# Algebraic Geometry

## — Basic Notions —

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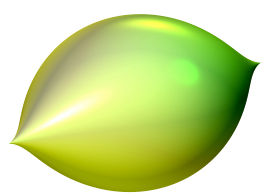
*L'algèbre n'est qu'une géométrie écrite, la géométrie n'est qu'une algèbre figure.*
*(Algebra is but written geometry, and geometry is but figured algebra.)*
[Marie-Sophie Germain, 1776–1831]



**Abstract**

Algebraic Geometry, as a modern mathematical discipline, has its roots in the study of sets of solutions of systems of polynomial equations in affine or projective spaces over algebraically closed fields. These sets are called affine and projective algebraic varieties, respectively. To understand their structure, aspects of commutative algebra, topology, differential geometry, number theory, and algorithmic algebra play a vital role.

## Contents

## 1 Introduction

**(1.1) Example: Solving polynomial equations.** Let $\mathcal{X} := \{X, Y, Z\}$ be indeterminates over $\mathbb{C}$, and let $\mathbb{C}[\mathcal{X}]$ be the associated polynomial $\mathbb{C}$-algebra. We consider the following polynomials in $f, g, h \in \mathbb{C}[\mathcal{X}]$:

$$\begin{array}{rcl} f & := & X + Y + Z - 1, \\ g & := & X^2 + Y^2 + Z^2 - 1, \\ h & := & X^3 + Y^3 + Z^3 - 1, \end{array}$$

and let $I := \langle f, g, h \rangle \trianglelefteq \mathbb{C}[\mathcal{X}]$ be the ideal generated by them. We aim at finding the solutions of the equations '$f = g = h = 0$', that is the **vanishing set**

$$\mathbf{V}(f, g, h) := \{[x, y, z] \in \mathbb{C}^3; f(x, y, z) = g(x, y, z) = h(x, y, z) = 0\}.$$

Then we have $\mathbf{V}(f, g, h) = \mathbf{V}(I) = \{v \in \mathbb{C}^3; p(v) = 0 \text{ for } p \in I\}$.

**i)** In order to determine $\mathbf{V}(I)$, we try to find suitable elements of $I$ by 'eliminating indeterminates': To this end, we order monomials **lexicographically** with respect to '$X > Y > Z > 1$', and apply polynomial division iteratively. (This falls short of applying the full **Buchberger algorithm**, but suffices here to actually find a **Gröbner basis**.) We successively get:

Dividing $h$ by $f$ we get $b' := h - X^2 f = -X^2 Y - X^2 Z + X^2 + Y^3 + Z^3 - 1$, and proceeding further we obtain

$$\begin{array}{rcl} b & := & \frac{1}{3}\big(b' + (XY + XZ - X - Y^2 - 2YZ - Z^2 + 2Y + 2Z - 1)f\big) \\ & = & -Y^2 Z + Y^2 - YZ^2 + 2YZ - Y + Z^2 - Z. \end{array}$$

Similarly, dividing $g$ by $f$ we get $c' := g - Xf = -XY - XZ + X + Y^2 + Z^2 - 1$, and proceeding further we obtain

$$c := \frac{1}{2}\big(c' + (Y + Z - 1)f\big) = Y^2 + YZ - Y + Z^2 - Z.$$

Finally, dividing $b$ by $c$ we get $d := b + (Z - 1)c = Z^3 - Z^2$.

Hence we have $I = \langle f, g, h \rangle = \langle c, b, f \rangle = \langle d, c, f \rangle \trianglelefteq \mathbb{C}[\mathcal{X}]$. Now let $[x, y, z] \in \mathbf{V}(I)$. Then from $d = Z(Z - 1)$ we get $z \in \{0, 1\}$. Next, from $c = Y^2 + Y(Z - 1) + Z(Z - 1)$, for $z = 1$ we get $y = 0$, and for $z = 0$ we get $y \in \{0, 1\}$. Finally, from $f = X + Y + Z - 1$, for $[y, z] \in \{[0, 1], [1, 0]\}$ we get $x = 0$, and for $[y, z] = [0, 0]$ we get $x = 1$. Hence we have $\mathbf{V}(I) = \{[0, 0, 1], [0, 1, 0], [1, 0, 0]\}$. (Note that we have three indeterminates, three equations, and a finite vanishing set.) ♯

**ii)** In the present case, next to the general attack using polynomial division, there is an approach taking advantage of the particular situation:

The symmetric group $\mathcal{S}_3$ acts on $\mathbb{C}[\mathcal{X}]$ by $\mathbb{C}$-algebra automorphisms given by permuting the indeterminates. Since $f, g, h \in \mathbb{C}[\mathcal{X}]$ are $\mathcal{S}_3$-invariant, so is $I = \langle f, g, h \rangle \trianglelefteq \mathbb{C}[\mathcal{X}]$, and thus $\mathcal{S}_3$ also acts on $\mathbf{V}(I)$. (We are tempted to say 'by automorphisms', but so far we do not have any structure on the set $\mathbf{V}(I)$.)

Appealing to the **Newton identities** for symmetric polynomials, entangling power sum polynomials and elementary symmetric polynomials, we infer that

$$e_2 = \frac{1}{2}(f^2 - g + 2f) = XY + XZ + YZ,$$

$$e_3 = \frac{1}{3}(h - f^3 + 3fe_2 - 3g - 3e_2 + 3f) = \frac{1}{6}(2h + f^3 - 3fg + 3f^2 - 3g) = XYZ.$$

Thus we have $I = \langle f, g, h\rangle = \langle f, e_2, e_3\rangle \trianglelefteq \mathbb{C}[\mathcal{X}]$. (Noting that $f = e_1 - 1$, we actually have $\mathbb{C}[f, g, h] = \mathbb{C}[f, e_2, e_3] = \mathbb{C}[e_1, e_2, e_3] = \mathbb{C}[\mathcal{X}]^{\mathcal{S}_3} \subseteq \mathbb{C}[\mathcal{X}]$, where $\mathbb{C}[\mathcal{X}]^{\mathcal{S}_3}$ denotes the $\mathbb{C}$-subalgebra of $\mathcal{S}_3$-invariants.)

Now let again $[x, y, z] \in \mathbf{V}(I)$. Then $e_3 = XYZ$ shows, using the $\mathcal{S}_3$-action, that we may assume $x = 0$. Next, $e_2 = XY + XZ + YZ$ shows, using the $\mathcal{S}_3$-action again, that we may assume $y = 0$ as well. Finally $f = X + Y + Z - 1$ yields $z = 1$. Thus in conclusion we get $\mathbf{V}(I) = \{[0, 0, 1], [0, 1, 0], [1, 0, 0]\}$. ♯

**iii)** We observe that the computations in both parts i) and ii) still hold true, if we replace the base field $\mathbb{C}$ by any (algebraically closed) field of characteristic different from $\{2, 3\}$. We treat the latter cases separately: (Note that in both cases we have two equations remaining, and the associated vanishing set essentially has a single free parameter.)

Let first $\mathbb{K}$ be an algebraically closed field of characteristic 3. Then we have $h = f^3 \in \mathbb{K}[\mathcal{X}]$, so that $I = \langle f, g\rangle$, entailing $I = \langle f, c\rangle$. Thus we may choose $z \in \mathbb{K}$ freely; then $y \in \mathbb{K}$ is a zero of $c(Y, z) = Y^2 + (z-1)Y + z(z-1)$, that is $y \in \{z - 1 \pm \sqrt{1-z}\}$; and finally we get $x = 1 - y - z = z - 1 \mp \sqrt{1-z} \in \mathbb{K}$. Thus $\mathbf{V}(I)$ has a surjective projection onto $\mathbb{K}$, given by the third coordinate, whose fibers over $\mathbb{K} \setminus \{1\}$ have cardinality 2, while the fiber over 1 equals $\{[0, 0, 1]\}$. ♯

Let now $K$ be a field of characteristic 2. Then we have $g = f^2 \in K[\mathcal{X}]$, so that $I = \langle f, h\rangle$, entailing $I = \langle f, b\rangle$. Thus we may choose $z \in \mathbb{K}$ freely; then $y \in \mathbb{K}$ is a zero of $b(Y, z) = (z+1)Y^2 + (z+1)^2 Y + z(z+1) = (z+1)(Y+1)(Y+z)$, that is $y \in \{1, z\} \subseteq \mathbb{K}$ if $z \neq 1$, while we may choose $y \in \mathbb{K}$ freely if $z = 1$; and finally we get $x = 1 + y + z \in \mathbb{K}$, thus $\{[z, 1, z], [1, z, z]\}$ if $z \neq 1$, while $\{[y, y, 1]\}$ if $z = 1$. Thus $\mathbf{V}(I)$ has a surjective projection onto $\mathbb{K}$, given by the third coordinate, whose fibers over $\mathbb{K} \setminus \{1\}$ have cardinality 2, while the fiber over 1 has cardinality $|\mathbb{K}|$ (which is not an entirely satisfying description). ♯

**(1.2) Example: The Sudoku game.** Let $N \in \mathbb{N}$. A **Sudoku problem** of size $N$ is an $N^2 \times N^2$ square tableau, covered by $N^2$ boxes of size $N \times N$, to be filled with numbers in $\mathcal{N} := \{1, \ldots, N^2\}$, such that the entries in each row are pairwise distinct, the entries in each column are pairwise distinct, and the entries in each box are pairwise distinct. To start with, a few entries are prescribed. A Sudoku problem is called **well-posed**, if the prescribed tableau can be completed uniquely. The classical Sudoku game is the case $N = 3$, but $N = 2$ is already suitable to do experiments, while $N = 1$ is trivial; see Table 1.

Let $\mathcal{X} := \{X_{ij}; i, j \in \mathcal{N}\}$ be indeterminates, and let $A := \mathbb{Q}[\mathcal{X}]$ be the associated polynomial $\mathbb{Q}$-algebra. Then filling in the entry $m_{ij}$ in position $[i, j]$ amounts to specializing $X_{ij} \mapsto m_{ij}$. Hence a completed tableau is translated into the maximal ideal $M := \langle X_{ij} - m_{ij}; i, j \in \mathcal{N} \rangle \lhd A$; note that $\dim_{\mathbb{Q}}(A/M) = 1$. Moreover, combinatorial conditions required to be fulfilled by the solutions of a Sudoku problem are translated into polynomial conditions as follows:

**i)** Let $X$ be an auxiliary indeterminate, and let $p := \prod_{k \in \mathcal{N}} (X - k) \in \mathbb{Q}[X]$.

Then the entries being in $\mathcal{N}$ gives rise to $\mathcal{S} := \{p(X_{ij}) \in A; i, j \in \mathcal{N}\}$.

Moreover, prescribing the entries in positions $\mathcal{M} \subseteq \mathcal{N} \times \mathcal{N}$ gives rise to

$$\mathcal{S}_p := \{X_{ij} - m_{ij} \in A; [i, j] \in \mathcal{M}\}.$$

**ii)** Let $Y$ be another auxiliary indeterminate. Then for $i \in \mathbb{N}_0$ we have $X^i - Y^i = (X - Y) \cdot \sum_{j=0}^{i-1} X^j Y^{i-j-1} \in \mathbb{Q}[X, Y]$. Writing $p := \sum_{i \geq 0} p_i X^i$, where $p_i \in \mathbb{Q}$, we get $p(X) - p(Y) = \sum_{i \geq 0} p_i \cdot (X^i - Y^i)$, showing that $(X - Y) \mid p(X) - p(Y) \in \mathbb{Q}[X, Y]$, so that $q(X, Y) := \frac{p(X) - p(Y)}{X - Y} \in \mathbb{Q}[X, Y]$ is a polynomial indeed. Moreover, specializing $Y \mapsto X$ yields $q(X, X) = \sum_{i \geq 1} p_i \cdot i X^{i-1} = (\partial p)(X)$, where $\partial$ denotes the formal derivative.

Thus for $x, y \in \mathcal{N}$ we get: If $x \neq y$, then $0 = p(x) - p(y) = (x - y)q(x, y)$, implying $q(x, y) = 0$. If $x = y$, then $q(x, x) = (\partial p)(x)$, where since $p$ has the zeroes $\mathcal{N}$, all of which are simple, we conclude that $(\partial p)(x) \neq 0$.

Hence, entries in the same row being pairwise different gives rise to

$$\mathcal{S}_r := \{q(X_{ij}, X_{ik}) \in A]; i, j, k \in \mathcal{N}, j < k\}.$$

Similarly, entries in the same column being pairwise different gives rise to

$$\mathcal{S}_c := \{q(X_{ij}, X_{kj}) \in A; i, j, k \in \mathcal{N}, i < k\}.$$

Finally, entries in the same box being pairwise different gives rise to

$$\mathcal{S}_b := \left\{ q(X_{aN+i, bN+j}, X_{aN+k, bN+l}) \in A; \begin{array}{l} a, b \in \{0, \ldots, N-1\}, \\ i, j, k, l \in \{1, \ldots, N\}, i < k, j \neq l \end{array} \right\}.$$

Now, let $I := \langle \mathcal{S}, \mathcal{S}_p, \mathcal{S}_r, \mathcal{S}_c, \mathcal{S}_b \rangle \unlhd A$. Then a completed tableau solves the given Sudoku problem if and only if the associated maximal ideal divides $I$.

Moreover, if $I \subseteq P \lhd A$ is a prime ideal dividing $I$, then $A/P$ is a domain extending $\mathbb{Q}$. Since $X_{ij} \in A/P$, for $i, j \in \mathcal{N}$, is a zero of $p = \prod_{k \in \mathcal{N}} (X - k)$, and the minimum polynomial of $X_{ij} \in A/P$ over $\mathbb{Q}$ exists and is irreducible, we conclude that there is $m_{ij} \in \mathcal{N}$ such that $X_{ij} - m_{ij} \in P$. Hence we have $P = \langle X_{ij} - m_{ij}; i, j \in \mathcal{N} \rangle \lhd A$, showing that $P$ is a maximal ideal. Thus the maximal ideals and prime ideals of $A/I$ coincide, of which there are only finitely many, and any such ideal provides a solution of the given Sudoku problem.

Hence we conclude that the solutions of the given Sudoku problem are precisely given by the maximal ideals of $A$ dividing $I$. In particular, the problem is

Table 1: Sudoku problems of size 2 and 3.

| | | | |
|---|---|---|---|
| **1** | 4 | 2 | 3 |
| **3** | 2 | 1 | 4 |
| 2 | 3 | 4 | **1** |
| 4 | 1 | 3 | **2** |

| | | | |
|---|---|---|---|
| **1** | | | 3 |
| **3** | | | 1 |
| 2 | | | **4** |
| 4 | | | **2** |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9 | **6** | 3 | **1** | 7 | **4** | 2 | **5** | 8 |
| | | **8** | **3** | 2 | **5** | **6** | 4 | 9 |
| **2** | 5 | 4 | 6 | 8 | 9 | 7 | 3 | **1** |
| **8** | 2 | 1 | **4** | 3 | 7 | 5 | 9 | 6 |
| 4 | 9 | **6** | 8 | 5 | 2 | **3** | 1 | 7 |
| | | 5 | **9** | 6 | **1** | 8 | 2 | **4** |
| **5** | 8 | 9 | 7 | 1 | 3 | 4 | 6 | **2** |
| | | 7 | **2** | 4 | **6** | **9** | 8 | 5 |
| 6 | **4** | 2 | **5** | 9 | **8** | 1 | **7** | 3 |

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unsolvable if and only if $I = A$, and it is well-posed if and only if $I \lhd A$ is maximal. In order to determine the solutions of the given Sudoku problem explicitly, we have to find a 'good' generating set of $I$, from which we are able to read off the solutions. (Again, **Gröbner bases** do the job, this time with respect to a **degree-driven** order on monomials.) For example, see Table 1:

**i)** For $N = 2$ with four suitable entries given (in bold face), there is a unique complete solution (depicted in normal font).

**ii)** Again for $N = 2$ with four, slightly different entries given, only four more entries are uniquely determined, while there are four complete solutions determined by specializing $x_{2,3} \in \{2,4\}$ and $x_{4,3} \in \{1,3\}$ (for example).

**iii)** For $N = 3$ with 26 suitable entries given, only 49 more entries are uniquely determined, while there are two complete solutions determined by $x_{8,2} \in \{1,3\}$.

**(1.3) Example: Curves and surfaces. a)** Let $\{X, Y\}$ be indeterminates, and let $\mathbb{R}[X,Y]$ be the associated polynomial $\mathbb{R}$-algebra. Given $f \in \mathbb{R}[X,Y] \setminus \mathbb{R}$, the set $\mathbf{V}_{\mathbb{R}}(f) := \{[x,y] \in \mathbb{R}^2; f(x,y) = 0\}$ is called a **plane curve**. A few examples are depicted in Tables 2 and 3, in particular exhibiting the geometrical phenomenon of **singularities** (of which there are at most finitely many):

**i)** $f := X(X^2-1) - Y^2$, **ii)** $f := X(X-1)^2 - Y^2$, **iii)** $f := X^3 - Y^2$,

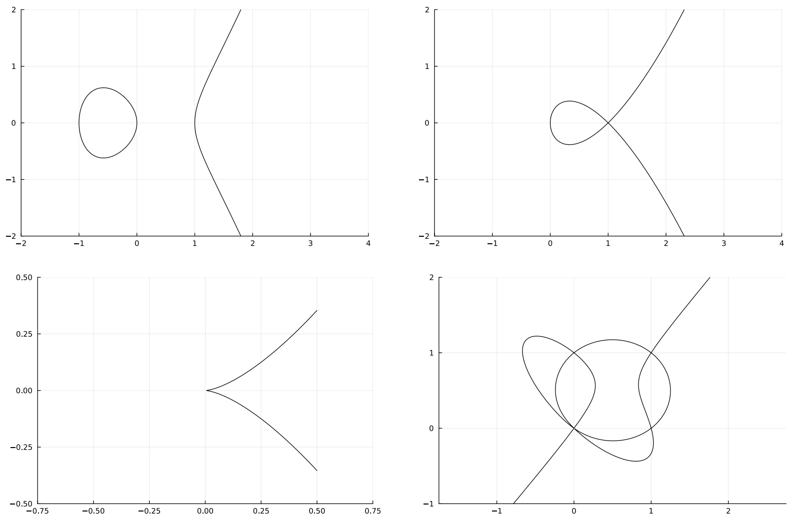
**iv)** $f := \begin{pmatrix} -3X^5 - 2X^4Y - 3X^3Y^2 + XY^4 + 3Y^5 + 6X^4 + 7X^3Y \\ +3X^2Y^2 - 2XY^3 - 6Y^4 - 3X^3 - 5X^2Y + XY^2 + 3Y^3 \end{pmatrix}$,

**v)** $f := X^2(X^2-1) + Y^2$, **vi)** $f := X^4(X+1) - Y^2$,

**vii)** $f := X^4(X^2-1) + Y^2$, **viii)** $f := (X^2+Y^2)^3 - 4X^2Y^2$.

**b)** Let $\{X, Y, Z\}$ be indeterminates, and let $\mathbb{R}[X,Y,Z]$ be the associated polynomial $\mathbb{R}$-algebra. Given $f \in \mathbb{R}[X,Y,Z] \setminus \mathbb{R}$, the set $\mathbf{V}_{\mathbb{R}}(f) := \{[x,y,z] \in \mathbb{R}^3; f(x,y,z) = 0\}$ is called a **(hyper-)surface**. For example, the surface given

Table 2: Some plane curves.



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by $f_\lambda := (X^2 - 1)^3 + \lambda \cdot (Y^2 + Z^2)$, where $\lambda > 0$, is depicted on the title page.

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# I Algebraic sets

## 2 Algebraic sets

**(2.1) Noetherian rings.** All rings and algebras occurring in the sequel will be commutative, associative, and unital, unless otherwise specified.

