dimension 35 and is uniserial of composition length 6. Hence it equals U_{35}, and all of its submodules are invariant even under the possibly larger Hecke algebra. Analogously, we find that the $12a$-local submodule U_{24} of M has dimension 24 and is the condensed module of a 5-modular reduction of $\chi_{14} = 7084$, whose modular decomposition is also known, namely $7084 = 1 + 896 + 896^* + 1771a + 3520$. We find that U_{24} is uniserial with constituents $1a$, $2a$, $2b$, $7a$, $12a$, and hence all of its submodules are invariant under the Hecke algebra.

Applying this chain of reasoning once again, there is a submodule U_{179} of M of dimension 179 which is the condensed module of a 5-modular reduction of 57960. It again turns out to be a local submodule, this time the $158a$-local one. Now we let $V_1 := U_{179} \cap U_{35}$ and $V_2 := U_{179} \cap U_{24}$. These have dimensions 11 and 10, respectively, and hence we conclude that U_{179} contains constituents $1a$, $2a$, $2b$, $7a$, $8a$, which are extendible to the Hecke algebra. Furthermore, $V := V_1 + V_2$ is also invariant under the Hecke algebra and U_{179}/V is uniserial with ascending composition series $1a$, $158a$. Hence it remains to show that the nontrivial submodule of U_{179}/V is invariant under the Hecke algebra. But if it is not, then there will be an irreducible module of dimension 159 for the Hecke algebra which splits uniserially under restriction to the condensation algebra. But the Zassenhaus Theorem shows that there is a submodule U_{60} of M of dimension 60 which is the condensed module of a 5-modular reduction of an ordinary module with character $1 + 7084 + 8855$, and it turns out that M/U_{60} has $158a$ as one of its socle constituents. If there now is an irreducible module of dimension 159 for the Hecke algebra, it hence will not split uniserially with head constituent $158a$ under restriction to the condensation algebra, a contradiction. Thus, we have proved that $\varphi_{15} = 52624$ is an irreducible Brauer character and that 57960 decomposes into irreducible Brauer characters as $57960 = 2 \cdot 1 + 896 + 896^* + 1771a + 1771b + 52624$.

3.5 A second condensation subgroup

The next task is an analysis of the permutation representation $P_{1311552}$, whose permutation character is given in Table 5. To make this feasible, we have to choose another, larger condensation subgroup, so as to yield a condensed module of a suitably small dimension. We let $K' := 3^5 : 11 < 3^5 : (2 \times M_{11}) < Co_3$. Again, this uniquely determines the subgroup fusion of $3^5 : 11$ into Co_3. The dimensions of condensed modules are then easily computed using GAP. Without reproducing this data here, we only mention that $P_{1311552}$ condenses to a module of dimension 544. We note that, while the first condensation subgroup was chosen not to condense the up to then known irreducible Brauer characters to zero, we now drop this requirement. In fact, it will turn out that $\varphi_3 = 230$ and $\varphi_{10} = 5290$ condense to zero;

Table 7: $\mathcal{PS}^3$

χ	Ψ^3																	
$\underline{1}$	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{2}$	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{4}$	.	$\underline{1}$	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{3}$	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{6}$	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{7}$	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.	.
$\underline{8}$	.	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.	.
$\underline{9}$	.	$\underline{1}$	$\underline{1}$	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.	.
$\underline{10}$	.	.	.	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.	.
$\underline{13}$	$\underline{1}$	.	.	$\underline{1}$	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.	.
$\underline{18}$	.	.	.	.	.	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.	.
$\underline{19}$	.	.	.	.	.	.	.	.	.	.	.	$\underline{1}$	.	.	.	.	.	.
21	.	.	.	$\underline{1}$	.	.	.	.	1	.	.	.	$\underline{1}$	.	.	.	.	.
$\underline{25}$	.	.	.	$\underline{1}$	.	.	.	.	.	$\underline{1}$	.	.	.	$\underline{1}$	.	.	.	.
$\underline{27}$	$\underline{2}$	.	.	.	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	.	.	.	.	.	.	$\underline{1}$	.	.	.
$\underline{32}$	.	$\underline{1}$	$\underline{1}$	$\underline{1}$	.	.	.	.	.	.	$\underline{1}$	$\underline{1}$	.	$\underline{1}$	.	$\underline{1}$	.	.
30	.	.	.	.	.	.	.	.	2	.	.	.	.	.	.	$\underline{1}$	$\underline{1}$	.
36	1	$\underline{1}$	.	$\underline{1}$	1	1	.	.	2	$\underline{1}$	$\underline{1}$	$\underline{1}$	.	$\underline{1}$	$\underline{1}$	.	.	$\underline{1}$