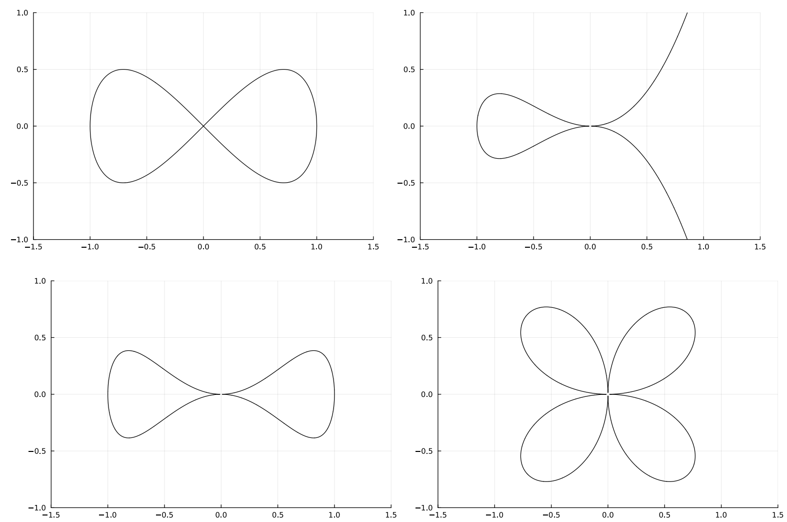
Let $R$ be a ring, let $M$ be a finitely generated $R$-module. Then $M$ is called **Noetherian** if any $R$-submodule of $M$ finitely generated. The ring $R$ is called **Noetherian** if it is so as an $R$-module, that is any ideal of $R$ is finitely generated.

This is equivalent to saying that $M$ fulfills the **ascending chain condition (A.C.C.)** on submodules, that is any strictly ascending chain of $R$-submodules terminates. Moreover, it is equivalent to the **maximum condition** on submodules, that is any set of $R$-submodules contains a maximal element.

This implies that any generating set of an ideal of $R$ contains a finite generating set. For example, any principal ideal ring, and thus any field is Noetherian. $\sharp$

For the proofs of these assertions, see Exercise (12.1). The following theorem is fundamental in algebraic geometry, providing the appropriate finiteness condi-

Table 3: Some more plane curves.



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tions to develop theory and to pursue computational approaches.

**(2.2) Theorem: Hilbert's Basis Theorem [1890].**
Let $R$ be a Noetherian ring, and let $X$ be an indeterminate. Then the polynomial ring $R[X]$ is Noetherian as well.

**Proof:** [Sarges, 1976]. Assume to the contrary that there is an ideal $I \trianglelefteq R[X]$ which is not finitely generated. Then there is a sequence $[f_1, f_2, \ldots] \subseteq I \subseteq R[X]$ such that $f_i \in I \setminus \langle f_1, \ldots, f_{i-1} \rangle$, being chosen such that its **degree** $d_i := \deg(f_i) \geq 0$ is minimal, for $i \geq 1$; note that $I \setminus \langle \rangle = I \setminus \{0\}$, and that $d_i \leq d_{i+1}$.

Let $a_i := \mathrm{lc}(f_i) \in R \setminus \{0\}$ be the **leading coefficient** of $f_i$, so that $f_i$ has **leading monomial** $\mathrm{lm}(f_i) := X^{d_i}$ and **leading term** $\mathrm{lt}(f_i) := a_i X^{d_i}$, for $i \geq 1$. (These notions appearing here is reminiscent of Gordan's proof [1900].)

Now let $J := \langle a_i; i \geq 1 \rangle \trianglelefteq R$. Then, since $R$ is Noetherian, there is $n \geq 0$ such that $J = \langle a_1, \ldots, a_n \rangle$. Thus we have $a_{n+1} = \sum_{i=1}^n a_i b_i$, for some $b_1, \ldots, b_n \in R$. Hence $g := \sum_{i=1}^n b_i f_i X^{d_{n+1}-d_i} \in \langle f_1, \ldots, f_n \rangle$ has degree $\deg(g) = d_{n+1}$ and leading coefficient $\mathrm{lc}(g) = \sum_{i=1}^n a_i b_i = a_{n+1}$, which implies that $f_{n+1} - g \in I \setminus \langle f_1, \ldots, f_n \rangle$ has degree $\deg(f_{n+1} - g) < d_{n+1}$, a contradiction. $\sharp$

**Corollary.** If $R$ is Noetherian, then so is any finitely generated $R$-algebra.

**Proof.** Any such algebra can be written as $R[\mathcal{X}]/I$, where $\mathcal{X}$ is a finite set of indeterminates, and $I \trianglelefteq R[\mathcal{X}]$. Now, by Hilbert's Basis Theorem and induction, $R[\mathcal{X}]$ is Noetherian, and so is $R[\mathcal{X}]/I$ by the Homomorphism Theorem. ♯

**(2.3) Algebraic sets.** Let $K \subseteq L$ be a field extension, let $\mathcal{X} := \{X_1, \ldots, X_n\}$ be indeterminates, where $n \in \mathbb{N}_0$, and let $A := K[\mathcal{X}]$ be the associated polynomial $K$-algebra. Recall that $\mathbf{A}^n(L) = L^n$ is also called the $n$-dimensional **affine space** over $L$, its elements are called **points**; in particular, $\mathbf{A}^1(L)$ and $\mathbf{A}^2(L)$ are called the **affine line** and the **affine plane** over $L$, respectively.

Then, letting $\mathcal{S} \subseteq A$, the set

$$\mathbf{V}_L(\mathcal{S}) := \{[x_1, \ldots, x_n] \in L^n; f(x_1, \ldots, x_n) = 0 \text{ for all } f \in \mathcal{S}\}$$

is called the **(affine) ($K$-)algebraic subset** given by the **defining set** $\mathcal{S}$; then $K$ and $L$ are called its **field of definition** and its **field of coordinates**, respectively. If $R \subseteq L$ is a subring, then $\mathbf{V}_R(\mathcal{S}) := \mathbf{V}_L(\mathcal{S}) \cap R^n$ is called the set of $R$**-rational** points of $\mathbf{V}_L(\mathcal{S})$.

We have $\mathbf{V}_L(\mathcal{S}) = \mathbf{V}_L(\langle \mathcal{S} \rangle)$. By Hilbert's Basis Theorem there are $f_1, \ldots, f_r \in \mathcal{S}$, for some $r \in \mathbb{N}_0$, such that $\mathbf{V}_L(\mathcal{S}) = \mathbf{V}_L(f_1, \ldots, f_r)$. Hence any algebraic set is defined by an ideal, or alternatively by finitely many polynomials. In particular, an algebraic set defined by a single non-constant polynomial is called a **hypersurface**; in the case $n = 2$ the latter is also called a **curve**.

For ideals $I \subseteq J \trianglelefteq A$ we have $\mathbf{V}_L(J) \subseteq \mathbf{V}_L(I)$. Given ideals $I_i \trianglelefteq A$, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, we have $\bigcap_{i \in \mathcal{I}} \mathbf{V}_L(I_i) = \mathbf{V}_L(\sum_{i \in \mathcal{I}} I_i)$, but in general we only have $\bigcup_{i \in \mathcal{I}} \mathbf{V}_L(I_i) \subseteq \mathbf{V}_L(\bigcap_{i \in \mathcal{I}} I_i)$.

For example, we have $\mathbf{V}_L(1) = \emptyset$ and $\mathbf{V}_L(0) = L^n$. Moreover, for $K = L$ and $n = 1$, we have $A = L[X]$, so that the associated algebraic sets are $L$ itself and its finite subsets; this shows that the infinite union of algebraic sets need not be algebraic again. (Finite unions of algebraic sets will be treated in (2.6) below.)

**(2.4) Vanishing ideals.** Letting $V \subseteq L^n$ be any subset, the set

$$\mathbf{I}_K(V) := \{f \in A; f(x_1, \ldots, x_n) = 0 \text{ for all } [x_1, \ldots, x_n] \in V\} \trianglelefteq A$$

is called the **vanishing ideal** of $V$; it is immediate that $\mathbf{I}_K(V)$ is an ideal.

For $V \subseteq W \subseteq L^n$ we have $\mathbf{I}_K(W) \subseteq \mathbf{I}_K(V)$. Given $V_i \subseteq L^n$, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, we have $\bigcap_{i \in \mathcal{I}} \mathbf{I}_K(V_i) = \mathbf{I}_K(\bigcup_{i \in \mathcal{I}} V_i)$, but in general (even for finite sums) we only have $\sum_{i \in \mathcal{I}} \mathbf{I}_K(V_i) \subseteq \mathbf{I}_K(\bigcap_{i \in \mathcal{I}} V_i)$.

For example, we have $\mathbf{I}_K(\emptyset) = A$, and if $L$ is infinite then (it is well-known that) we have $\mathbf{I}_K(L^n) = \{0\}$. Moreover, for $K = L$ infinite and $n = 1$, if $V_1, V_2 \subseteq L$ are infinite such that $V_1 \cap V_2$ is finite, then we have $\mathbf{I}_K(V_1) = \{0\} = \mathbf{I}_K(V_2)$, but $\mathbf{I}_K(V_1 \cap V_2) \neq \{0\}$, so that we have $\mathbf{I}_K(V_1) + \mathbf{I}_K(V_2) \neq \mathbf{I}_K(V_1 \cap V_2)$. (Intersections of algebraic sets will be treated in (2.6) below.)

**(2.5) Radical ideals. a)** Let $R$ be a ring, and let $I \trianglelefteq A$ be an ideal. Then the **radical** of $I$ is given as $\sqrt{I} := \{f \in R; f^k \in I \text{ for some } k \in \mathbb{N}\}$. It is immediate that for $f, g \in \sqrt{I}$ we have $f \cdot R \subseteq \sqrt{I}$; and from $(f+g)^k = \sum_{l=1}^{k} \binom{k}{l} f^l g^{k-l}$ we infer that $\sqrt{I}$ is additively closed; hence $\sqrt{I} \trianglelefteq R$ is an ideal again.

We have $\sqrt{I} = R$ if and only if $I = R$. Moreover, we have $I \subseteq \sqrt{I} = \sqrt{\sqrt{I}}$, and if $\sqrt{I} = I$ then $I$ is called a **radical ideal**. In particular, if $P \triangleleft R$ is prime, then $f^k \in P$ implies $f \in P$, saying that $\sqrt{P} = P$, that is $P$ is radical.

In particular, the ideal $\mathrm{nil}(R) := \sqrt{\{0\}} \trianglelefteq R$, consisting of the **nilpotent** elements of $R$, is called the **nilradical** of $R$. The ring $R$ is called **reduced** if $\mathrm{nil}(R) = \{0\}$.

We have $\mathrm{nil}(R/\mathrm{nil}(R)) = \{0\}$, that is $R_{\mathrm{red}} := R/\mathrm{nil}(R)$ is reduced, being called the associated **reduced ring**. For any ideal $I \trianglelefteq R$ we have $\mathrm{nil}(R/I) = \sqrt{I}/I$, that is $(R/I)_{\mathrm{red}} \cong R/\sqrt{I}$; thus $R/I$ is reduced if and only if $I = \sqrt{I}$ is radical.

**(2.6) Algebraic sets and their ideals. a)** The relevance of the above notions is given by the following observations: For any subset $V \subseteq L^n$ we have $\mathbf{I}_K(V) = \sqrt{\mathbf{I}_K(V)}$, and for any ideal $I \trianglelefteq A$ we have $\mathbf{V}_L(\sqrt{I}) = \mathbf{V}_L(I)$:

Since there are no non-zero nilpotent elements of $L$, from $f^k \in \mathbf{I}_K(V)$, for some $f \in A$ and $k \in \mathbb{N}$, we get $f \in \mathbf{I}_K(V)$. Similarly, if $v \in \mathbf{V}_L(I)$, then for $f \in \sqrt{I}$ we have $f^k(v) = 0$, for some $k \in \mathbb{N}$, implying $f(v) = 0$, hence $v \in \mathbf{V}_L(\sqrt{I})$. ♯

**b)** We consider the interplay between the operators $\mathbf{V}_L$ and $\mathbf{I}_K$: By definition, for $V \subseteq L^n$ we have $V \subseteq \mathbf{V}_L(\mathbf{I}_K(V))$, and for $I \trianglelefteq A$ we have $I \subseteq \mathbf{I}_K(\mathbf{V}_L(I))$.

For three-fold compositions this yields: The inclusion $I \subseteq \mathbf{I}_K(\mathbf{V}_L(I))$ implies $\mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_L(I))) \subseteq \mathbf{V}_L(I)$, so that the inclusion $\mathbf{V}_L(I) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_L(I)))$ implies equality; that is we have

$$\mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_L(I))) = \mathbf{V}_L(I), \quad \text{for any } I \trianglelefteq A.$$

Similarly, the inclusion $V \subseteq \mathbf{V}_L(\mathbf{I}_K(V))$ implies $\mathbf{I}_K(\mathbf{V}_L(\mathbf{I}_K(V))) \subseteq \mathbf{I}_K(V)$, so that the inclusion $\mathbf{I}_K(V) \subseteq \mathbf{I}_K(\mathbf{V}_L(\mathbf{I}_K(V)))$ implies equality; that is we have

$$\mathbf{I}_K(\mathbf{V}_L(\mathbf{I}_K(V))) = \mathbf{I}_K(V), \quad \text{for any } V \subseteq L^n.$$

**c)** Let $\mathbf{V}, \mathbf{W} \subseteq L^n$ be algebraic. Then we have $\mathbf{V} \subseteq \mathbf{W}$ if and only if $\mathbf{I}_K(\mathbf{W}) \subseteq \mathbf{I}_K(\mathbf{V})$, and $\mathbf{V} = \mathbf{W}$ if and only if $\mathbf{I}_K(\mathbf{W}) = \mathbf{I}_K(\mathbf{V})$: From $\mathbf{I}_K(\mathbf{W}) \subseteq \mathbf{I}_K(\mathbf{V})$ we get $\mathbf{V} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{W})) = \mathbf{W}$; similarly, equality $\mathbf{I}_K(\mathbf{W}) = \mathbf{I}_K(\mathbf{V})$ implies $\mathbf{V} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})) = \mathbf{V}_L(\mathbf{I}_K(\mathbf{W})) = \mathbf{W}$.

We consider (arbitrary) intersections and (finite) unions of algebraic sets:

Firstly, for algebraic sets $\mathbf{V}_i \subseteq L^n$, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, we have $\mathbf{V}_L(\sum_{i \in \mathcal{I}} \mathbf{I}_K(\mathbf{V}_i)) = \bigcap_{i \in \mathcal{I}} \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_i)) = \bigcap_{i \in \mathcal{I}} \mathbf{V}_i$. In particular, an arbitrary intersection of algebraic sets is algebraic again. Secondly:

**Proposition.** We have $\mathbf{V} \cup \mathbf{W} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}) \cap \mathbf{I}_K(\mathbf{W})) = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}) \cdot \mathbf{I}_K(\mathbf{W}))$. In particular, a finite union of algebraic sets is algebraic again.

**Proof.** We have $\mathbf{I}_K(\mathbf{V}) \cdot \mathbf{I}_K(\mathbf{W}) \subseteq \mathbf{I}_K(\mathbf{V}) \cap \mathbf{I}_K(\mathbf{W}) \subseteq \mathbf{I}_K(\mathbf{V} \cup \mathbf{W})$, implying

$$\mathbf{V} \cup \mathbf{W} \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V} \cup \mathbf{W})) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}) \cap \mathbf{I}_K(\mathbf{W})) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}) \cdot \mathbf{I}_K(\mathbf{W})).$$

Conversely, let $v \in L^n \setminus (\mathbf{V} \cup \mathbf{W})$. Since $\mathbf{V} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}))$ and $\mathbf{W} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{W}))$, there are $f \in \mathbf{I}_K(\mathbf{V})$ and $g \in \mathbf{I}_K(\mathbf{W})$ such that $f(v) \neq 0 \neq g(v)$. Hence we have $fg \in \mathbf{I}_K(\mathbf{V}) \cdot \mathbf{I}_K(\mathbf{W})$ such that $(fg)(v) \neq 0$, thus $v \notin \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}) \cdot \mathbf{I}_K(\mathbf{W}))$. ♯

Thus the smallest algebraic set containing $V$ is given as the intersection $\overline{V} := \bigcap\{\mathbf{W} \subseteq L^n \text{ algebraic}; V \subseteq \mathbf{W}\}$. Hence we have $V \subseteq \mathbf{V}_L(\mathbf{I}_K(V)) \subseteq \overline{V}$. Since conversely for any algebraic set $V \subseteq \mathbf{V}_L(I)$ we already have $\overline{V} = \mathbf{V}_L(\mathbf{I}_K(V)) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_L(I))) = \mathbf{V}_L(I)$, in conclusion we get equality $\overline{V} = \mathbf{V}_L(\mathbf{I}_K(V))$. (The notation of a 'closure operator' will be explained in (3.4) below.)

Thus, for subsets $V \subseteq W$ we have $\overline{V} \subseteq \overline{W}$, and $V$ is algebraic if and only if $V = \overline{V}$. (The case of $\mathbf{I}_K(\mathbf{V}_L(I)) \trianglelefteq A$ will be dealt with in (2.8) below.)

**(2.7) Algebraic-geometric correspondence.** We conclude that the operator $\mathbf{I}_K$ induces an inclusion-reversing (with respect to set-theoretic inclusion) injective correspondence

$$\mathbf{I}_K \colon \{\mathbf{V} \subseteq L^n \text{ affine } K\text{-algebraic}\} \to \{I \trianglelefteq A \text{ radical}\},$$

whose inverse on the image of $\mathbf{I}_K$ is given by the operator $\mathbf{V}_L$.

In general, $\mathbf{I}_K$ is not surjective: For example, let $K = L = \mathbb{R}$ and $n = 1$. Then $f := X^2 + 1 \in \mathbb{R}[X]$ is irreducible, hence $\langle f \rangle \lhd \mathbb{R}[X]$ is prime, thus radical. But $\mathbf{V}_\mathbb{R}(f) = \emptyset$, so that $\mathbf{I}_\mathbb{R}(\mathbf{V}_\mathbb{R}(f)) = \mathbf{I}_\mathbb{R}(\emptyset) = \mathbb{R}[X]$, implying that $\langle f \rangle \notin \operatorname{im}(\mathbf{I}_\mathbb{R})$. ♯

But it follows from the geometric form of Hilbert's Nullstellensatz, see (2.8) below, that if $L$ is algebraically closed then $\mathbf{I}_K$ is surjective as well. In this case, the bijective correspondence given by $\mathbf{I}_K$ and $\mathbf{V}_L$ provides an **algebraic-geometric dictionary**, allowing us to translate between geometric properties of algebraic sets and algebraic properties of ideals.

**(2.8) Hilbert's Nullstellensatz.** We prove the following 'Theorem of Zeroes' of fundamental importance in algebraic geometry:

**Theorem: Nullstellensatz (strong form)** [Hilbert, 1893]**.**
Let $L$ be algebraically closed, and let $I \lhd A$. Then we have $\mathbf{V}_L(I) \neq \emptyset$.

**Theorem: Nullstellensatz (field-theoretic form).**
Let the field $L$ be finitely generated as a $K$-algebra. Then $K \subseteq L$ is algebraic.

In order to show the above assertions, we proceed as follows: We first prove the equivalence of the strong form and the field theoretic form. Then we prove two lemmas, from which the field theoretic form follows:

**Proof: Equivalence of strong and field theoretic form.**
**i)** Assume first that the strong form holds, and let $\overline{K}$ be an algebraic closure of $K$. Since $L$ is finitely generated as a $K$-algebra, by $n \in \mathbb{N}_0$ elements say, there is $P \lhd A$ maximal such that $L \cong A/P$ as $K$-algebras. Let $[x_1, \ldots, x_n] \in \mathbf{V}_{\overline{K}}(P)$, and let $\varphi \colon A \to \overline{K}$ be the (non-zero) homomorphism of $K$-algebras defined by $X_i \mapsto x_i$. Then for any $f \in P$ we have

$$\varphi(f) = \varphi(f(X_1, \ldots, X_n)) = f(\varphi(X_1), \ldots, \varphi(X_n)) = f(x_1, \ldots, x_n) = 0 \in \overline{K},$$

hence $\ker(\varphi) = P$. Thus there is an embedding $L \cong A/P \to \overline{K}$, implying that $K \subseteq L$ is an algebraic field extension.

**ii)** Assume now that the field-theoretic form holds. Let $I \subseteq P \lhd A$ be maximal. Then $A/P$ is a field extension of $K$, which is finitely generated as a $K$-algebra. Hence by assumption $K \subseteq (A/P)$ is algebraic. Since $L$ is algebraically closed, there is a homomorphism $\varphi \colon A \to L$ of $K$-algebras, inducing an embedding $\overline{\varphi} \colon A/P \to L$. Let $v := [\varphi(X_1), \ldots, \varphi(X_n)] \in L^n$. Then for any $f \in P$ we have

$$f(v) = f(\varphi(X_1), \ldots, \varphi(X_n)) = \varphi(f(X_1, \ldots, X_n)) = \varphi(f) = 0 \in L,$$

hence $v \in \mathbf{V}_L(P) \subseteq \mathbf{V}_L(I)$. Thus we have $\mathbf{V}_L(I) \neq \emptyset$. ♯

**Lemma.** Let $R$ be a Noetherian ring, let $T$ be a finitely generated $R$-algebra, and let $S \subseteq T$ be an $R$-subalgebra, such that $T$ is a finitely generated $S$-module. Then $S$ is finitely generated as an $R$-algebra as well.

**Proof.** Let $T = R[\mathcal{G}]$, where $\mathcal{G}$ is finite, and let $T = \langle \mathcal{T} \rangle_S$ where $\mathcal{G} \subseteq \mathcal{T} = \{t_1, \ldots, t_r\}$, for some $r \in \mathbb{N}_0$. Then we have $t_i t_j = \sum_{k=1}^r t_k s_{ijk}$, for suitable $s_{ijk} \in S$. Let $S' := R[s_{ijk}; i, j, k \in \{1, \ldots, r\}] \subseteq S$. Then $S'$ is a finitely generated $R$-algebra, such that $T = R[\mathcal{G}] \subseteq \langle \mathcal{T} \rangle_{S'} \subseteq T$, implying that $T$ is a finitely generated $S'$-module.

Since $R$ is Noetherian, by Hilbert's Basis Theorem $S'$ is Noetherian as well. Thus $T$ is a Noetherian $S'$-module. Hence $S \subseteq T$ is a finitely generated $S'$-module as well. Thus, since $S'$ is a finitely generated $R$-algebra, $S$ is a finitely generated $R$-algebra as well. ♯

**Lemma.** Let $R := K(\mathcal{X})$ be the associated rational function field, where $n \geq 1$. Then $R$ is not finitely generated as a $K$-algebra.

**Proof.** Assume to the contrary that $R = K[\frac{f_i}{g}; i \in \{1, \ldots, r\}]$, where $f_i \in A := K[\mathcal{X}]$ and $0 \neq g \in A$, for some $r \in \mathbb{N}_0$; note that we may indeed assume the elements of the generating set to have the same denominator. This implies that any element of $R$ can be written as $\frac{f}{g^k}$, for some $f \in A$ and $k \in \mathbb{N}_0$. Since $A$ is factorial, $g$ has only finitely many irreducible divisors, up to associates.

Now, $A$ has infinitely many irreducible polynomials, up to associates: If $K$ is infinite, there are $X - a \in K[X]$ for $a \in K$; if $K$ is finite, there are irreducible univariate polynomials of any degree $d$. Thus there is an irreducible polynomial $p \in A$ coprime to $g$. Then $\frac{1}{p} \in R$ is not of the required form, a contradiction. ♯

**Proof: Field theoretic form** [Zariski, 1947; Artin–Tate, 1951]**.** Assume that $K \subseteq L$ is not algebraic. Then, since $K \subseteq L$ is a finitely generated field extension, let $\mathcal{Y} := \{Y_1, \ldots, Y_r\}$ be a **transcendence basis** of $L$ over $K$, for some $r \in \mathbb{N}$; that is $K(\mathcal{Y})$ is a rational function field, such that $K(\mathcal{Y}) \subseteq L$ is algebraic. Since $K(\mathcal{Y}) \subseteq L$ is a finitely generated field extension, we conclude that $L$ is a finitely generated $K(\mathcal{Y})$-vector space. Now, since $L$ is a finitely generated $K$-algebra, the first lemma implies that $K(\mathcal{Y})$ is a finitely generated $K$-algebra as well, contradicting the second lemma. ♯

**(2.9) Hilbert's Nullstellensatz, cont.** We draw a couple of corollaries from the strong form of Hilbert's Nullstellensatz. For the second one we need an independent criterion to decide whether a ring element is contained in the radical of an ideal; this is also called the **Rabinowitsch Trick**, see also (7.2).

**Corollary: Nullstellensatz (weak form).**
Let $K$ be algebraically closed, and let $P \lhd A$ be maximal. Then there is $v = [x_1, \ldots, x_n] \in K^n$ such that $P = \langle X_1 - x_1, \ldots, X_n - x_n \rangle \lhd A$.

**Proof.** By the strong form of Hilbert's Nullstellensatz, applied to $K = L$, the ideal $P$ has a zero $v = [x_1, \ldots, x_n] \in K^n$. Hence for any $f \in A \setminus P$ we have $f(v) \neq 0$: Assume that $f(v) = 0$, then $A = \langle P, f \rangle \subseteq \mathbf{I}_K(v) \lhd A$, a contradiction.

We conclude that $I := \langle X_1 - x_1, \ldots, X_n - x_n \rangle \subseteq P \lhd A$. Now polynomial division shows that $\dim_K(A/I) \leq 1$, thus we have $I = P \lhd A$. ♯

**Proposition: Radical membership test.** Let $R$ be a domain, let $I \unlhd R$, and let $f \in R$. Moreover, let $T$ be an indeterminate, and let $J := \langle I, fT - 1 \rangle \unlhd R[T]$. Then we have $f \in \sqrt{I}$ if and only if $J = R[T]$.

**Proof. i)** Let first $f \in \sqrt{I}$, and let $k \in \mathbb{N}$ such that $f^k \in I \subseteq J$. Then we have $1 = (fT)^k = f^k T^k = 0 \in R[T]/J$, which implies $J = R[T]$. (Note that $f \in R[T]/J$ is 'forced' to be a unit, and $f \in \sqrt{I}$ implies that it is nilpotent.)

**ii)** Let conversely $J = R[T]$. Then there are $g, g_1, \ldots, g_r \in R[T]$ and $f_1, \ldots, f_r \in I$, for some $r \in \mathbb{N}_0$, such that $1 = g \cdot (fT - 1) + \sum_{i=1}^{r} g_i f_i \in R[T]$.

We may assume that $f \neq 0$. Then there is an $R$-algebra homomorphism $\varphi\colon R[T] \to \mathrm{Q}(R)\colon T \mapsto \frac{1}{f}$. Thus we have $\varphi(fT - 1) = 0$, so that $\sum_{i=1}^{r} \varphi(g_i) f_i = 1 \in \mathrm{Q}(R)$. We may assume that $\varphi(g_i) = \frac{g_i'}{f^k} \in \mathrm{Q}(R)$, where $g_i' \in R$ and $k \in \mathbb{N}$. This yields $f^k = \sum_{i=1}^{r} g_i' f_i \in R \subseteq \mathrm{Q}(R)$, showing that $f^k \in I$, thus $f \in \sqrt{I}$. ♯

**Corollary: Nullstellensatz (geometric form).**
Let $L$ be algebraically closed, and let $I \unlhd A$. Then we have $\mathbf{I}_K(\mathbf{V}_L(I)) = \sqrt{I} \unlhd A$.

**Proof.** We have already seen that $I \subseteq \sqrt{I} \subseteq \mathbf{I}_K(\mathbf{V}_L(I)) = \sqrt{\mathbf{I}_K(\mathbf{V}_L(I))} \trianglelefteq A$. Hence let $f \in \mathbf{I}_K(\mathbf{V}_L(I))$. We apply the radical membership test: Let $T$ be an indeterminate, and let $J := \langle I, fT-1\rangle \trianglelefteq A[T]$. We have to show that $J = A[T]$:

Assume there is $[x_1, \ldots, x_n, t] \in \mathbf{V}_L(J) \subseteq L^{n+1}$, then $[x_1, \ldots, x_n] \in \mathbf{V}_L(I)$, thus we have $f(x_1, \ldots, x_n) = 0$, which implies that $0 = (fT-1)(x_1, \ldots, x_n, t) = f(x_1, \ldots, x_n) \cdot t - 1 = -1$, a contradiction. Hence we have $\mathbf{V}_L(J) = \emptyset$, thus by the strong form of Hilbert's Nullstellensatz we infer that $J = A[T]$ indeed. $\sharp$

## 3 Affine varieties

**(3.1) Topological spaces.** We recall some notions from general topology: A collection of subsets of a set $V$, being called **open**, is called a **topology** on $V$, provided the following properties hold: Both $\emptyset$ and $V$ are open; if $U, U' \subseteq V$ are open, then $U \cap U' \subseteq V$ is open as well; and if $U_i \subseteq V$ are open, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, then $\bigcup_{i\in\mathcal{I}} U_i \subseteq V$ is open as well.

Then $V$ together with a topology on it is called a **topological space**. For example, the collection $\{\emptyset, V\}$ is a topology, being called the **trivial topology**; and the set of all subsets of $V$ is a topology, being called the **discrete topology**.

A subset $W \subseteq V$ is called **closed** if its complement $V \setminus W \subseteq V$ is open. Hence by taking complements a topology is equivalently given by a collection of closed subsets, having the following properties: Both $\emptyset$ and $V$ are closed; if $W, W' \subseteq V$ are closed, then $W \cup W' \subseteq V$ is closed; and if $W_i \subseteq V$ are closed, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, then $\bigcap_{i\in\mathcal{I}} W_i \subseteq V$ is closed.

Given a subset $U \subseteq V$, its **closure** in $V$ is defined as the closed subset $\overline{U} := \bigcap\{W \subseteq V \text{ closed}; U \subseteq W\} \subseteq V$; note that the latter set of sets contains $V$ as an element, thus is non-empty, so that the intersection is well-defined. In other words, $\overline{U} \subseteq V$ is the smallest closed subset (with respect to set-theoretic inclusion) containing $U$. The subset $U \subseteq V$ is called **dense** if $\overline{U} = V$.

Any subset $V' \subseteq V$ carries the **induced topology**, whose open subsets are given as $U \cap V' \subseteq V'$, where $U \subseteq V$ is open; likewise, its closed subsets are given as $W \cap V' \subseteq V'$, where $W \subseteq V$ is closed.

**(3.2) Irreducible spaces.** A topological space $V \neq \emptyset$ is called **reducible**, if $V = V' \cup V''$ is a union of proper closed subsets $V', V'' \subset V$; note that we do not require $V'$ and $V''$ to be disjoint. If $V \neq \emptyset$ is not reducible, that is whenever $V = V' \cup V''$ is a union of closed subsets we necessarily have $V' = V$ or $V'' = V$, then $V$ is called **irreducible**. A subset $\emptyset \neq W \subseteq V$ is called **(ir)reducible** if it is so with respect to the induced topology.

We present various characterizations of irreducible topological spaces:

**i)** By taking complements, it follows that $V$ is irreducible if and only if whenever $U', U'' \subseteq V$ are open such that $U' \cap U'' = \emptyset$, then we have $U' = \emptyset$ or $U'' = \emptyset$.

**ii)** This can be rephrased as follows: $V$ is irreducible if and only if whenever $\emptyset \neq U', U'' \subseteq V$ are open, then we have $U' \cap U'' \neq \emptyset$.

In particular, any irreducible topological space is **connected**, that is cannot be written as the disjoint union of two non-empty open (hence closed) subsets. (But the converse does not hold in general.)

**iii)** Thus $V$ is irreducible if and only if any non-empty open subset of $V$ is dense:

Let $V$ be irreducible, and let $\emptyset \neq U \subseteq V$ be open; then $V \setminus \overline{U} \subseteq V$ is open, and we have $U \cap (V \setminus \overline{U}) = \emptyset$; thus we have $V \setminus \overline{U} = \emptyset$, that is $\overline{U} = V$.

Conversely, let $V$ be such that any non-empty open subset is dense, and assume there are $\emptyset \neq U', U'' \subseteq V$ open such that $U' \cap U'' = \emptyset$; then $V \setminus U'' \subset V$ is closed, so that $U' \subseteq V \setminus U''$ implies $U' \subseteq \overline{U}' \subseteq V \setminus U'' \subset V$, a contradiction. ♯

**Proposition.** A subset $W \subseteq V$ is irreducible if and only if $\overline{W} \subseteq V$ is.

**Proof.** Let first $U \subseteq V$ be open such $U \cap W = \emptyset$. Then $V \setminus U$ is closed and contains $W$, so that we have $W \subseteq \overline{W} \subseteq V \setminus U$, thus $U \cap \overline{W} = \emptyset$ as well. (Conversely, if $U \cap \overline{W} = \emptyset$ then we trivially have $U \cap W = \emptyset$ as well.)

**i)** Now let $W$ be irreducible, and let $U', U'' \subseteq V$ be open such that $U' \cap \overline{W} \neq \emptyset \neq U'' \cap \overline{W}$. Then we have $U' \cap W \neq \emptyset \neq U'' \cap W$ as well, so that by irreducibility of $W$ we have $\emptyset \neq U' \cap U'' \cap W \subseteq U' \cap U'' \cap \overline{W}$. Thus $\overline{W}$ is irreducible.

**ii)** Let conversely $\overline{W}$ be irreducible, and let $U', U'' \subseteq V$ be open such that $U' \cap W \neq \emptyset \neq U'' \cap W$. Then we trivially have $U' \cap \overline{W} \neq \emptyset \neq U'' \cap \overline{W}$ as well, so that by irreducibility of $\overline{W}$ we have $U' \cap U'' \cap \overline{W} \neq \emptyset$. This in turn implies $U' \cap U'' \cap W \neq \emptyset$ as well. Thus $W$ is irreducible. ♯

**(3.3) Noetherian spaces.** A topological space $V$ is called **Noetherian**, if any strictly descending chain of closed subsets of $V$ terminates, or equivalently if any non-empty subset of closed subsets of $V$ has a minimal element.

By taking complements, $V$ is Noetherian if and only if any strictly ascending chain of open subsets of $V$ terminates, or equivalently if any non-empty subset of open subsets of $V$ has a maximal element.

**Proposition.** Any Noetherian topological space $V$ is **quasi-compact**, that is any open covering of $V$ has a finite subcovering.

**Proof.** Let $\{U_i; i \in \mathcal{I}\}$ be an open covering of $V$, where $\mathcal{I}$ is an index set, and let $\mathcal{M}$ be the set of all finite unions of some of the $U_i$. Since $V$ is Noetherian, we conclude that $\mathcal{M}$ has a maximal element, $U$ say. Assume that $U \neq V$, then there is $i \in \mathcal{I}$ such that $U \subset U \cup U_i$, contradicting the maximality of $U$. Hence we have $U = V$, saying that the given covering has a finite subcovering. ♯

**Proposition.** Let $V \neq \emptyset$ be a Noetherian topological space. Then $V$ can written as an irredundant finite union $V = \bigcup_{i=1}^r V_i$, for some $r \in \mathbb{N}$, of irreducible closed subsets, that is we have $V_i \not\subseteq \bigcup_{j \neq i} V_j$ for all $i$. Moreover, the subsets $V_1, \ldots, V_r$ are precisely the (finitely many) maximal irreducible (closed) subsets of $V$, being called its **irreducible components**.

**Proof.** Assume first that $V$ cannot be written as a finite union of irreducible closed subsets. Hence $V$ is reducible, so that it is a union $V = W_1 \cup W_1'$ of proper closed subsets, where we may assume that $W_1$ is reducible again. Iterating this process yields an infinite strictly descending chain $V \supset W_1 \supset W_2 \supset \cdots$ of closed subsets, a contradiction. Hence we may write $V = \bigcup_{i=1}^r V_i$, for some $r \in \mathbb{N}$, as an irredundant union of irreducible closed subsets.

Then any irreducible closed subset $V' \subseteq V$ is contained in (at least) one of the $V_i$: We have $V' = V' \cap (\bigcup_{i=1}^r V_i) = \bigcup_{i=1}^r (V' \cap V_i)$, where the sets $V' \cap V_i \subseteq V'$ are closed; since $V'$ is irreducible, we have $V' \cap V_i = V'$, for some $i$, thus $V' \subseteq V_i$. Hence the $V_i$ are precisely the maximal irreducible closed subsets of $V$. ♯

**(3.4) Zariski topology.** Let again $K \subseteq L$ be a field extension, and let $n \in \mathbb{N}_0$. We have seen that we have quite a rich supply of algebraic sets: Both $\emptyset$ and $L^n$ are algebraic, and both arbitrary intersections and finite unions of algebraic sets are algebraic again. Thus the set of algebraic subsets of $L^n$ can be considered as the closed subsets of a topology on $L^n$, being called its **($K$-)Zariski topology**. Thus henceforth algebraic subsets will just be called **($K$-)closed**.

The Zariski topology is Noetherian: Let $L^n \supseteq \mathbf{V}_1 \supset \mathbf{V}_2 \supset \cdots \supset \emptyset$ be an infinite strictly descending chain of closed subsets; then the chain $\{0\} \subseteq \mathbf{I}_K(L^n) \subseteq \mathbf{I}_K(V_1) \subset \mathbf{I}_K(V_2) \subset \cdots \trianglelefteq A$ of ideals is strictly increasing, a contradiction. ♯

In particular, the closure of any subset $V \subseteq L^n$, that is the smallest closed set containing $V$, is given as $\overline{V} = \mathbf{V}_L(\mathbf{I}_K(V))$, justifying the notation already used earlier. Moreover, any closed set is the finite union of its irreducible components. Thus the study of closed sets can often be reduced to the case of irreducible ones. We characterize irreducible closed sets algebraically:

**Proposition.** A closed set $\mathbf{V} \subseteq L^n$ is irreducible if and only if $\mathbf{I}_K(\mathbf{V})$ is prime.

**Proof. i)** Let first $\mathbf{V}$ be irreducible. Then $\mathbf{V} \neq \emptyset$ implies that $I := \mathbf{I}_K(\mathbf{V}) \lhd A$ is proper. In order to show that $I$ is prime, let $f, g \in A$ such that $fg \in I$. Then we have $\mathbf{V} = \mathbf{V}_L(I) \subseteq \mathbf{V}_L(fg) = \mathbf{V}_L(f) \cup \mathbf{V}_L(g)$, implying that $\mathbf{V} = (\mathbf{V}_L(f) \cap \mathbf{V}) \cup (\mathbf{V}_L(g) \cap \mathbf{V})$. Since $\mathbf{V}$ is irreducible, we have $\mathbf{V}_L(f) \cap \mathbf{V} = \mathbf{V}$, say, that is $\mathbf{V} \subseteq \mathbf{V}_L(f)$, or equivalently $f \in I$.

**ii)** Let now $I$ be prime. Then $I \lhd A$ being proper implies $\mathbf{V} \neq \emptyset$. In order to show that $\mathbf{V}$ is irreducible, assume to the contrary that $\mathbf{V} = \mathbf{V}' \cup \mathbf{V}''$, where $\mathbf{V}', \mathbf{V}'' \subset \mathbf{V}$ are closed and proper. Letting $I' := \mathbf{I}_K(\mathbf{V}') \trianglelefteq A$ and $I'' := \mathbf{I}_K(\mathbf{V}'') \trianglelefteq A$, we have $I = \mathbf{I}_K(\mathbf{V}) = \mathbf{I}_K(\mathbf{V}' \cup \mathbf{V}'') = \mathbf{I}_K(\mathbf{V}') \cap \mathbf{I}_K(\mathbf{V}'') = I' \cap I''$, where both

$I \subset I'$ and $I \subset I''$. Hence there are $f \in I' \setminus I$ and $g \in I'' \setminus I$. But now we have $fg \in I' \cdot I'' \subseteq I' \cap I'' = I$, a contradiction. ♯

**(3.5) Coordinate algebras. a)** Let $\mathbf{V} \subseteq L^n$ be closed. Then the vanishing ideal $\mathbf{I}_K(\mathbf{V}) \trianglelefteq A$ is radical, thus the (finitely generated) **coordinate algebra** $K[\mathbf{V}] := A/\mathbf{I}_K(\mathbf{V})$ of $\mathbf{V}$ is reduced.

A reduced finitely generated $K$-algebra is called an **affine $K$-algebra**. Actually, any affine $K$-algebra $R$ is the coordinate algebra of a $K$-closed set: Let $L$ be an algebraic closure of $K$. Since $R$ is finitely generated, by $n \in \mathbb{N}_0$ elements say, we have $R \cong A/I$, where since $R$ is reduced we have $I = \sqrt{I}$; hence letting $\mathbf{V} := \mathbf{V}_L(I) \subseteq L^n$ we get $K[\mathbf{V}] = A/\mathbf{I}_K(\mathbf{V}) = A/\mathbf{I}_K(\mathbf{V}_L(I)) = A/\sqrt{I} = A/I \cong R$.