but since we already know the corresponding projective indecomposable characters, the information we obtain using condensation with $3^5 : 11$ is sufficient for our analysis. Using similar techniques as in Section 3.4, we look at submodules coming from the ordinary constituents $\chi_{21} = 26082$, $\chi_{32} = 93312$, $\chi_{38} = 226688$ of $P_{1311552}$. Besides several improvements on different entries in the decomposition matrix, this in particular allows us to determine the decomposition of 93312 into irreducible Brauer characters, see Table 7, and hence to show that we have an irreducible Brauer character $\varphi_{16} = 25255$.

3.6 Finding $\mathcal{PS}^3$

All the results found by condensation analysis in Sections 3.4, 3.5 are now fed back into MOC, which finds a new basis $\mathcal{PS}^3$, see Table 7. Furthermore, MOC finds $\mathcal{BS}^3 = \{\varphi_1, \ldots, \varphi_{12}, \theta_{13}, \varphi_{14}, \varphi_{15}, \varphi_{16}, \theta_{17}, \theta_{18}\}$ as a basic set of Brauer characters, where $\theta_{13} = \chi_{21} - \varphi_4 = 26082 - 253$, $\theta_{17} = \chi_{30} = 80960$, and $\theta_{18} = \chi_{36} - (\varphi_2 + \varphi_4 + \varphi_{10} + \varphi_{11} + \varphi_{12} + \varphi_{14} + \varphi_{15}) = 184437 - (23 + 253 + 5290 + 20608 + 20608^* + 26335 + 52624)$. Note that at first Ψ^3_{16} is not yet known to be projective indecomposable. But next we check the decomposability of tensor products of characters found so far, and MOC filters out that the principal block component of $\varphi_2 \otimes \varphi_{16} = 23 \otimes 25255$ equals $[2, 2, 1, 2, 2, 2, 0, 0, 3, 0, 3, 3, 0, 0, 1, -1, 2, 0] \cdot \mathcal{BS}^3$. From this we conclude that $\theta'_{17} := \theta_{17} - \varphi_{16} = 80960 - 25255$ is a Brauer character, and hence Ψ^3_{16} indeed is a projective indecomposable character.

3.7 Applying TensorCondense

Next we turn to the determination of the irreducible Brauer characters φ_{13} and φ_{18}, using the TensorCondense package of the MeatAxe. Using the results obtained in Section 3.6, we find that θ_{13} at most might contain the constituent 3520, and θ_{18} at most might contain the constituents 1, 896, 896^*, $2 \cdot 3520$. Furthermore, we find that $\varphi_3 \otimes \varphi_5 = 230 \otimes 896$ equals $[0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 1, 1, 1, 1, 1, 0, 1] \cdot \mathcal{BS}^3$. The explicit construction of the irreducible modular representations 230 and 896 from the representations available so far is a standard application of the MeatAxe. Application of TensorCondense with respect to the first condensation subgroup $K = SL_2(7)$ yields a module N of dimension 610, which by application of the MeatAxe is found to have the constituents

$$1a, 2a, 2b, 2c, 12a, 58a, 64a, 77a, 79a, 155a, 158a.$$

From this we conclude by a consideration of dimensions and using the information collected in Section 3.4, that $58a$, $77a$, and $158a$ are the condensed modules of the irreducible modular representations 20608^*, 25255, and 52624, respectively. It turns out that the $58a$-local submodule L_{58a} of N is the unique submodule having $58a$ in its head. Hence it follows that L_{58a} is invariant under the Hecke algebra. We further observe that N/L_{58a} has the constituents $77a$, $155a$, $158a$. This means that that φ_{18} condenses to $155a$, hence we have $\varphi_{18} = \theta_{18} - 1 - 896 - 896^* - 3520 = 53383$.