**b)** The closed set $\mathbf{V} \subseteq L^n$, together with its affine coordinate algebra $K[\mathbf{V}]$, is called an **affine ($K$-)variety**. Note that the (induced) Zariski topology on $\mathbf{V}$ can be recovered from $K[\mathbf{V}]$, but that all of this depends on the embedding $\mathbf{V} \subseteq L^n$. The coordinate algebra is decisive for the structure of $\mathbf{V}$:

Any $f \in K[\mathbf{V}]$ induces a 'polynomial' **regular function** $f^\bullet \colon \mathbf{V} \to L \colon v \mapsto f(v)$, where $f(v)$ is given by choosing any representative of $f$ in $A$ modulo $\mathbf{I}_K(\mathbf{V})$. (We will show in (3.6) below that regular functions are actually continuous with respect to the Zariski topology.)

This gives rise to the homomorphism of $K$-algebras $K[\mathbf{V}] \to \mathrm{Maps}(\mathbf{V}, L) \colon f \mapsto f^\bullet$. Since for $g, h \in A$ we have $g^\bullet = h^\bullet$ if and only if $g - h \in \mathbf{I}_K(\mathbf{V})$, we conclude that the latter map is injective. Hence $K[\mathbf{V}]$ can be identified with the algebra of functions it induces, so that $K[\mathbf{V}]$ is also called the **algebra of regular functions** on $\mathbf{V}$. In particular, for $X_i \in K[\mathbf{V}]$ we get the **coordinate function** $(X_i)^\bullet$ mapping $[x_1, \ldots, x_n] \in \mathbf{V}$ to its $i$-th coordinate $x_i \in L$.

**c)** We have the following **algebraic-geometric correspondence**: For any subset $\mathcal{S} \subseteq K[\mathbf{V}]$ we get the (closed) subset

$$\mathbf{V}_{\mathbf{V}}(\mathcal{S}) := \{v \in \mathbf{V}; f(v) = 0 \text{ for all } f \in \mathcal{S}\} = \mathbf{V}_L(\mathcal{S} + \mathbf{I}_K(\mathbf{V})) \subseteq \mathbf{V}.$$

Conversely, for any subset $W \subseteq \mathbf{V}$ we get the (radical) vanishing ideal

$$\mathbf{I}_{\mathbf{V}}(W) := \{f \in K[\mathbf{V}]; f(w) = 0 \text{ for all } w \in W\} = \mathbf{I}_K(W)/\mathbf{I}_K(\mathbf{V}) \trianglelefteq K[\mathbf{V}].$$

Then we have $\overline{W} = \mathbf{V}_{\mathbf{V}}(\mathbf{I}_{\mathbf{V}}(W)) \subseteq \mathbf{V}$. For any closed subset $\mathbf{W} \subseteq \mathbf{V}$ we have

$$K[\mathbf{W}] = A/\mathbf{I}_K(\mathbf{W}) \cong (A/\mathbf{I}_K(\mathbf{V}))/(\mathbf{I}_K(\mathbf{W})/\mathbf{I}_K(\mathbf{V})) = K[\mathbf{V}]/\mathbf{I}_{\mathbf{V}}(\mathbf{W}),$$

where the natural epimorphism $K[\mathbf{V}] \to K[\mathbf{W}]$ with kernel $\mathbf{I}_{\mathbf{V}}(\mathbf{W})$ is given by restricting regular functions on $\mathbf{V}$ to $\mathbf{W}$. Moreover, $\mathbf{W}$ is irreducible if and only if $K[\mathbf{W}]$ is a domain, or equivalently if and only if $\mathbf{I}_{\mathbf{V}}(\mathbf{W}) \trianglelefteq K[\mathbf{V}]$ is prime.

If $\mathbf{V} \subseteq L^n$ is closed, the operator $\mathbf{I}_{\mathbf{V}}$ induces an inclusion-reversing (with respect to set-theoretic inclusion) injective correspondence

$$\{\mathbf{W} \subseteq \mathbf{V} \ K\text{-closed}\} \to \{I \trianglelefteq K[\mathbf{V}] \text{ radical}\},$$

whose inverse on the image of $\mathbf{I}_{\mathbf{V}}$ is the operator $\mathbf{V}_{\mathbf{V}}$. If $L$ is algebraically closed, then $\mathbf{I}_{\mathbf{V}}$ is surjective, and for $I \trianglelefteq K[\mathbf{V}]$ we have $\mathbf{I}_{\mathbf{V}}(\mathbf{V}_{\mathbf{V}}(I)) = \sqrt{I} \trianglelefteq K[\mathbf{V}]$.

**Example. i)** If $L$ is infinite, then we have $\mathbf{I}_K(L^n) = \{0\} \lhd A = K[\mathcal{X}]$, where $\mathcal{X} = \{X_1, \ldots, X_n\}$ are indeterminates, thus $K[L^n] = A/\{0\} \cong A$; since $A$ is a domain, $L^n$ is irreducible.

**ii)** If $K = L$, for $v = [x_1, \ldots, x_n] \in K^n$ letting $I := \langle X_1 - x_1, \ldots, X_n - x_n \rangle \unlhd A$, we have $\mathbf{V}_K(I) = \{v\}$, showing that all singleton sets are closed and irreducible. Moreover, polynomial division shows that $\dim_K(A/I) \leq 1$. Hence from $I \subseteq \mathbf{I}_K(v) \lhd A$ we get $\mathbf{I}_K(v) = I \lhd A$, being maximal, and $K[\{v\}] = A/I \cong K$.

**iii)** If moreover $K = L$ is finite, then all subsets of $K^n$ are closed, the irreducible ones being the singleton subsets. Thus in this case the Zariski topology coincides with the discrete topology; in particular, $K^n$ is reducible for $n \geq 1$.

**(3.6) Regular maps. a)** Let additionally $B := K[\mathcal{Y}]$, where $\mathcal{Y} := \{Y_1, \ldots, Y_m\}$ are (further) indeterminates for some $m \in \mathbb{N}_0$. Moreover, let $\mathbf{V} \subseteq L^n$ and $\mathbf{W} \subseteq L^m$ be closed. A map $\varphi \colon \mathbf{V} \to \mathbf{W}$ is called **regular** or a **($K$-)morphism (of affine varieties)**, if there are 'polynomials' $f_1, \ldots, f_m \in K[\mathbf{V}]$ such that $\varphi(v) = [f_1(v), \ldots, f_m(v)] \in \mathbf{W}$, for all $v \in \mathbf{V}$.

Let $\mathrm{Mor}_K(\mathbf{V}, \mathbf{W})$ be the set of all regular maps from $\mathbf{V}$ to $\mathbf{W}$. In particular, for $\mathbf{W} = L$ we have $\mathrm{Mor}_K(\mathbf{V}, L) = K[\mathbf{V}]$ as sets. Moreover, $\varphi \in \mathrm{Mor}_K(\mathbf{V}, \mathbf{W})$ is called an **isomorphism (of affine varieties)**, if it is bijective and its inverse is regular again; in particular, $\mathrm{id}_{\mathbf{V}} \in \mathrm{Mor}_K(\mathbf{V}, \mathbf{V})$ is an isomorphism.

**b)** Let $\mathrm{Hom}(K[\mathbf{W}], K[\mathbf{V}])$ be the set of all $K$-algebra homomorphisms from $K[\mathbf{W}]$ to $K[\mathbf{V}]$; recall that an algebra homomorphism is an isomorphism if and only if it is bijective. Letting $\varphi \colon \mathbf{V} \to \mathbf{W}$ be regular, by pre-composition we get the **dual** $K$-algebra homomorphism or **comorphism** of coordinate $K$-algebras

$$\varphi^* \colon K[\mathbf{W}] \to K[\mathbf{V}] \colon g \mapsto g \circ \varphi = g(f_1, \ldots, f_m).$$

We only have to show that $\varphi^*$ is well-defined: For $g \in K[\mathbf{W}]$ and $v \in \mathbf{V}$ we get $g(\varphi(v)) = g(f_1(v), \ldots, f_m(v)) = g(f_1, \ldots, f_m)(v)$, where $g(f_1, \ldots, f_m) \in K[\mathbf{V}]$; note that this independent of the choice of representatives for $g \in K[\mathbf{W}]$ and the $f_j \in K[\mathbf{V}]$ in $K[\mathcal{Y}]$ and $K[\mathcal{X}]$, respectively.

**(3.7) Functoriality.** Now we vary the regular map: Let $\psi \in \mathrm{Mor}_K(\mathbf{U}, \mathbf{V})$, where $\mathbf{U}$ is an affine variety. Then it is immediate (by concatenation of 'polynomial' functions) that $\psi\varphi \in \mathrm{Mor}_K(\mathbf{U}, \mathbf{W})$. For $g \in K[\mathbf{W}]$ we have $g \circ (\varphi \circ \psi) = (g \circ \varphi) \circ \psi$, so that we get $(\psi\varphi)^* = \varphi^*\psi^*$. Moreover, we have $(\mathrm{id}_{\mathbf{W}})^* = \mathrm{id}_{K[\mathbf{W}]}$.

**Theorem.** The map $?^* \colon \mathrm{Mor}_K(\mathbf{V}, \mathbf{W}) \to \mathrm{Hom}(K[\mathbf{W}], K[\mathbf{V}]) \colon \varphi \mapsto \varphi^*$ is a bijection. Moreover, $\varphi$ is an isomorphism (of affine varieties) if and only if $\varphi^*$ is an isomorphism (of $K$-algebras), in which case we have $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

**Proof.** Let $\varphi \colon \mathbf{V} \to \mathbf{W}$ be regular, given by $f_1, \ldots, f_m \in K[\mathbf{V}]$. Then for $Y_j \in K[\mathbf{W}]$, where $j \in \{1, \ldots, m\}$, we have $\varphi^*(Y_j) = f_j$, implying injectivity.

Let $\alpha\colon K[\mathbf{W}] \to K[\mathbf{V}]$ be a homomorphism of $K$-algebras. For $j \in \{1, \ldots, m\}$ let $f_j := \alpha(Y_j + \mathbf{I}_K(\mathbf{W})) \in K[\mathbf{V}]$, and let $\varphi\colon \mathbf{V} \to L^m\colon v \mapsto [f_1(v), \ldots, f_m(v)]$ be the associated regular map. Then we have $\varphi^*\colon B \to K[\mathbf{V}]\colon Y_j \mapsto f_j$. Thus by construction $\varphi^*$ factors through $B/\mathbf{I}_K(\mathbf{W}) = K[\mathbf{W}]$. Hence for $f \in \mathbf{I}_K(\mathbf{W})$ we have $\varphi^*(f) = 0$, which for $v \in \mathbf{V}$ implies $f(\varphi(v)) = (\varphi^*(f))(v) = 0$. This says that $\varphi(v) \in \mathbf{V}_L(\mathbf{I}_K(\mathbf{W})) = \mathbf{W}$. Thus we have $\varphi(\mathbf{V}) \subseteq \mathbf{W}$. Hence, slightly abusing notation, we have a regular map $\varphi\colon \mathbf{V} \to \mathbf{W}$, having comorphism $\varphi^*\colon K[\mathbf{W}] \to K[\mathbf{V}]\colon Y_j \mapsto f_j$. This shows $\varphi^* = \alpha$, implying that $?^*$ is surjective.

If $\varphi\colon \mathbf{V} \to \mathbf{W}$ is an isomorphism with inverse $\psi\colon \mathbf{W} \to \mathbf{V}$, then we have $\varphi^*\psi^* = (\psi\varphi)^* = (\mathrm{id}_{\mathbf{W}})^* = \mathrm{id}_{K[\mathbf{W}]}$ and $\psi^*\varphi^* = (\varphi\psi)^* = (\mathrm{id}_{\mathbf{V}})^* = \mathrm{id}_{K[\mathbf{V}]}$, showing that $\varphi^*$ is an isomorphism such that $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

Conversely, if $\varphi^*\colon K[\mathbf{W}] \to K[\mathbf{V}]$ is an isomorphism, then by surjectivity there is a regular map $\psi\colon \mathbf{W} \to \mathbf{V}$ such that $\psi^* = (\varphi^*)^{-1}\colon K[\mathbf{V}] \to K[\mathbf{W}]$. Hence we get $(\varphi\psi)^* = \psi^*\varphi^* = \mathrm{id}_{K[\mathbf{V}]} = (\mathrm{id}_{\mathbf{V}})^*$ and $(\psi\varphi)^* = \varphi^*\psi^* = \mathrm{id}_{K[\mathbf{W}]} = (\mathrm{id}_{\mathbf{W}})^*$, by injectivity implying $\varphi\psi = \mathrm{id}_{\mathbf{V}}$ and $\psi\varphi = \mathrm{id}_{\mathbf{W}}$; thus $\varphi$ is an isomorphism. ♯

**Example.** Being an isomorphism of affine varieties is a property which is strictly stronger than being a homeomorphism, let alone being bijective:

Let $q$ be a prime power, let $\mathbb{F}_q \subseteq \mathbf{F}$ be an algebraic closure, and let $A := \mathbb{F}_q[X] = \mathbb{F}_q[\mathbf{F}]$. The **Frobenius map** $\Phi = \Phi_q\colon \mathbf{F} \to \mathbf{F}\colon x \mapsto x^q$ is a $\mathbb{F}_q$-regular, whose comorphism is the $\mathbb{F}_q$-algebra homomorphism $\Phi^*\colon A \to A\colon X \mapsto X^q$. Then we have $\Phi^*(f) = f(X^q) = f^q$, for $f \in A$, implying that $\Phi^*$ is injective, but not surjective. Thus $\Phi^*$ is not an isomorphism, so $\Phi$ neither is.

Still, since any $x \in \mathbf{F}$ has a unique $q$-th root in $\mathbf{F}$, the map $\Phi$ is bijective, with (non-regular) inverse $\Phi^{-1}\colon \mathbf{F} \to \mathbf{F}\colon x \mapsto x^{\frac{1}{q}}$. Moreover, $\Phi$ fixes all closed subsets of $\mathbf{F}$: Let $\emptyset \neq \mathbf{V} \subset \mathbf{F}$ be closed, thus there is $f \in A$ such that $\mathbf{V} = \mathbf{V}_{\mathbf{F}}(f)$, hence we have $\Phi^{-1}(\mathbf{V}) = \{x^{\frac{1}{q}} \in \mathbf{F}; x \in \mathbf{V}\} = \mathbf{V}_{\mathbf{F}}(\Phi^*(f)) = \mathbf{V}_{\mathbf{F}}(f) = \mathbf{V}$. Thus $\Phi$ is continuous (which also follows from regularity), and is a **closed map**, that is it maps closed sets to closed sets, which implies that $\Phi^{-1}$ is continuous as well, so that $\Phi$ is a homeomorphism. ♯

**(3.8) Theorem: Topological properties.** Let $\varphi\colon \mathbf{V} \to \mathbf{W}$ be regular.

**a)** Then $\varphi$ is continuous (with respect to the Zariski topology). In particular, if $\mathbf{V}$ is irreducible, then so are both $\varphi(\mathbf{V}) \subseteq \mathbf{W}$ and $\overline{\varphi(\mathbf{V})} \subseteq \mathbf{W}$.

**b)** $\varphi^*$ is injective if and only if $\varphi$ is **dominant**, that is $\varphi(\mathbf{V}) \subseteq \mathbf{W}$ is dense.

**c)** $\varphi^*$ is surjective if and only if $\varphi$ is a **closed embedding**, that is $\varphi(\mathbf{V}) \subseteq \mathbf{W}$ is closed such that $\varphi$ induces an isomorphism $\varphi'\colon \mathbf{V} \to \varphi(\mathbf{V})$.

**Proof. a)** Firstly, let $\mathbf{U} \subseteq \mathbf{W}$ be closed, and let $I := \mathbf{I}_{\mathbf{W}}(\mathbf{U}) \trianglelefteq K[\mathbf{W}]$. Then $\mathbf{V}_{\mathbf{W}}(I) = \mathbf{U}$ yields $\varphi^{-1}(\mathbf{U}) = \{v \in \mathbf{V}; \varphi(v) \in \mathbf{U}\} = \{v \in \mathbf{V}; f(\varphi(v)) = 0 \text{ for all } f \in I\} = \{v \in \mathbf{V}; (\varphi^*(f))(v) = 0 \text{ for all } f \in I\} = \{v \in \mathbf{V}; (\varphi^*(I))(v) = 0\} = \mathbf{V}_{\mathbf{V}}(\varphi^*(I))$, which is closed in $\mathbf{V}$ indeed.

Secondly, recall that $U := \varphi(\mathbf{V})$ is irreducible if and only if $\mathbf{U} := \overline{U}$ is. Now let $\mathbf{W}', \mathbf{W}'' \subseteq \mathbf{U}$ be closed such that $\mathbf{U} = \mathbf{W}' \cup \mathbf{W}''$. Then we have $\mathbf{V} = \varphi^{-1}(\mathbf{W}') \cup \varphi^{-1}(\mathbf{W}'')$. Since $\varphi$ is continuous, and $\mathbf{V}$ is irreducible, we may assume that $\varphi^{-1}(\mathbf{W}') = \mathbf{V}$. This implies $U = \varphi(\mathbf{V}) \subseteq \mathbf{W}'$, hence $\mathbf{W}' \subseteq \mathbf{U} = \overline{U} \subseteq \mathbf{W}'$.

**b)** Let $\varphi$ be dominant, and let $f \in \ker(\varphi^*) \trianglelefteq K[\mathbf{W}]$. Then we have $f(\varphi(v)) = (\varphi^*(f))(v) = 0$, for $v \in \mathbf{V}$, implying that $f|_{\varphi(\mathbf{V})} = 0$. Since $\{0\} \subseteq L$ is ($K$-)closed and $f$ is continuous, we infer that $f^{-1}(\{0\}) \subseteq \mathbf{W}$ is closed, hence $\varphi(\mathbf{V}) \subseteq f^{-1}(\{0\})$ entails $\mathbf{W} = \overline{\varphi(\mathbf{V})} \subseteq f^{-1}(\{0\}) \subseteq \mathbf{W}$, thus $f = 0 \in K[\mathbf{W}]$.

Conversely, let $\varphi^*$ be injective, and let $\mathbf{U} := \overline{\varphi(\mathbf{V})} \subseteq \mathbf{W}$. We show that $\mathbf{I}_{\mathbf{W}}(\mathbf{U}) = \{0\}$, implying $\mathbf{U} = \mathbf{V}_{\mathbf{W}}(\mathbf{I}_{\mathbf{W}}(\mathbf{U})) = \mathbf{W}$: Let $f \in \mathbf{I}_{\mathbf{W}}(\mathbf{U})$, then we have $(\varphi^*(f))(v) = f(\varphi(v)) = 0$, for $v \in \mathbf{V}$, implying $\varphi^*(f) = 0$, thus $f = 0$.

**c)** Let $\varphi$ be a closed embedding. Then $\mathbf{U} := \varphi(\mathbf{V}) \subseteq \mathbf{W}$ is closed such that $\varphi = \varphi' \cdot \iota_{\mathbf{U}}^{\mathbf{W}}$, where $\varphi' \colon \mathbf{V} \to \mathbf{U}$ is an isomorphism, and $\iota_{\mathbf{U}}^{\mathbf{W}} \colon \mathbf{U} \to \mathbf{W}$ is the natural inclusion. Moreover, $(\iota_{\mathbf{U}}^{\mathbf{W}})^* = \rho_{\mathbf{U}}^{\mathbf{W}} \colon K[\mathbf{W}] \to K[\mathbf{U}]$ is the restriction epimorphism. Then we have $\varphi^* = (\iota_{\mathbf{U}}^{\mathbf{W}})^* \cdot (\varphi')^* = \rho_{\mathbf{U}}^{\mathbf{W}} \cdot (\varphi')^*$. Since $\rho_{\mathbf{U}}^{\mathbf{W}}$ is surjective, and $(\varphi')^* \colon K[\mathbf{U}] \to K[\mathbf{V}]$ is bijective, $\varphi^*$ is surjective.

Conversely, let $\varphi^*$ be surjective. Then let $\mathbf{U} := \overline{\varphi(\mathbf{V})} \subseteq \mathbf{W}$, and let $\varphi' \colon \mathbf{V} \to \mathbf{U}$ such that $\varphi = \varphi' \cdot \iota_{\mathbf{U}}^{\mathbf{W}}$. Then we have $\varphi^* = (\iota_{\mathbf{U}}^{\mathbf{W}})^* \cdot (\varphi')^* = \rho_{\mathbf{U}}^{\mathbf{W}} \cdot (\varphi')^*$. Since $\varphi^*$ is surjective, so is $(\varphi')^*$. Moreover, $\varphi'$ is dominant, thus $(\varphi')^*$ is injective. Thus $(\varphi')^*$ is an isomorphism, hence so is $\varphi'$, in particular $\varphi(\mathbf{V}) = \mathbf{U}$ is closed. ♯

**Example.** The image of a regular map needs neither be closed nor open:

Let $K = L = \mathbb{C}$, let $\mathbf{V} = \mathbb{C}^2$, let $A := \mathbb{C}[X, Y] \cong \mathbb{C}[\mathbf{V}]$, and let $\varphi \colon \mathbf{V} \to \mathbf{V} \colon [x, y] \mapsto [xy, y]$, being regular with $f_x = XY \in A$ and $f_y = Y \in A$. The associated comorphism is the $\mathbb{C}$-algebra homomorphism given by $\varphi^* \colon A \to A \colon X \mapsto XY, Y \mapsto Y$. Hence $\varphi^*(\sum_{i,j \geq 0} a_{ij} X^i Y^j) = \sum_{i,j \geq 0} a_{ij} X^i Y^{i+j}$ shows that $\varphi^*$ is injective, but $X \notin \varphi^*(A)$ shows that $\varphi^*$ is not surjective.

We have $\varphi(\mathbf{V}) = \{[0, 0]\} \,\dot{\cup}\, (\mathbb{C} \times \mathbb{C}^*) \subseteq \mathbf{V}$, that is $\mathbf{V} \setminus \varphi(\mathbf{V}) = \mathbb{C}^* \times \{0\}$. Since $\mathbf{V}$ is irreducible, so is $\varphi(\mathbf{V})$. The injectivity of $\varphi^*$ implies that $\varphi$ is dominant, in particular $\varphi(\mathbf{V}) \subset \mathbf{V}$ is not closed. We show that $\varphi(\mathbf{V}) \subset \mathbf{V}$ is not open: Assume to the contrary that $\varphi(\mathbf{V}) \subset \mathbf{V}$ is open, then $\mathbf{V} \setminus \varphi(\mathbf{V}) \subset \mathbf{V}$ is closed; since $\{[0, 0]\} \subseteq \varphi(\mathbf{V})$ is closed as well, there is $f \in A$ such that $f(x, 0) = 0$ for all $x \neq 0$, but $f(0, 0) \neq 0$, a contradiction. ♯

**(3.9) Example: Coordinate algebras of curves.** Let $K = L = \mathbb{C}$, and let $n = 2$, hence $A := \mathbb{C}[X, Y]$. (We leave out the subscripts in the notation for the operators $\mathbf{V}_{\mathbb{C}}$ and $\mathbf{I}_{\mathbb{C}}$.)

**a)** We consider the curve $\mathbf{C} := \mathbf{V}(f) \subseteq \mathbb{C}^2$ given by $f := Y^2 - X^2 \in A$. We have $f = (Y - X)(Y + X) \in A$, so that $\mathbf{C}$ is reducible. We have $\mathbf{C} = \mathbf{V}((Y - X)(Y + X)) = \mathbf{V}(Y - X) \cup \mathbf{V}(Y + X)$.

Since both $Y \pm X \in A$ are irreducible, both $\mathbf{V}(Y \pm X)$ are irreducible, and we have $\mathbb{C}[\mathbf{V}(Y \pm X)] = A/\langle Y \pm X \rangle \cong \mathbb{C}[X]$, so that $\mathbf{V}(Y \pm X) \cong \mathbb{C}$ is isomorphic to the affine line. Since the above union is irredundant, we conclude that $\mathbf{V}(Y \pm X)$ are the irreducible components of $\mathbf{C}$; note that $\mathbf{V}(Y-X) \cap \mathbf{V}(Y+X) = \{0\} \neq \emptyset$.

Moreover, $\langle f \rangle \trianglelefteq A$ is radical: Let $g \in \sqrt{\langle f \rangle}$, then $f \mid g^k$ for some $k \in \mathbb{N}$; since $Y \pm X \in A$ are non-associate irreducible, and $A$ is factorial, we conclude that both $Y \pm X \mid g$; thus $f \mid g$, that is $g \in \langle f \rangle$. Hence we have $\mathbb{C}[\mathbf{C}] = A/\langle f \rangle = \mathbb{C}[X,Y]/\langle (Y-X)(Y+X) \rangle$; actually $Y \pm X \in \mathbb{C}[\mathbf{C}]$ are zero-divisors.

**b)** We now consider the curve $\mathbf{C} := \mathbf{V}(f) \subseteq \mathbb{C}^2$, where $f := Y^2 - X^3 \in A$. Since $X^3 \in \mathbb{C}[X]$ is not a square, $f$ is irreducible, hence so is $\mathbf{C}$. We have $\mathbb{C}[\mathbf{C}] = A/\langle f \rangle = \mathbb{C}[X,Y]/\langle Y^2 - X^3 \rangle \cong \mathbb{C}[X, \sqrt{X^3}] \cong \mathbb{C}[Y, \sqrt[3]{Y^2}]$, which is a domain. (But it is not a polynomial algebra, which is not too easily proved.)

We consider the regular map $\varphi \colon \mathbb{C} \to \mathbf{C} \colon t \mapsto [t^2, t^3]$, which since $f(\varphi(t)) = 0$ is well-defined indeed. Moreover, let $\psi \colon \mathbf{C} \to \mathbb{C}$ be given by $\psi(0,0) = 0$ and $\psi(x,y) = \frac{y}{x}$, for $[x,y] \neq [0,0]$. Then we have $\psi(\varphi(t)) = \frac{t^3}{t^2} = t$ for $t \neq 0$, and $\psi(\varphi(0)) = 0$; and the other way around $\varphi(\psi(x,y)) = [\frac{y^2}{x^2}, \frac{y^3}{x^3}] = [\frac{x^3}{x^2}, \frac{y^3}{y^2}] = [x,y]$ for $[x,y] \neq [0,0]$, and $\varphi(\psi(0,0)) = [0,0]$. Thus $\varphi$ is bijective. More generally, a dominant regular map $\varphi \colon \mathbb{C} \to \mathbf{C}$ is also called a **parametrisation** of $\mathbf{C}$.

The associated comorphism is given as $\varphi^* \colon \mathbb{C}[\mathbf{C}] \to \mathbb{C}[T] \colon X \mapsto T^2, Y \mapsto T^3$. Since $\varphi$ is dominant, we conclude that $\varphi^*$ is injective. But since $T \notin \varphi^*(\mathbb{C}[\mathbf{C}])$ we conclude that $\varphi^*$ is not surjective. Hence $\varphi$ is not an isomorphism, where $\varphi^{-1}$ actually is a **rational map**.

## 4 Projective varieties

**(4.1) Projective spaces.** Let $L$ be a field, and let $n \in \mathbb{N}_0$. Then the **($n$-dimensional) projective space** $\mathbf{P} := \mathbf{P}^n(L)$ over $L$ is defined as the set of equivalence classes in $L^{n+1} \setminus \{0_{n+1}\}$ with respect to the equivalence relation given by $v \sim \lambda v$, for all $\lambda \in L^\sharp$. In particular, $\mathbf{P}^0$ is a singleton set, while $\mathbf{P}^1$ and $\mathbf{P}^2$ are called the **projective line** and the **projective plane**, respectively.

The equivalence class in $\mathbf{P}$ containing the **point** $0_{n+1} \neq [x_0, x_1, \ldots, x_n] \in L^{n+1}$ is denoted by $[x_0 \colon x_1 \colon \ldots \colon x_n]$, where the entries $x_i \in L$ are called the associated **homogeneous coordinates**. Thus $\mathbf{P}$ can be identified with the set of 1-dimensional $L$-subspaces of $L^{n+1}$, via $[x_0 \colon \ldots \colon x_n] \mapsto \langle [x_0, \ldots, x_n] \rangle_L$.

For $i \in \{0, \ldots, n\}$ let $D_i := \{[x_0 \colon \ldots \colon x_n] \in \mathbf{P}; x_i \neq 0\} \subseteq \mathbf{P}$. Then we have $\mathbf{P} = \bigcup_{i=0}^n D_i$, where $D_i$ can be identified with the $n$-dimensional affine space $L^n$, via **dehomogenizing** and **homogenizing** at position $i$, respectively:

$$D_i \to L^n \colon [x_0 \colon \ldots \colon x_n] \mapsto [\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}],$$

$$L^n \to D_i \colon [x_0, \ldots, \widehat{x}_i, \ldots, x_n] \mapsto [x_0 \colon \ldots \colon x_{i-1} \colon 1 \colon x_{i+1}, \ldots, x_n],$$

where we write $[x_0, \ldots, \widehat{x}_i, \ldots, x_n] := [x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \in L^n$.

Thus for $n = 1$ we have $\mathbf{P}^1 = D_0 \,\dot\cup\, \{[0 \colon 1]\}$, where (for historical reasons) the point $[0 \colon 1]$ is abbreviated by '$\infty$', so that we may write $\mathbf{P}^1(L) = L \,\dot\cup\, \{\infty\}$. In particular, for $L = \mathbb{C}$ we get the **Riemann sphere** $\mathbf{P}^1(\mathbb{C}) = \mathbb{C} \,\dot\cup\, \{\infty\}$ (which with respect to the complex topology becomes a compact Riemannian surface).

**(4.2) Homogeneous ideals. a)** Let $K$ be a field. A $K$-algebra $A$ is called **(non-negatively) graded**, if there is a decomposition $A = \bigoplus_{d \geq 0} A_d$ as $K$-vector spaces, such that $A_i A_j \subseteq A_{i+j}$, for $i, j \geq 0$. The $K$-subspace $A_d$ is called the **homogeneous component** of **degree** $d$. Moreover, for $c \in \mathbb{N}_0$ we abbreviate $A_{\leq c} := \bigoplus_{d=0}^{c} A_d$ and $A_{<c} := \bigoplus_{d=0}^{c-1} A_d$.

Then we have $1 \in A_0$: Writing $1 = \sum_{i \geq 0} e_i$, from $f_d = 1 \cdot f_d = \sum_{i \geq 0} e_i f_d$, where $f_d \in A_d$, we get $f_d = e_0 f_d$; thus we have $f = e_0 f$ for all $f \in A$, implying $e_0 = 1$. Hence if $A \neq \{0\}$, then we have $K \cong K \cdot 1 \leq A_0$; if $A_0 \cong K$, then $A$ is called an **indecomposable** graded $K$-algebra.

**b)** An ideal $I \trianglelefteq A$ is called **homogeneous** if $I = \bigoplus_{d \geq 0} I_d$, where $I_d := I \cap A_d$; in other words, an element of $A$ belongs to $I$ if and only if all its homogeneous components belong to $I$. We may characterize homogeneous ideals as follows:

**i)** An ideal $I$ is homogeneous if and only if it is generated by homogeneous elements: If $I$ is homogeneous, then it is generated by its homogeneous elements. Conversely, if $I$ is generated by homogeneous elements, $g \in A_d$ say, then the equation $g \cdot (\sum_{i \geq 0} f_i) = \sum_{i \geq 0} (g f_i)$ implies that the homogeneous components of any element of $I$ belong to $I$ as well, thus $I$ is homogeneous. ♯

**ii)** An ideal $I$ is homogeneous if and only if $A/I$ is a graded $K$-algebra again, with respect to the inherited grading: If $I$ is homogeneous, then we have $A/I = (\bigoplus_{d \geq 0} A_d)/(\bigoplus_{d \geq 0} I_d) \cong \bigoplus_{d \geq 0} A_d/I_d$, where $(A_i/I_i) \cdot (A_j/I_j) \subseteq A_{i+j}/I_{i+j}$, for $i, j \geq 0$. Conversely, if $A/I$ is naturally graded again, that is we have $A/I \cong \bigoplus_{d \geq 0} (A_d + I)/I \cong \bigoplus_{d \geq 0} A_d/I_d$, where the second isomorphism follows from the Homomorphism Theorem, then for $f = \sum_{i \geq 0} f_i \in A$ we have $f \in I$ if and only if $f_i \in I_i$ for all $i \geq 0$; thus $I$ is homogeneous. ♯

In particular, if $A$ is indecomposable, then $A_+ := \bigoplus_{d \geq 1} A_d \lhd A$ is the unique maximal homogeneous ideal, being called the **irrelevant ideal** (for a reason becoming clear soon); we have $A/A_+ \cong A_0 \cong K$, hence $A_+$ is maximal.

**c)** We collect a few straightforward properties of homogeneous ideals:

It is immediate that arbitrary sums, arbitrary intersections, and (finite) products of homogeneous ideals are homogeneous. Moreover, for $I \trianglelefteq A$ homogeneous and $J \trianglelefteq A$ we have the following: If $J$ is homogeneous, then so is $(J+I)/I \trianglelefteq A/I$; if moreover $I \subseteq J$ such that $J/I \trianglelefteq A/I$ is homogeneous, then $J$ is homogeneous.

Slightly more subtle are the following properties:

**i)** If $I \trianglelefteq A$ is homogeneous, then so is $\sqrt{I} \trianglelefteq A$: Going over to the graded $K$-algebra $A/I$, it suffices to show that $\mathrm{nil}(A) \trianglelefteq A$ is homogeneous. Hence let $0 \neq f = \sum_{i=0}^{d} f_i \in A$, for some $d \in \mathbb{N}_0$ such that $f_d \neq 0$, be nilpotent, that is there is $k \in \mathbb{N}$ such that $0 = f^k = (f_d)^k \in A/A_{<kd}$. We conclude that $(f_d)^k = 0$, thus $f_d$ is nilpotent as well. Going over to $f - f_d$, the assertion now follows from induction with respect to $d$. ♯

**ii)** Finally, given $I \lhd A$ homogeneous, in order to check whether $I$ is prime, it suffices to show for $f, g \in A$ homogeneous only, that $fg \in I$ implies $f \in I$ or $g \in I$: Assume that $I$ has the above property, and let $0 \neq f = \sum_{i=0}^{d} f_i \in A$ and $0 \neq g = \sum_{i=0}^{c} g_i \in A$, where $f_d \neq 0 \neq g_c$, be arbitrary elements such that $fg \in I$. Then we have $fg = f_d g_c \in A/A_{<(c+d)}$, thus since $I$ is homogeneous we have $f_d g_c \in I$ as well. By assumption, we have $f_d \in I$, say. Going over to $f - f_d$, it follows from induction with respect to $c + d$ that $f \in I$ or $g \in I$. $\sharp$

**Example.** Let $\mathcal{X} := \{X_1, \ldots, X_n\}$ and $\mathcal{X}^\sharp := \{X_0\} \,\dot\cup\, \mathcal{X}$ be indeterminates, for $n \in \mathbb{N}_0$. Then $A^\sharp := K[\mathcal{X}^\sharp]$ becomes an indecomposable graded $K$-algebra with respect to the total degree, where the homogeneous component $A^\sharp_d$ is the finite-dimensional $K$-subspace generated by the monomials of degree $d$.

**(4.3) Projective algebraic sets.** Let $K \subseteq L$ be a field extension. The elements of $A^\sharp$ do *not* induce functions on $\mathbf{P}$. But still, whenever $f \in A^\sharp$ is homogeneous of degree $d$, for $[x_0, \ldots, x_n] \in L^{n+1}$ we have $f(\lambda \cdot [x_0, \ldots, x_n]) = \lambda^d \cdot f(x_0, \ldots, x_n)$, for all $0 \neq \lambda \in L$, so that $f(\lambda \cdot [x_0, \ldots, x_n]) \neq 0$ if and only if $f([x_0, \ldots, x_n]) \neq 0$. Thus the vanishing set in $\mathbf{P}$ of $f$ is well-defined, by taking representatives in $L^{n+1}$ of the equivalence classes in $\mathbf{P}$.

Hence, let $\mathcal{S} \subseteq A^\sharp$ be a subset consisting of homogeneous polynomials. Then

$$\mathbf{V}^\sharp_L(\mathcal{S}) := \{v \in \mathbf{P}; f(v) = 0 \text{ for all } f \in \mathcal{S}\} \subseteq \mathbf{P}$$

is called the **(projective) ($K$-)algebraic subset** being **defined** by $\mathcal{S}$; then $K$ and $L$ are called its **field of definition** and **field of coordinates**, respectively.

If $I \unlhd A^\sharp$ is homogeneous, let $\mathbf{V}^\sharp_L(I) := \mathbf{V}^\sharp_L(\{f \in I; f \text{ homogeneous}\})$. In particular, we have $\mathbf{V}^\sharp_L(\mathcal{S}) = \mathbf{V}^\sharp_L(\langle \mathcal{S} \rangle)$. Thus by Hilbert's Basis Theorem there are $f_1, \ldots, f_r \in \mathcal{S}$, for some $r \in \mathbb{N}_0$, such that $\mathbf{V}^\sharp_L(\mathcal{S}) = \mathbf{V}^\sharp_L(f_1, \ldots, f_r)$. Hence any projective algebraic set is defined by a homogeneous ideal, or alternatively by finitely many homogeneous polynomials. Moreover, for $I \subseteq J \unlhd A^\sharp$ homogeneous we have $\mathbf{V}^\sharp_L(J) \subseteq \mathbf{V}^\sharp_L(I)$, and $\mathbf{V}^\sharp_L(\sqrt{I}) = \mathbf{V}^\sharp_L(I)$.

For example, we have $\mathbf{V}^\sharp_L(0) = \mathbf{P}$, and $\mathbf{V}^\sharp_L(1) = \emptyset = \mathbf{V}^\sharp_L(\mathcal{X}^\sharp) = \mathbf{V}^\sharp_L(A^\sharp_+)$. More interestingly, an algebraic set defined by a single homogeneous polynomial of degree $d \geq 1$ is called a **(projective) hypersurface** of **degree** $d$; for $n = 2$ the latter is also called a **(projective) curve**.

A hypersurface of degree 1 is called a **(projective) hyperplane**; in particular, we have $\mathbf{H}_i = \mathbf{H}_{X_i} := \mathbf{V}^\sharp_L(X_i) = \{[x_0 \colon \ldots \colon x_n] \in \mathbf{P}; x_i = 0\} = \mathbf{P} \setminus D_i$, for $i \in \{0, \ldots, n\}$. The elements of $\mathbf{H}_0 = \mathbf{P} \setminus D_0$ are called (for historical reasons) the **points at infinity** of $\mathbf{P}$; note that this depends on the coordinates chosen, so that any hyperplane can be deemed to be at infinity. Moreover, for $n \geq 1$ the hyperplane $\mathbf{H}_0$ can be identified with the projective space $\mathbf{P}^{n-1}$, via $[0 \colon x_1 \colon \ldots \colon x_n] \mapsto [x_1 \colon \ldots \colon x_n]$, so that, using the identification of $D_0$ with $L^n$ as well, we get $\mathbf{P} = D_0 \,\dot\cup\, \mathbf{H}_0 = L^n \,\dot\cup\, \mathbf{P}^{n-1}$.

**(4.4) Projective algebraic sets and their ideals.** Let $V \subseteq \mathbf{P}$ be any subset. Then its **vanishing ideal** is given as

$$\mathbf{I}_K^\sharp(V) := \langle \{f \in A^\sharp \text{ homogeneous}; f(v) = 0 \text{ for all } v \in V\} \rangle \trianglelefteq A^\sharp.$$

Then we have $\mathbf{I}_K^\sharp(V) = \sqrt{\mathbf{I}_K^\sharp(V)}$, and for $V \subseteq W$ we get $\mathbf{I}_K^\sharp(W) \subseteq \mathbf{I}_K^\sharp(V)$; in particular we have $\mathbf{I}_K^\sharp(\emptyset) = A^\sharp$.

We consider the interplay between the operators $\mathbf{V}_L^\sharp$ and $\mathbf{I}_K^\sharp$, where entirely similar to the affine case treated in (2.6) we observe the following:

For any $V \subseteq \mathbf{P}$ and any $I \trianglelefteq A^\sharp$ homogeneous we get $V \subseteq \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(V))$ and $I \subseteq \mathbf{I}_K^\sharp(\mathbf{V}_L^\sharp(I))$, which entails $\mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{V}_L^\sharp(I))) = \mathbf{V}_L^\sharp(I)$ and $\mathbf{I}_K^\sharp(\mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(V))) = \mathbf{I}_K^\sharp(V)$. In particular, for algebraic subsets $\mathbf{V}, \mathbf{W} \subseteq \mathbf{P}$ we get $\mathbf{V} \subseteq \mathbf{W}$ if and only if $\mathbf{I}_K^\sharp(\mathbf{W}) \subseteq \mathbf{I}_K^\sharp(\mathbf{V})$, and $\mathbf{V} = \mathbf{W}$ if and only if $\mathbf{I}_K^\sharp(\mathbf{W}) = \mathbf{I}_K^\sharp(\mathbf{V})$; in particular, if $\mathbf{V} \neq \emptyset$ then from $\mathbf{V}_L^\sharp(A_+^\sharp) = \emptyset$ we get $\mathbf{I}_K^\sharp(\mathbf{V}) \subset A_+^\sharp$.