Furthermore, we conclude that $64a$ is a constituent of the condensation of the irreducible modular representation φ_{13}. If $64a$ is not extendible to the Hecke algebra, then $12a + 64a$ will be the restriction of an irreducible module for the Hecke algebra. Next we find that $64a$ is a socle constituent of N, but $12a$ is not. A consideration of the contragradient dual module $230 \otimes 896^*$ then shows, similarly to the argument given at the end of Section 3.4, that $64a$ is extendible to the Hecke algebra. Hence we have $\varphi_{13} = \theta_{13} = 22309$.

3.8 Analysis of $P_{1416800}$

To find the last unknown irreducible Brauer character φ_{17} we construct a permutation representation $P_{1416800}$ on the cosets of a subgroup U of index 2 in a maximal subgroup $3_+^{1+4} : 4S_6$. There are three of such subgroups in a fixed $3_+^{1+4} : 4S_6$; having fixed the subgroup fusion from $3_+^{1+4} : 4S_6$ to Co_3, these can be distinguished as follows. One of the candidate subgroups contains a normal subgroup $3_+^{1+4} : 4$ and for exactly one of the remaining two subgroups the irreducible ordinary character $\chi_{30} = 80960$ of Co_3 occurs in the corresponding permutation character, as is seen using GAP. This is our choice of $U := 3_+^{1+4} : 2 \cdot A_6.2$.

To construct this permutation representation we proceed as follows. We use the permutation representation P_{37950} to find a generating set, as words in the given generators for Co_3, for the corresponding one-point stabilizer, which is isomorphic to $U_4(3) : 2^2$. As $3_+^{1+4} : 4S_6$ is the normalizer in Co_3 of a $3A$ element, we first find a generating set for the normalizer in $U_4(3) : 2^2$ of a $3A$ element, the latter group being of isomorphism type $3_+^{1+4} : 2S_4 : 2^2$. This subgroup is used together with the vector permutation technique to find the permutation representation of degree 708400 on the cosets of $3_+^{1+4} : 4S_6$, which in turn allows us to find a generating set for a subgroup isomorphic to $3_+^{1+4} : 4S_6$. (We thank the referee for pointing out that a generating set can also be accessed from the library [20].) Then we finally find a representation of Co_3 such that its restriction to U has a trivial socle constituent on which $3_+^{1+4} : 4S_6$ acts linearly but non-trivially. Applying the vector permutation technique again yields the desired permutation representation $P_{1416800}$.

Now $P_{1416800}$ is condensed with respect to the first condensation subgroup $K = SL_2(7)$, giving a module of dimension 4247. Note that we will be content with finding the constituents of a quotient of this module, which indeed is feasible for modules of this size. But of course this module would be too large for a closer structural analysis along the lines of [11]. Now, $P_{1416800}$ has an ordinary direct sum decomposition reflecting the decomposition $1_U \uparrow^{3_+^{1+4}:4S_6} \cong 1 \oplus 1^-$. As we are interested only in the '1^-'-summand of $P_{1416800}$, we first identify the condensed module of the '1'-summand as a submodule of the condensed module and by the MeatAxe compute the corresponding quotient module, which only has dimension 2119. The MeatAxe finds the constituents of the latter module, and it turns out that its largest constituent has dimension 171. From this it follows directly that $\varphi_{17} = \theta'_{17} = 55705$ holds and we are done.

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