We consider (arbitrary) intersections and (finite) unions of algebraic sets:

Firstly, for algebraic sets $\mathbf{V}_i \subseteq \mathbf{P}$, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, we have $\mathbf{V}_L^\sharp(\sum_{i \in \mathcal{I}} \mathbf{I}_K^\sharp(\mathbf{V}_i)) = \bigcap_{i \in \mathcal{I}} \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{V}_i)) = \bigcap_{i \in \mathcal{I}} \mathbf{V}_i$; in particular, an arbitrary intersection of algebraic sets is algebraic again.

Secondly, for algebraic sets $\mathbf{V}, \mathbf{W} \subseteq \mathbf{P}$ we have $\mathbf{V} \cup \mathbf{W} = \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{V}) \cap \mathbf{I}_K^\sharp(\mathbf{W})) = \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{V}) \cdot \mathbf{I}_K^\sharp(\mathbf{W}))$; in particular, a finite union of algebraic sets is algebraic again. (This again is obtained entirely similar to (2.6), by observing that the polynomials occurring can be chosen homogeneous.)

Thus the smallest algebraic set containing $V \subseteq \mathbf{P}$ is given as $\overline{V} := \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(V))$.

**(4.5) Projective varieties. a)** By the properties proved above, the set of algebraic subsets of $\mathbf{P}$ form the closed subsets of a topology, called the **($K$-)Zariski topology**. Thus an algebraic subset $\mathbf{V} \subseteq \mathbf{P}$ will just be called **($K$-)closed**, and it carries the induced topology, also being called its Zariski topology.

The reduced finitely generated graded $K$-algebra $K[\mathbf{V}] := A^\sharp/\mathbf{I}_K^\sharp(\mathbf{V})$ is called the **homogeneous coordinate algebra** of $\mathbf{V}$. Then $\mathbf{V}$, together with its homogeneous coordinate algebra $K[\mathbf{V}]$, is (for the time being) called a **projective ($K$-)variety**. Note that the Zariski topology on $\mathbf{V}$ can be recovered from $K[\mathbf{V}]$, but that all of this depends on the embedding $\mathbf{V} \subseteq \mathbf{P}$.

Actually, since $K[\mathbf{V}]$ does *not* induce functions on $\mathbf{V}$, it is not a suitable invariant attached to $\mathbf{V}$; in particular, even its isomorphism type depends on the embedding $\mathbf{V} \subseteq \mathbf{P}$. Thus we will have to define a suitable $K$-algebra of 'regular functions' on $\mathbf{V}$ to remedy this, see (8.4) below. Anyway:

Going over to vanishing ideals in $A^\sharp$, we conclude that the Zariski topology is Noetherian, so that any closed subset of $\mathbf{P}$ is the finite union of its irreducible components. In view of this, we observe that a closed subset $\mathbf{V} \subseteq \mathbf{P}$ is irreducible if and only if its vanishing ideal $\mathbf{I}_K^\sharp(\mathbf{V}) \trianglelefteq A^\sharp$ is prime:

We have $\mathbf{V} \neq \emptyset$ if and only if $I := \mathbf{I}_K^\sharp(\mathbf{V}) \subset A^\sharp$. To show that $I$ is prime, it suffices to check for $f, g \in A^\sharp$ homogeneous that $fg \in I$ implies $f \in I$ or $g \in I$; thus we may proceed entirely similar to the affine case treated in (3.4). $\sharp$

**b)** We have the following **algebraic-geometric correspondence**: The operator $\mathbf{I}_K^\sharp$ induces an inclusion-reversing (with respect to set-theoretic inclusion) injective correspondence

$$\mathbf{I}_K^\sharp \colon \{\mathbf{V} \subseteq \mathbf{P} \text{ projective } K\text{-closed}\} \to \{A_+^\sharp \neq I \trianglelefteq A^\sharp \text{ radical homogeneous}\},$$

whose inverse on the image of $\mathbf{I}_K^\sharp$ is given by the operator $\mathbf{V}_L^\sharp$. Moreover, a closed subset $\mathbf{V} \subseteq \mathbf{P}$ is irreducible if and only if $\mathbf{I}_K^\sharp(\mathbf{V}) \trianglelefteq A^\sharp$ is prime. In addition, if $L$ is algebraically closed then $\mathbf{I}_K^\sharp$ is surjective, which follows from:

**(4.6) Theorem: Hilbert's Nullstellensatz (projective version).**
Let $L$ be algebraically closed, let $I \trianglelefteq A^\sharp$ be homogeneous, and let $\mathbf{V} := \mathbf{V}_L^\sharp(I)$. Then precisely one of the following cases occurs:
**i)** We have $\mathbf{V} = \emptyset$ and $A_+^\sharp \subseteq \sqrt{I} \trianglelefteq A^\sharp = \mathbf{I}_K^\sharp(\mathbf{V})$.
**ii)** We have $\mathbf{V} \neq \emptyset$ and $\mathbf{I}_K^\sharp(\mathbf{V}) = \sqrt{I} \subset A_+^\sharp \triangleleft A^\sharp$.

**Proof.** A subset $\emptyset \neq V \subseteq L^{n+1}$ is called an **(affine) cone** if for any $v \in V$ we have $\langle v \rangle_L \subseteq V$ as well; in particular we have $0_{n+1} \in V$, and $\{0_{n+1}\}$ is a cone.

We may assume that $I \triangleleft A^\sharp$, and let $\widetilde{\mathbf{V}} := \mathbf{V}_L(I) \subseteq L^{n+1}$ be the associated affine closed subset. Then by Hilbert's Nullstellensatz we have $\widetilde{\mathbf{V}} \neq \emptyset$, such that $\mathbf{I}_K(\widetilde{\mathbf{V}}) = \sqrt{I}$. Then we conclude that $\widetilde{\mathbf{V}}$ is a closed cone, where $\widetilde{\mathbf{V}} \setminus \{0_{n+1}\}$ consists precisely of the equivalence classes with respect to $\sim$ belonging to $\mathbf{V}$.

We have $\widetilde{\mathbf{V}} = \{0_{n+1}\}$ if and only if $\sqrt{I} = \langle \mathcal{X}^\sharp \rangle = A_+^\sharp \triangleleft A^\sharp$. In this case we have $\mathbf{V} = \emptyset$ and $\mathbf{I}_K^\sharp(\mathbf{V}) = A^\sharp$; otherwise, $\mathbf{V} \neq \emptyset$ and $\mathbf{I}_K^\sharp(\mathbf{V}) = \mathbf{I}_K(\widetilde{\mathbf{V}}) = \sqrt{I} \subseteq A_+^\sharp$. $\sharp$

**(4.7) Homogenization of ideals. a)** Given $f \in A^\sharp$, the specialization epimorphism of $K$-algebras $A^\sharp \to A$ defined by $X_0 \mapsto 1$ yields the **dehomogenization** $f' := f(1, X_1, \ldots, X_n) \in A$. Conversely, for $0 \neq f \in A$ we let $f^\sharp := X_0^{\deg(f)} \cdot f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}) \in A^\sharp$ be its **homogenization** with respect to $X_0$; for completeness we let $0^\sharp := 0$. Then we have $X_0 \nmid f^\sharp$ for $0 \neq f \in A$, and multiplicativity $(fg)^\sharp = f^\sharp g^\sharp$ holds for $f, g \in A$.

For $0 \neq f \in A$ we have $(f^\sharp)' = (X_0^{\deg(f)} \cdot f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}))' = f(X_1, \ldots, X_n) = f$. Conversely, for $0 \neq f \in A^\sharp$ homogeneous, letting $\nu(f) = \nu_{X_0}(f) := \deg(f) - \deg(f') \in \mathbb{N}_0$, that is $X_0^{\nu(f)}$ is the highest power of $X_0$ dividing $f$, we have $(f')^\sharp = f(1, X_1, \ldots, X_n)^\sharp = X_0^{\deg(f')} \cdot f(1, \frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}) = X_0^{-\nu(f)} \cdot f(X_0, X_1, \ldots, X_n)$.

**b)** For $I \trianglelefteq A^\sharp$ homogeneous let $I' := \{f' \in A; f \in I\} \trianglelefteq A$ be its **dehomogenization**, and for $I \trianglelefteq A$ let $I^\sharp := \langle f^\sharp \in A^\sharp; f \in I \rangle \trianglelefteq A^\sharp$ be its **homogenization**.

In particular, let $I \trianglelefteq A$ be homogeneous. Then we have $I^\sharp = \langle f^\sharp \in A^\sharp; f \in I \text{ homogeneous}\rangle \trianglelefteq A^\sharp$. Now, let $I = \langle f_1, \ldots, f_r\rangle \trianglelefteq A$ for some $r \in \mathbb{N}_0$, where the $f_i$ are homogeneous; hence we have $f_i^\sharp = f_i$. Then the homogeneous elements of $I$ are of the form $\sum_{i=1}^r f_i g_i \in I$, where the $g_i \in A$ are homogeneous such that $\deg(f_i) + \deg(g_i)$ is constant, for $i \in \{1, \ldots, r\}$. Hence we have $(\sum_{i=1}^r f_i g_i)^\sharp = \sum_{i=1}^r f_i^\sharp g_i^\sharp = \sum_{i=1}^r f_i g_i^\sharp$. Thus in this case we have $I^\sharp = \langle f_1, \ldots, f_r\rangle \trianglelefteq A^\sharp$.

For $I \trianglelefteq A$ we have $(I^\sharp)' = (\langle f^\sharp \in A^\sharp; f \in I\rangle_{A^\sharp})' = \langle (f^\sharp)' \in A; f \in I\rangle_A = I$. Conversely, for $I \trianglelefteq A^\sharp$ homogeneous we have $(I')^\sharp = \{f' \in A; f \in I\}^\sharp = \langle (f')^\sharp \in A^\sharp; f \in I\rangle_{A^\sharp} = \langle X_0^{-\nu(f)} \cdot f \in A^\sharp; f \in I \text{ homogeneous}\rangle_{A^\sharp}$; in particular $I \subseteq (I')^\sharp$, such that for any $f \in (I')^\sharp$ homogeneous we have $X_0^k \cdot f \in I$ for some $k \in \mathbb{N}_0$.

**Proposition. a) i)** An ideal $I \trianglelefteq A$ is radical if and only if $I^\sharp \trianglelefteq A^\sharp$ is radical.
**ii)** If a homogeneous ideal $I \trianglelefteq A^\sharp$ is radical, then $I' \trianglelefteq A$ is radical as well.
**iii)** If $I = (I')^\sharp \trianglelefteq A^\sharp$ and $I' \trianglelefteq A$ is radical, then $I$ is radical.

**b) i)** An ideal $I \trianglelefteq A$ is prime if and only if $I^\sharp \trianglelefteq A^\sharp$ is prime.
**ii)** If a homogeneous ideal $I \trianglelefteq A^\sharp$ is prime, then $I' = A$ or $I' \trianglelefteq A$ is prime.
**iii)** If $I = (I')^\sharp \trianglelefteq A^\sharp$ and $I' \trianglelefteq A$ is prime, then $I$ is prime.

**Proof. a) i)** Let $I^\sharp \trianglelefteq A^\sharp$ be radical, and let $f \in \sqrt{I} \trianglelefteq A$. Then we have $f^k \in I$ for some $k \in \mathbb{N}$. Thus we get $(f^\sharp)^k = (f^k)^\sharp \in I^\sharp = \sqrt{I^\sharp}$. This implies $f^\sharp \in I^\sharp$, so that $f = (f^\sharp)' \in (I^\sharp)' = I$.

Conversely, let $I \trianglelefteq A$ be radical, and let $f \in \sqrt{I^\sharp} \trianglelefteq A^\sharp$ homogeneous. Then we have $f^k \in I^\sharp$ for some $k \in \mathbb{N}$. Thus we get $(f')^k = (f^k)' \in (I^\sharp)' = I = \sqrt{I}$. This implies $f' \in I$, so that $(f')^\sharp = X_0^{-\nu(f)} \cdot f \in I^\sharp$, thus $f \in I^\sharp$.

**ii)** Let $I \trianglelefteq A^\sharp$ be radical, and let $f \in \sqrt{I'} \trianglelefteq A$. Then we have $f^k \in I'$ for some $k \in \mathbb{N}$. Thus we get $(f^\sharp)^k = (f^k)^\sharp \in (I')^\sharp$. Since $\nu(f^\sharp) = 0$ we conclude that $(f^\sharp)^k \in I = \sqrt{I}$. This implies $f^\sharp \in I$, so that $f = (f^\sharp)' \in I'$.

**iii)** Conversely, let $I' \trianglelefteq A$ be radical, and let $f \in \sqrt{I} \trianglelefteq A^\sharp$ homogeneous. Then we have $f^k \in I$ for some $k \in \mathbb{N}$. Thus we get $(f')^k = (f^k)' \in I' = \sqrt{I'}$. This implies $f' \in I'$, so that $(f')^\sharp = X_0^{-\nu(f)} \cdot f \in (I')^\sharp$, hence by assumption $f \in (I')^\sharp = I$.

**b) i)** Let $I^\sharp \trianglelefteq A^\sharp$ be prime. Assume that $I = A$, then $1 \in I^\sharp$, a contradiction. Hence $I \triangleleft A$, and we let $f, g \in A$ such that $fg \in I$. Then we have $f^\sharp g^\sharp = (fg)^\sharp \in I^\sharp$. Thus we may assume that $f^\sharp \in I^\sharp$. This implies $f = (f^\sharp)' \in (I^\sharp)' = I$.

Conversely, let $I \trianglelefteq A$ be prime. Assume that $I^\sharp = A^\sharp$, then $1 = 1' \in (I^\sharp)' = I$, a contradiction. Hence we have $I^\sharp \triangleleft A^\sharp$, and we let $f, g \in A^\sharp$ be homogeneous such that $fg \in I^\sharp$. Then we have $f'g' = (fg)' \in (I^\sharp)' = I$. Thus we may assume that $f' \in I$. This implies $(f')^\sharp = X_0^{-\nu(f)} \cdot f \in I^\sharp$, hence $f \in I^\sharp$.

**ii)** Let $I \trianglelefteq A^\sharp$ be prime. If $X_0 \in I$, then we have $1 \in I' = A$. Thus we may assume that $X_0 \notin I$ and $I' \triangleleft A$. Now let $f, g \in A$ such that $fg \in I'$. Then we have $f^\sharp g^\sharp = (fg)^\sharp \in (I')^\sharp$, hence $X_0^k f^\sharp g^\sharp \in I$ for some $k \in \mathbb{N}_0$. Since $X_0 \notin I$, we may assume that $f^\sharp \in I$, implying $f = (f^\sharp)' = I'$. Thus $I'$ is prime.

**iii)** Conversely, let $I' \unlhd A$ be prime. Assume that $I = A$, then $1 = 1' \in I'$, a contradiction. Hence we have $I \lhd A$, and we let $f, g \in A^\sharp$ be homogeneous such that $fg \in I$. Then we have $f'g' = (fg)' \in I'$. Thus we may assume that $f' \in I'$. This implies $(f')^\sharp = X_0^{-\nu(f)} \cdot f \in (I')^\sharp$, hence by assumption $f \in (I')^\sharp = I$. $\sharp$

**(4.8) Affine open subsets.** We consider $D_0 = \mathbf{P} \setminus \mathbf{H}_0$, which is open with respect to the Zariski topology on $\mathbf{P}$. Recall that $D_0$ can be identified with $L^n$ via (de)homogenization $L^n \to D_0 \colon v = [x_1, \ldots, x_n] \mapsto [1\colon x_1\colon \ldots\colon x_n] =: v^\sharp$ and $D_0 \to L^n \colon v = [x_0\colon \ldots\colon x_n] \mapsto [\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] =: v'$.

We show that the topology on $L^n$ induced by the Zariski topology on $\mathbf{P}$ and the Zariski topology on $L^n$ coincide; thus the identification $L^n \to D_0$ is a homeomorphism, and $D_0 \subseteq \mathbf{P}$ is (for the time being) called **affine open**:

For $I \unlhd A^\sharp$ homogeneous we have $\mathbf{V}_L^\sharp(I) \cap L^n = \{v \in L^n; v^\sharp \in \mathbf{V}_L^\sharp(I)\} = \{v \in L^n; f'(v) = 0 \text{ for all } f \in I \text{ homogeneous}\} = \mathbf{V}_L(I')$; thus any closed subset of $L^n$ with respect to the induced topology is Zariski closed. Conversely, for $I \unlhd A$ we have $\mathbf{V}_L^\sharp(I^\sharp) \cap L^n = \mathbf{V}_L((I^\sharp)') = \mathbf{V}_L(I)$; thus any Zariski closed subset of $L^n$ is closed with respect to the induced topology. $\sharp$

In particular, if $L$ is infinite, from $\mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{P})) = \mathbf{P}$ we get $\mathbf{V}_L(\mathbf{I}_K^\sharp(\mathbf{P})') = L^n$, so that $\mathbf{I}_L^\sharp(\mathbf{P})' = \{0\}$, entailing $\mathbf{I}_L^\sharp(\mathbf{P}) = \{0\}$; thus we have $K[\mathbf{P}] \cong A^\sharp$.

**(4.9) Projective closure. a)** We compare affine closed sets and projective closed sets: For $I \unlhd A$ and $V := \mathbf{V}_L(I) \subseteq L^n$ affine closed, let $\overline{V} \subseteq \mathbf{P}$ be its **projective closure**, that is the smallest projective closed subset containing $V$.

Letting $\mathbf{W} := \mathbf{V}_L^\sharp(I^\sharp) \subseteq \mathbf{P}$, being projective closed such that $\mathbf{W} \cap L^n = \mathbf{V}_L((I^\sharp)') = \mathbf{V}_L(I) = V$, we get $\overline{V} \subseteq \mathbf{W}$, hence $V \subseteq \overline{V} \cap L^n \subseteq \mathbf{W} \cap L^n = V$, so that $\overline{V} \cap L^n = V$. (We will comment on the relationship $\overline{V} \subseteq \mathbf{W}$ below.) The elements of $\overline{V} \setminus V \subseteq \mathbf{P} \setminus L^n = \mathbf{H}_0$ are called the **points at infinity** of $V$; recall that $\overline{V}$ is irreducible if and only if $V$ is irreducible.

Conversely, if $\mathbf{V} \subseteq \mathbf{P}$ is projective closed and irreducible, such that $\mathbf{V} \not\subseteq \mathbf{H}_0$, we have $\emptyset \neq \mathbf{V} \cap L^n \subseteq \mathbf{V}$ open, and hence dense, entailing $\overline{\mathbf{V} \cap L^n} = \mathbf{V}$. In conclusion, we have shown that mapping $V \mapsto \overline{V}$ yields a bijection

$$\{V \subseteq L^n \text{ closed, irreducible}\} \to \{\mathbf{V} \subseteq \mathbf{P} \text{ closed, irreducible}; \mathbf{V} \not\subseteq \mathbf{H}_0\},$$

whose inverse is given by $\mathbf{V} \mapsto \mathbf{V} \cap L^n$.

In general, if $V = \bigcup_{i=1}^r V_i \subseteq L^n$ are the irreducible components of $V$, then we have $\overline{V} = \bigcup_{i=1}^r \overline{V}_i \subseteq \mathbf{P}$, where the $\overline{V}_i$ are irreducible, and since $\overline{V}_i \cap L^n = V_i$ the decomposition of $\overline{V}$ is irredundant, so that the $\overline{V}_i$ are the irreducible components of $\overline{V}$. Hence the above bijections extend to bijections between the set of affine closed subsets of $L^n$, and the set of projective closed subsets of $\mathbf{P}$ having no irreducible component being contained in the hyperplane at infinity.

In particular, if $L$ is infinite, then both $L^n$, having coordinate algebra $A$, and $\mathbf{P}$, having homogeneous coordinate algebra $A^\sharp$, are irreducible, so that $\mathbf{P} \cap L^n = L^n$

implies that the projective closure of $L^n$ is $\overline{L^n} = \mathbf{P}$. (In contrast, if $L$ is finite and $K = L$, then singleton sets are closed, so that $\mathbf{P}$ carries the discrete topology.)

**b)** We proceed to describe the vanishing ideals of projective closures: To this end, let $\mathbf{V} \subseteq \mathbf{P}$ be projective closed having no irreducible component contained in $\mathbf{H}_0$, and let $V := \mathbf{V} \cap L^n \subseteq L^n$, being affine closed such that $\overline{V} = \mathbf{V}$.

**i)** If $I \trianglelefteq A^\sharp$ is homogeneous such that $\mathbf{V} = \mathbf{V}_L^\sharp(I)$, then we have $V = \mathbf{V} \cap L^n = \mathbf{V}_L(I')$, saying that a defining ideal of $V$ is given as the dehomogenization of any defining ideal of $\mathbf{V}$. In particular, if $L$ is algebraically closed, the vanishing ideal of $V$ is given as the dehomogenization of the vanishing ideal of $\mathbf{V}$.

**ii)** Let conversely $I \trianglelefteq A$ such that $V = \mathbf{V}_L(I)$, and let $\mathbf{W} := \mathbf{V}_L^\sharp(I^\sharp) \subseteq \mathbf{P}$. Then we have already seen that $\mathbf{W} \cap L^n = \mathbf{V}_L((I^\sharp)') = \mathbf{V}_L(I) = V$, implying that $\mathbf{V} = \overline{V} \subseteq \mathbf{W}$. If $L$ is algebraically closed we show that actually $\mathbf{V} = \mathbf{W}$, by showing that any projective closed set containing $V$ already contains $\mathbf{W}$:

Let $J \trianglelefteq A^\sharp$ homogeneous such that $\mathbf{U} := \mathbf{V}_L^\sharp(J)$ contains $V$, that is we have $V \subseteq \mathbf{U} \cap L^n = \mathbf{V}_L(J')$. Thus we have $J' \subseteq \mathbf{I}_K(V) = \sqrt{I} \trianglelefteq A$. Hence for any $f \in J$ we have $(f^k)' = (f')^k \in I$, for some $k \in \mathbb{N}$. This implies $(X_0^{-\nu(f)} \cdot f)^k = ((f^k)')^\sharp \in I^\sharp$, thus $X_0^{-\nu(f)} \cdot f \in \sqrt{I^\sharp} = \mathbf{I}_K^\sharp(\mathbf{W})$. This entails $f \in \mathbf{I}_K^\sharp(\mathbf{W})$, hence $J \subseteq \mathbf{I}_K^\sharp(\mathbf{W})$, thus $\mathbf{W} = \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{W})) \subseteq \mathbf{V}_L^\sharp(J) = \mathbf{U}$. ♯

Thus, if $L$ is algebraically closed, a defining ideal of $\mathbf{V}$ is given as the homogenization of any defining ideal of $V$. In particular, the vanishing ideal of $\mathbf{V}$ is given as the homogenization of the vanishing ideal of $V$. Note that the latter assertions do not necessarily hold if $L$ is not algebraically closed:

**Example.** Let $K = L = \mathbb{R}$ and $n = 2$, hence $A^\sharp = \mathbb{R}[T, X, Y]$. Let $I = \langle X^2 + Y^4 \rangle \trianglelefteq A$. Then we have $V = \mathbf{V}(I) = \{[0,0]\}$, so that $\overline{V} = V = \{[1\colon 0\colon 0]\}$. But we have $I^\sharp = \langle T^2X^2 + Y^4 \rangle$, so that $\mathbf{V}^\sharp(I^\sharp) = \{[1\colon 0\colon 0], [0\colon 1\colon 0]\}$.

**(4.10) Example: The twisted cubic.** Let $K = L = \mathbb{C}$, and let $n = 3$, hence we have $A = \mathbb{C}[X, Y, Z]$ and $A^\sharp = \mathbb{C}[W, X, Y, Z]$. We consider the projective closed set $\mathbf{W} := \mathbf{V}^\sharp(I) \subseteq \mathbf{P} = \mathbf{P}^3(\mathbb{C})$, where $I := \langle XZ - Y^2, YW - X^2 \rangle \trianglelefteq A^\sharp$.

Note that the graded $\mathbb{C}$-algebra automorphism of $A^\sharp$ given by $X \leftrightarrow Y$ and $W \leftrightarrow Z$ leaves $I$ invariant, so that it induces a graded $\mathbb{C}$-algebra automorphism of $A^\sharp/I$ and thus of $\mathbb{C}[\mathbf{W}] = A^\sharp/\mathbf{I}^\sharp(\mathbf{W}) = A^\sharp/\sqrt{I}$. (But we do not know yet what an 'automorphism' of a projective variety should be.)

**i)** We observe that $\mathbf{W} \cap \mathbf{H}_0 = \mathbf{V}^\sharp(XZ - Y^2, YW - X^2, W) = \mathbf{V}^\sharp(W, X, Y) = \{[0\colon 0\colon 0\colon 1]\}$. Hence $\mathbf{W}$ either has no irreducible component contained in $\mathbf{H}_0$, or $\{[0\colon 0\colon 0\colon 1]\}$ is the only one. We will exclude the latter case by showing (on the fly) that $\{[0\colon 0\colon 0\colon 1]\}$ is contained in a larger closed irreducible subset.

We proceed to examine the affine closed set $\mathbf{W} \cap \mathbb{C}^3 = \mathbf{V}(I')$, where we have $I' = \langle XZ - Y^2, Y - X^2 \rangle = \langle X(Z - X^3), Y - X^2 \rangle \trianglelefteq A$. Then both $P := \langle Z - X^3, Y - X^2 \rangle \trianglelefteq A$ and $Q := \langle X, Y - X^2 \rangle = \langle X, Y \rangle \trianglelefteq A$ divide $I'$.

**ii)** We show that $P$ is prime: To this end, we consider the homomorphism of $\mathbb{C}$-algebras $\varphi^*\colon A/P \to \mathbb{C}[T]\colon X \mapsto T, Y \mapsto T^2, Z \mapsto T^3$, which is well-defined by the definition of $P$. Moreover, we have a homomorphism of $\mathbb{C}$-algebras $\psi^*\colon \mathbb{C}[T] \to A/P\colon T \mapsto X$. Then we have $\psi^*(\varphi^*(X)) = X$, and $\psi^*(\varphi^*(Y)) = X^2 = Y$, and $\psi^*(\varphi^*(Z)) = X^3 = Z$, thus $\varphi^*\psi^* = \mathrm{id}_{A/P}$. Similarly, we get $\varphi^*(\psi^*(T)) = T$, thus $\psi^*\varphi^* = \mathrm{id}_{\mathbb{C}[T]}$. This shows that $A/P \cong \mathbb{C}[T]$ is a domain.

Let $V := \mathbf{V}(P) \subseteq \mathbb{C}^3$, which is irreducible. Then $\varphi^*$ is the comorphism associated with the isomorphism $\varphi\colon \mathbb{C} \to V\colon t \mapsto [t, t^2, t^3]$.

Similarly, $A/Q \cong \mathbb{C}[Z]$ is a domain, hence $Q$ is prime. Let $U := \mathbf{V}(Q) = \{0\} \times \{0\} \times \mathbb{C} \subseteq \mathbb{C}^3$, that is the '$z$-axis'; we have the isomorphism $\mathbb{C} \to U\colon z \mapsto [0, 0, z]$.

We show that $P \cap Q = I'$: We have $I' \subseteq P \cap Q$. From $P = \mathbf{I}(V)$ and $Q = \mathbf{I}(U)$ we get $\mathbf{I}(V \cup U) = P \cap Q = PQ$. Since $Y - X^2 \in I'$ we have $PQ = \langle X(Z - X^3)\rangle = \{0\} \trianglelefteq A/I'$. Thus we infer that $PQ \subseteq I'$, so that $I' = PQ = P \cap Q = \mathbf{I}(V \cup U)$.

Hence we have the decomposition $\mathbf{W} \cap \mathbb{C}^3 = \mathbf{V}(I') = \mathbf{V}(\mathbf{I}(V \cup U)) = V \cup U$ into irreducible components, where $V \cap U = \mathbf{V}(P + Q) = \mathbf{V}(\langle X, Y, Z\rangle) = \{[0, 0, 0]\}$.

Letting $\mathbf{V} := \overline{V} = \mathbf{V}^\sharp(P^\sharp)$ and $\mathbf{U} := \overline{U} = \mathbf{V}^\sharp(Q^\sharp)$ we have the decomposition $\mathbf{W} = \mathbf{V} \cup \mathbf{U} \cup \{[0\colon 0\colon 0\colon 1]\}$ into irreducible closed subsets, where it remains to be decided whether the last piece is redundant. Being homogenizations of prime ideals, both $P^\sharp$ and $Q^\sharp$ are prime. In order to determine $P^\sharp$ and $Q^\sharp$ explicitly, we apply homogenization, recalling that we have to apply homogenization not only to a generating set of the ideal in question, but to all its elements:

**iii)** We determine $Q^\sharp \trianglelefteq A^\sharp$: Since $Q = \langle X, Y\rangle \trianglelefteq A$ is homogeneous, we get $Q^\sharp = \langle X, Y\rangle \trianglelefteq A^\sharp$. Thus we have

$$\mathbf{U} = (\mathbf{U} \cap \mathbb{C}^3) \,\dot\cup\, (\mathbf{U} \cap \mathbf{H}_0) = \{[1\colon 0\colon 0\colon z] \in \mathbf{P}; z \in \mathbb{C}\} \,\dot\cup\, \{[0\colon 0\colon 0\colon 1]\},$$

having homogeneous coordinate algebra $\mathbb{C}[\mathbf{U}] = A^\sharp/Q^\sharp \cong \mathbb{C}[W, Z] = \mathbb{C}[\mathbf{P}^1]$. Indeed we have the homeomorphism $\mathbf{P}^1 \to \mathbf{U}\colon [w\colon z] \mapsto [w\colon 0\colon 0\colon z]$. (We are tempted to call it an 'isomorphism', but so far we do not even have a definition of a 'morphism' between projective varieties.)

**iv)** We proceed to determine $P^\sharp \trianglelefteq A^\sharp$: We have $\langle YW - X^2, ZW^2 - X^3\rangle \subseteq P^\sharp$. But we have $Z = X^3 = XY \in A/P$ and $Y^2 = X^4 = XZ \in A/P$, which implies that $Z - XY \in P$ and $Y^2 - XZ \in P$. Hence we have $ZW - XY \in P^\sharp$ and $Y^2 - XZ \in P^\sharp$ as well. Letting

$$J := \langle Y^2 - XZ, ZW - XY, YW - X^2\rangle \subseteq P^\sharp \trianglelefteq A^\sharp,$$

we observe that $ZW^2 - X^3 = XYW - XYW = 0 \in A^\sharp/J$, making this generator redundant. We guess that we actually have $J = P^\sharp$, and set out to show this:

To this end, we consider the epimorphism of $\mathbb{C}$-algebras

$$\alpha\colon A^\sharp \to \mathbb{C}[S, T]_{3\mathbb{N}_0} := \bigoplus_{d \in \mathbb{N}_0} \mathbb{C}[S, T]_{3d}\colon W \mapsto S^3, X \mapsto S^2T, Y \mapsto ST^2, Z \mapsto T^3.$$

Then we observe that $J \subseteq \ker(\alpha) \trianglelefteq A^\sharp$. We show that equality holds:

Since $\alpha$ is a homomorphism of graded algebras, with respect to the grading of $\mathbb{C}[S,T]_{3\mathbb{N}_0}$ indicated above, we infer that $\ker(\alpha) \trianglelefteq A^\sharp$ is a homogeneous ideal. Since $J \trianglelefteq A^\sharp$ is homogeneous as well, both $A^\sharp/J$ and $A^\sharp/\ker(\alpha)$ are graded algebras. Thus we have $J = \ker(\alpha)$, if and only if for all $d \geq 0$ we have

$$\dim_\mathbb{C}((A^\sharp)_d/J_d) \leq \dim_\mathbb{C}((A^\sharp)_d/\ker(\alpha)_d) = \dim_\mathbb{C}(\mathbb{C}[S,T]_{3d}) = 3d+1.$$

Now $(A^\sharp/J)_d$ is generated as a $\mathbb{C}$-vector space by the cosets of the monomials in $A^\sharp$ of degree $d$. Taking the (**binomial**) generators of $J$ into account, it is immediate that the following cosets suffice:

$$\{W^iZ^j; i+j=d\} \,\dot\cup\, \{W^iZ^jX; i+j=d-1\} \,\dot\cup\, \{W^iZ^jY; i+j=d-1\}.$$

This set has cardinality $(d+1)+2d = 3d+1$, showing $\dim_\mathbb{C}((A^\sharp)_d/J_d) \leq 3d+1$.

From $J = \ker(\alpha) \trianglelefteq A^\sharp$, since $\mathbb{C}[S,T]_{3\mathbb{N}_0} \subseteq \mathbb{C}[S,T]$ is a domain, we conclude that $J \trianglelefteq A^\sharp$ is prime. Thus $J' = P = (P^\sharp)'$ yields $\mathbf{V}^\sharp(J) = \overline{\mathbf{V}(J')} = \overline{\mathbf{V}(P)} = \overline{\mathbf{V}((P^\sharp)')} = \mathbf{V}^\sharp(P^\sharp) = \mathbf{V}$, entailing $J = \mathbf{I}^\sharp(\mathbf{V}^\sharp(J)) = \mathbf{I}^\sharp(\mathbf{V}^\sharp(P^\sharp)) = P^\sharp$.

From this we get the **twisted cubic** (space curve)

$$\mathbf{V} = (\mathbf{V} \cap \mathbb{C}^3) \,\dot\cup\, (\mathbf{V} \cap \mathbf{H}_0) = \{[1\colon t\colon t^2\colon t^3] \in \mathbf{P}; t \in \mathbb{C}\} \,\dot\cup\, \{[0\colon 0\colon 0\colon 1]\},$$

having homogeneous coordinate algebra $\mathbb{C}[\mathbf{V}] = A^\sharp/P^\sharp \cong \mathbb{C}[S,T]_{3\mathbb{N}_0} \not\cong \mathbb{C}[S,T]$.

But still we have the homeomorphism $\mathbf{P}^1 \to \mathbf{V}\colon [s\colon t] \mapsto [s^3\colon s^2t\colon st^2\colon t^3]$, see also Exercise (12.32); note that $[s\colon t] = [s^3\colon s^2t]$ if $s \neq 0$, and $[s\colon t] = [st^2\colon t^3]$ if $t \neq 0$. (Again we are tempted to call it an 'isomorphism'.)

**v)** In conclusion, we have $\mathbf{W} = \mathbf{V} \cup \mathbf{U}$, the latter being the irreducible components of $\mathbf{W}$, where $\mathbf{V} \cap \mathbf{U} = \{[1\colon 0\colon 0\colon 0], [0\colon 0\colon 0\colon 1]\}$.

Moreover, we have $\mathbf{I}^\sharp(\mathbf{W}) = \mathbf{I}^\sharp(\overline{\mathbf{W} \cap \mathbb{C}^3}) = \mathbf{I}(\mathbf{W} \cap \mathbb{C}^3)^\sharp = (I')^\sharp$ and $\mathbf{I}^\sharp(\mathbf{W}) = \mathbf{I}^\sharp(\mathbf{V}) \cap \mathbf{I}^\sharp(\mathbf{U}) = P^\sharp \cap Q^\sharp = P^\sharp Q^\sharp$, thus we get $(I')^\sharp = P^\sharp \cap Q^\sharp = P^\sharp Q^\sharp$.

Finally, since $I \subseteq (I')^\sharp = P^\sharp \cap Q^\sharp$, computing in $A^\sharp/I = A^\sharp/\langle XZ-Y^2, YW-X^2\rangle$ yields $P^\sharp Q^\sharp = \langle Y^2 - XZ, ZW - XY, YW - X^2\rangle\langle X, Y\rangle = \langle ZW - XY\rangle\langle X, Y\rangle \subseteq A^\sharp/I$, where $XZW - X^2Y = Y^2W - Y^2W = 0 \in A^\sharp/I$ and $YZW - XY^2 = X^2Z - X^2Z = 0 \in A^\sharp/I$ yields $P^\sharp Q^\sharp = \langle XZW - X^2Y, YZW - XY^2\rangle = \{0\} \subseteq A^\sharp/I$. Thus we get $I = (I')^\sharp = P^\sharp \cap Q^\sharp = P^\sharp Q^\sharp = \mathbf{I}^\sharp(\mathbf{W})$. ♯

**Remark.** A couple of comments concerning part (iv) is in order:

**a)** The ideal $\widetilde{J} := \langle YW - X^2, ZW^2 - X^3\rangle \trianglelefteq A^\sharp$ encountered as the 'first approximation' of $P^\sharp$ is indeed properly contained in $P^\sharp$: We have $\mathbf{V}^\sharp(\widetilde{J}) \cap \mathbf{H}_0 = \mathbf{V}^\sharp(YW - X^2, ZW^2 - X^3, W) = \mathbf{V}^\sharp(X, W) = \{[0\colon 0\colon y\colon z] \in \mathbf{P}; [y\colon z] \in \mathbf{P}^1\}$, while $\mathbf{V}^\sharp(P^\sharp) \cap \mathbf{H}_0 = \mathbf{V} \cap \mathbf{H}_0 = \{[0\colon 0\colon 0\colon 1]\}$. (Recall that $(\widetilde{J})' = P = (P^\sharp)'$, so that $\mathbf{V}^\sharp(\widetilde{J}) \cap \mathbb{C}^3 = \mathbf{V}(P) = \mathbf{V}^\sharp(P^\sharp) \cap \mathbb{C}^3$.)

**b)** In order to avoid a specially tailored argument to determine a generating set of $P^\sharp$ from a generating set of $P$, we may proceed computationally as follows: We compute a Gröbner basis of $P$ with respect to a degree-driven monomial order, then by [1, Sect.8.4] its homogenization generates $P^\sharp$; actually it is a Gröbner basis of $P^\sharp$ with respect to a certain extension of the given monomial order from $\mathcal{X}$ to $\mathcal{X}^\sharp$. Here, we get $P = \langle Y^2 - XZ, XY - Z, X^2 - Y\rangle \trianglelefteq A$, so that we again obtain $P^\sharp = \langle Y^2 - XZ, XY - ZW, X^2 - YW\rangle \trianglelefteq A^\sharp$.

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# II Varieties

## 5 Categories

**(5.1) Categories.** A **category** $\mathcal{C}$ consists of a **class** $\mathrm{Ob}(\mathcal{C})$ of **objects** (which is not necessarily a set), together with sets of **morphisms** $\mathrm{Hom}_\mathcal{C}(A, B)$, also denoted by $\mathrm{Mor}_\mathcal{C}(A, B)$, for all $A, B \in \mathcal{C}$, such that
**i)** for all $A \in \mathcal{C}$ there is an **identity** $\mathrm{id}_A \in \mathrm{End}_\mathcal{C}(A) := \mathrm{Hom}_\mathcal{C}(A, A)$, and
**ii)** for all $A, B, C \in \mathcal{C}$ there is a **concatenation** map

$$\mathrm{Hom}_\mathcal{C}(A, B) \times \mathrm{Hom}_\mathcal{C}(B, C) \to \mathrm{Hom}_\mathcal{C}(A, C)\colon [\alpha, \beta] \mapsto \alpha\beta,$$

fulfilling the following conditions, for all $A, B, C, D \in \mathcal{C}$:
**i)** For all $\alpha\colon A \to B$ and $\beta\colon B \to A$ we have $\mathrm{id}_A \cdot \alpha = \alpha$ and $\beta \cdot \mathrm{id}_A = \beta$, and
**ii)** for all $\alpha\colon A \to B$, $\beta\colon B \to C$, $\gamma\colon C \to D$ we have $(\alpha\beta)\gamma = \alpha(\beta\gamma)\colon A \to D$.

Here and further on, for objects $A \in \mathrm{Ob}(\mathcal{C})$ we abbreviate by writing $A \in \mathcal{C}$, and for $\alpha \in \mathrm{Hom}_\mathcal{C}(A, B)$ we write an **arrow** $\alpha\colon A \to B$.

**Example.** The class **Sets** of all sets, together with the set of all maps between pairs of sets as morphisms, forms a category. Note that the class of all sets is not a set, so that we have to be careful with these constructions. (We will not go into any detail here, but we will just take this for granted.)

The following is an example of a **small** category, that is one whose class of objects is a set: Let $\mathcal{M}$ be a set. Then the set $\mathbf{Sets}(\mathcal{M})$ of all subsets of $\mathcal{M}$, together with all maps between subsets of $\mathcal{M}$ as morphisms, forms a category. This can be varied, for example by allowing only for injective maps, or only for surjective maps morphisms; or by going down to a smaller set of objects, such as the set of all finite subsets of $\mathcal{M}$; or both.

Further examples are the class **Top** of all topological spaces, together with all continuous maps as morphisms; the class **Ab** of abelian groups, together with all group homomorphisms as morphisms; the class **Mod**-$R$ of all $R$-modules, where $R$ is a ring, or the class **mod**-$R$ of all finitely generated $R$-modules, together with all $R$-module homomorphisms as morphisms. $\sharp$

**(5.2) Functors. a)** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **(covariant) functor** $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is an assignment $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})\colon A \to \mathcal{F}(A)$, together with maps

$$\mathcal{F}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A),\mathcal{F}(B))\colon \alpha \mapsto \mathcal{F}_{A,B}(\alpha) = \mathcal{F}(\alpha),$$

fulfilling the following conditions, for $A, B, C \in \mathcal{C}$: **i)** We have $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$, and **ii)** for $\alpha\colon A \to B$ and $\beta\colon B \to C$ we have $\mathcal{F}(\alpha) \cdot \mathcal{F}(\beta) = \mathcal{F}(\alpha\beta)$.

Similarly, a **contravariant functor** $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ consists of a map $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, together with **arrow reversing** maps

$$\mathcal{F}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(B),\mathcal{F}(A)),$$

such that **i)** $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$, and **ii)** $\mathcal{F}_{B,C}(\beta) \cdot \mathcal{F}_{A,B}(\alpha) = \mathcal{F}_{A,C}(\alpha\beta)\colon C \to A$.

For example, we have the (covariant) **identity functor** $\mathrm{Id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$, mapping $A \mapsto A$ for $A \in \mathcal{C}$, and $\alpha \mapsto \alpha$ for $\alpha\colon A \to B$.

**b)** If $\mathcal{E}$ is a category, and $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{E}$ are (covariant) functors, then the **concatenation** $\mathcal{G} \circ \mathcal{F}\colon \mathcal{C} \to \mathcal{E}$ is the (covariant) functor given by

$$\mathcal{G} \circ \mathcal{F}\colon A \mapsto \mathcal{G}(\mathcal{F}(A)) \quad \text{and} \quad (\mathcal{G} \circ \mathcal{F})(\alpha) = \mathcal{G}(\mathcal{F}(\alpha))\colon \mathcal{G}(\mathcal{F}(A)) \to \mathcal{G}(\mathcal{F}(B)),$$

for $A, B \in \mathcal{C}$, and $\alpha\colon A \to B$ a morphism in $\mathcal{C}$; it is immediately checked that this is a functor indeed. The concatenation of contravariant functors, or of covariant and contravariant functors is defined analogously.

Categories $\mathcal{C}$ and $\mathcal{D}$ are called **isomorphic**, if there are functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that $\mathcal{G} \circ \mathcal{F} = \mathrm{Id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{Id}_{\mathcal{D}}$. Note that in this case either both $\mathcal{C}$ and $\mathcal{D}$ are covariant, or both $\mathcal{C}$ and $\mathcal{D}$ are contravariant.

**c)** Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{C} \to \mathcal{D}$ be (covariant) functors. Then a **natural transformation** $\mathcal{N}\colon \mathcal{F} \Rightarrow \mathcal{G}$ is a map assigning a morphism $\mathcal{N}_A\colon \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathcal{D}$ to all $A \in \mathcal{C}$, such that for any morphism $\alpha\colon A \to B$ in $\mathcal{C}$ we have

$$\big(\mathcal{F}(\alpha) \cdot \mathcal{N}_B\colon \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{G}(B)\big) = \big(\mathcal{N}_A \cdot \mathcal{G}(\alpha)\colon \mathcal{F}(A) \to \mathcal{G}(A) \to \mathcal{G}(B)\big).$$

If the morphisms $\mathcal{N}_A$ are isomorphisms in $\mathcal{D}$, for all $A \in \mathcal{C}$, then $\mathcal{N}$ is called a **natural isomorphism**; in this case we write $\mathcal{F} \cong \mathcal{G}$. Natural transformations and isomorphisms between contravariant functors are defined analogously.

Categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent** if there are (covariant or contravariant) functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that $\mathcal{G} \circ \mathcal{F} \cong \mathrm{Id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{Id}_{\mathcal{D}}$; in this case we write $\mathcal{C} \cong \mathcal{D}$. In particular, isomorphic categories are (covariantly or contravariantly) equivalent. Typically, 'equivalence', rather than 'isomorphism', is the appropriate notion to say that two categories are 'essentially equal'.

**(5.3) Limits.** Let $\mathcal{I}$ be a set carrying a **partial order**, that is a reflexive, anti-symmetric and transitive relation $\leq$. Then $\mathcal{I}$ becomes a category by letting $\mathrm{Hom}_{\mathcal{I}}(i,j) = \{\rho_{ij}\}$ be a singleton set if $i \leq j$, and $\mathrm{Hom}_{\mathcal{I}}(i,j) = \emptyset$ otherwise, concatenation being given by $\rho_{ij}\rho_{jk} = \rho_{ik}$ for $i \leq j \leq k$; in particular $\rho_{ii} = \mathrm{id}_i$.

Now let $\mathcal{C}$ be a category. Then an **inverse system** or **projective system** in $\mathcal{C}$ is a contravariant functor $\mathcal{I} \to \mathcal{C}$; that is it specifies a set of objects $\{A_i \in \mathcal{C}; i \in \mathcal{I}\}$, together with morphisms $\{\psi_{ji} \in \mathrm{Hom}_{\mathcal{C}}(A_j, A_i); i, j \in \mathcal{I}, j \geq i\}$, such that $\psi_{kj}\psi_{ji} = \psi_{ki}$, and $\psi_{ii} = \mathrm{id}_{A_i}$, for $k \geq j \geq i$.

Given $A \in \mathcal{C}$, let $\mathrm{Hom}_{\mathcal{C}}(A, \{A_i\})$ consist of all sets $\{\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(A, A_i); i \in \mathcal{I}\}$ such that $\varphi_j\psi_{ji} = \varphi_i$, for $j \geq i$. Then the assignment

$$\mathrm{Hom}_{\mathcal{C}}(?, \{A_i\})\colon \mathcal{C} \to \mathbf{Sets}\colon A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, \{A_i\})$$

is a contravariant functor, where a morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is mapped to the map $\varphi^*\colon \mathrm{Hom}_{\mathcal{C}}(B, \{A_i\}) \to \mathrm{Hom}_{\mathcal{C}}(A, \{A_i\})$ induced by pre-composition.

An object $P \in \mathcal{C}$, together with a **universal morphism** $\{\pi_i\} \in \mathrm{Hom}_{\mathcal{C}}(P, \{A_i\})$, is called an **(inverse) limit** or **projective limit**, if $P$ has the following universal property: For any $A \in \mathcal{C}$ and $\{\varphi_i\} \in \mathrm{Hom}_{\mathcal{C}}(A, \{A_i\})$ there is a unique $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, P)$ such that $\varphi\pi_i = \varphi_i$, for all $i \in \mathcal{I}$.

It is immediate that the limit is unique up to unique isomorphism in $\mathcal{C}$, if it exists at all; in this case we write $P = \varprojlim\{A_i\}$. (For proofs of these assertions, see Exercise (13.3).) Then we have the natural bijection, given by post-composition,

$$\mathrm{Hom}_{\mathcal{C}}(A, P) \to \mathrm{Hom}_{\mathcal{C}}(A, \{A_i\})\colon \varphi \mapsto \{\varphi\pi_i\}.$$

Recall that $\mathrm{Hom}_{\mathcal{C}}(?, P)\colon \mathcal{C} \to \mathbf{Sets}\colon A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, P)$ is a contravariant functor, where $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is mapped to $\varphi^*\colon \mathrm{Hom}_{\mathcal{C}}(B, P) \to \mathrm{Hom}_{\mathcal{C}}(A, P)$, induced by pre-composition. Then the universal property says that the universal morphism induces an equivalence of functors $\mathrm{Hom}_{\mathcal{C}}(?, P) \cong \mathrm{Hom}_{\mathcal{C}}(?, \{A_i\})$. In other words, $\mathrm{Hom}_{\mathcal{C}}(?, \{A_i\})$ is a **representable functor**, being represented by $P$ and the universal morphism $\{\pi_i\} \in \mathrm{Hom}_{\mathcal{C}}(P, \{A_i\})$.

**(5.4) Limits in the category of sets. a)** Let $\mathcal{C} := \mathbf{Sets}$. Given sets $A_i$ and maps $\psi_{ji}\colon A_j \to A_i$, for $j \geq i$, let $\mathcal{P} := \prod_{i \in \mathcal{I}} A_i := \{\rho\colon \mathcal{I} \to \bigcup_{i \in \mathcal{I}} A_i; \rho(i) \in A_i \text{ for } i \in \mathcal{I}\}$ be the **(Cartesian) product** of the $A_i$, whose elements may be written as $\mathcal{I}$-tuples, and let $\pi_i\colon \mathcal{P} \to A_i\colon x = [x_j; j \in \mathcal{I}] \mapsto x_i$ be the natural projections, for $i \in \mathcal{I}$. Then it is immediate that the subset

$$P := \{x \in \mathcal{P}; x_i = \psi_{ji}(x_j) \text{ for } j \geq i\} = \{x \in \mathcal{P}; \pi_i(x) = \psi_{ji}(\pi_j(x)) \text{ for } j \geq i\},$$

together with the induced maps $\{\pi_i\}$, is a limit in $\mathcal{C}$.

In particular, if $\mathcal{I}$ carries the **trivial** partial order, that is the only relations are $i \geq i$, for $i \in \mathcal{I}$, then the product $\mathcal{P}$ is the associated limit. If $\mathcal{I} = \emptyset$, then the (empty) product $\mathcal{P}$ boils down to a singleton set, which is a **terminal object** in $\mathcal{C}$, that is to which any set has a unique map.

**b)** Let $\mathcal{I} := \{0, 1, 2\}$ be partially ordered by the non-trivial relations $1 \geq 0$ and $2 \geq 0$. Given maps $\psi_1\colon A_1 \to A_0$ and $\psi_2\colon A_2 \to A_0$ between sets, the associated **fibre product**, also called the **pullback** of $\psi_1$ and $\psi_2$, is defined as the associated limit in the category of sets

$$A_1 \times_{A_0} A_2 := \varprojlim\{A_0, A_1, A_2\} = \{[x_0, x_1, x_2] \in \mathcal{P}; \psi_1(x_1) = x_0 = \psi_2(x_2)\}.$$

Hence we have $A_1 \times_{A_0} A_2 \cong \{[x_1, x_2] \in A_1 \times A_2; \psi_1(x_1) = \psi_2(x_2)\}$. In particular, if $A_0$ is a singleton set, the product $A_1 \times A_2$ is the limit. Note that if both $\psi_i$ are surjective, then so are the induced maps $\pi_i \colon A_1 \times_{A_0} A_2 \to A_i$.

**Example: Completions of $\mathbb{Z}$. i)** Let $\mathcal{I} := \mathbb{N}$, and let the partial order be given by divisibility $n \mid m$. Let $\mathbb{Z}_n := \mathbb{Z}/\langle n\rangle$, let $\psi_{mn} \colon \mathbb{Z}_m \to \mathbb{Z}_n$ be the natural ring epimorphism, for $n \mid m$, and let $\widehat{\mathbb{Z}} := \varprojlim\{\mathbb{Z}_n\} \subseteq \prod_{n\in\mathbb{N}} \mathbb{Z}_n$ be the associated limit in the category of sets.

Then $\prod_{n\in\mathbb{N}} \mathbb{Z}_n$ becomes a ring with componentwise addition and multiplication. Since the maps $\psi_{mn}$ and $\pi_n$ are ring homomorphisms, $\widehat{\mathbb{Z}}$ becomes a ring. Hence $\widehat{\mathbb{Z}}$ is a limit in the category of rings, called the **profinite completion** of $\mathbb{Z}$.

**ii)** Similarly, let $p \in \mathbb{Z}$ be a prime, let $\mathcal{I}_p := \{p^n \in \mathbb{N}; n \in \mathbb{N}_0\}$, and let $\widehat{\mathbb{Z}}_p := \varprojlim\{\mathbb{Z}_{p^n}\} \subseteq \prod_{n\in\mathbb{N}_0} \mathbb{Z}_{p^n}$ be the associated limit in the category of sets.

Then again $\prod_{n\in\mathbb{N}_0} \mathbb{Z}_{p^n}$ becomes a ring, and hence $\widehat{\mathbb{Z}}_p$ is a limit in the category of rings, called the **$p$-adic completion** of $\mathbb{Z}$, or the ring of **$p$-adic integers**. (Actually, $\widehat{\mathbb{Z}}_p$ is a **complete local domain** with maximal ideal $\langle p\rangle$; its field of fractions $\widehat{\mathbb{Q}}_p := \mathrm{Q}(\widehat{\mathbb{Z}}_p)$ is called the field of **$p$-adic numbers**. By the Chinese Remainder Theorem we have $\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \widehat{\mathbb{Z}}_p$; in particular $\widehat{\mathbb{Z}}$ is not a domain.)

**(5.5) Colimits.** A **direct system** or **inductive system** in $\mathcal{C}$ is a (covariant) functor $\mathcal{I} \to \mathcal{C}$; that is it specifies objects $\{A_i \in \mathcal{C}; i \in \mathcal{I}\}$, together with morphisms $\{\psi_{ij} \in \mathrm{Hom}_{\mathcal{C}}(A_i, A_j); i, j \in \mathcal{I}, i \leq j\}$, such that $\psi_{ij}\psi_{jk} = \psi_{ik}$, and $\psi_{ii} = \mathrm{id}_{A_i}$, for $i \leq j \leq k$.

Given $A \in \mathcal{C}$, let $\mathrm{Hom}_{\mathcal{C}}(\{A_i\}, A)$ consist of all sets $\{\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, A); i \in \mathcal{I}\}$ such that $\psi_{ij}\varphi_j = \varphi_i$, for $i \leq j$. Then the assignment

$$\mathrm{Hom}_{\mathcal{C}}(\{A_i\}, ?) \colon \mathcal{C} \to \mathbf{Sets} \colon A \mapsto \mathrm{Hom}_{\mathcal{C}}(\{A_i\}, A)$$

is a (covariant) functor, where a morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is mapped to the map $\varphi_* \colon \mathrm{Hom}_{\mathcal{C}}(\{A_i\}, A) \to \mathrm{Hom}_{\mathcal{C}}(\{A_i\}, B)$ induced by post-composition.

An object $I \in \mathcal{C}$, together with a universal morphism $\{\iota_i\} \in \mathrm{Hom}_{\mathcal{C}}(\{A_i\}, I)$, is called a **colimit** or **direct limit** or **inductive limit**, if $I$ has the following universal property: For any $A \in \mathcal{C}$ and $\{\varphi_i\} \in \mathrm{Hom}_{\mathcal{C}}(\{A_i\}, A)$ there is a unique $\varphi \in \mathrm{Hom}_{\mathcal{C}}(I, A)$ such that $\iota_i\varphi = \varphi_i$, for all $i \in \mathcal{I}$.

It is immediate that the colimit is unique up to unique isomorphism in $\mathcal{C}$, if it exists at all; in this case we write $I = \varinjlim\{A_i\}$. (For proofs, see Exercise (13.4).) Then we have the natural bijection, given by pre-composition,

$$\mathrm{Hom}_{\mathcal{C}}(I, A) \to \mathrm{Hom}_{\mathcal{C}}(\{A_i\}, A) \colon \varphi \mapsto \{\iota_i\varphi\}.$$

Recall that $\mathrm{Hom}_{\mathcal{C}}(I,?)\colon \mathcal{C} \to \mathbf{Sets}\colon A \mapsto \mathrm{Hom}_{\mathcal{C}}(I,A)$ is a (covariant) functor, where $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is mapped to $\varphi_*\colon \mathrm{Hom}_{\mathcal{C}}(I,A) \to \mathrm{Hom}_{\mathcal{C}}(I,B)$, induced by post-composition. Then the universal property says that the universal morphism induces an equivalence of functors $\mathrm{Hom}_{\mathcal{C}}(I,?) \cong \mathrm{Hom}_{\mathcal{C}}(\{A_i\},?)$. In other words, $\mathrm{Hom}_{\mathcal{C}}(\{A_i\},?)$ is a representable functor, being represented by $I$ and the universal morphism $\{\iota_i\} \in \mathrm{Hom}_{\mathcal{C}}(I,\{A_i\})$.

**(5.6) Colimits in the category of sets. a)** Let $\mathcal{C} := \mathbf{Sets}$. Given sets $A_i$ and maps $\psi_{ij}\colon A_i \to A_j$, for $i \leq j$, let $\mathcal{Q} := \coprod_{i\in\mathcal{I}} A_i := \{[i,x] \in \mathcal{I} \times (\bigcup_{i\in\mathcal{I}} A_i); [i,x] \in \{i\} \times A_i\}$ be the **coproduct** or **disjoint union** of the $A_i$, and let $\iota_i\colon A_i \to \mathcal{Q}\colon x \mapsto [i,x]$ be the natural inclusion, for $i \in \mathcal{I}$.

Let $\sim$ be the equivalence relation on $\mathcal{Q}$ generated by transitive and symmetric closure from $\iota_i(x) \sim \iota_j(\psi_{ij}(x))$ for $i \leq j$ and $x \in A_i$; note that reflexivity is fulfilled anyway. Then it is immediate that the set of equivalence classes $I := \mathcal{Q}/_\sim$, together with the induced maps $\{\iota_i\}$, is a colimit in $\mathcal{C}$.

In particular, if $\mathcal{I}$ carries the **trivial** partial order, that is the only relations are $i \leq i$, for $i \in \mathcal{I}$, then the coproduct $\mathcal{Q}$ is the associated colimit. If $\mathcal{I} = \emptyset$, then the (empty) coproduct $\mathcal{Q}$ boils down to the empty set, which is the **initial object** in $\mathcal{C}$, that is which has a unique map to any set.

**b)** Let $\mathcal{I} := \{0,1,2\}$ be partially ordered by the non-trivial relations $0 \leq 1$ and $0 \leq 2$. Given maps $\psi_1\colon A_0 \to A_1$ and $\psi_2\colon A_0 \to A_2$ between sets, the associated **fibre sum**, also called the **pushout** of $\psi_1$ and $\psi_2$, is defined as the associated colimit in the category of sets

$$A_1 \sqcup_{A_0} A_2 := \varinjlim\{A_0, A_1, A_2\} = (\coprod_{i\in\{0,1,2\}} A_i)/_\sim,$$

where $\sim$ is the equivalence relation generated by $\iota_0(x) \sim \iota_1(\psi_1(x)) \sim \iota_2(\psi_2(x))$, for $x \in A_0$. Hence we have $A_1 \sqcup_{A_0} A_2 \cong (A_1 \sqcup A_2)/_\sim$, where $\sim$ is the relation generated by reflexive closure from $\iota_1(\psi_1(x)) \sim \iota_2(\psi_2(x))$, for $x \in A_0$. In particular, if $A_0 = \emptyset$ then the coproduct $A_1 \sqcup A_2$ is the colimit. Note that if both $\psi_i$ are injective, then so are the induced maps $\iota_i\colon A_i \to A_1 \sqcup_{A_0} A_2$.

## 6 Sheaves

**(6.1) Presheaves. a)** Let $V$ be a topological space. We get a category $\mathcal{T} = \mathcal{T}_V$, consisting of the open subsets of $V$, whose morphism sets are defined as follows: Letting $W,U \subseteq V$ be open, if $W \not\subseteq U$ then $\mathrm{Hom}_{\mathcal{T}}(W,U) := \emptyset$, if $W \subseteq U$ then $\mathrm{Hom}_{\mathcal{T}}(W,U) := \{\iota_W^U\}$, where $\iota_W^U\colon W \to U$ is the natural inclusion map; in particular we have $\iota_U^U = \mathrm{id}_U$.

Let $\mathcal{A} \subseteq \mathbf{Sets}$ be a category; typically, we will have $\mathcal{A} \subseteq \mathbf{Ab}$. Then a **presheaf** on $V$ with **values** in $\mathcal{A}$ is a contravariant functor $\mathcal{F}\colon \mathcal{T} \to \mathcal{A}$.

Thus a presheaf $\mathcal{F}$ on $V$ assigns a set $\Gamma(U,\mathcal{F}) := \mathcal{F}(U) \in \mathcal{A}$ to any open subset $U \subseteq V$, whose elements are called the **sections** of $\mathcal{F}$ over $U$; in particular, the elements of $\Gamma(\mathcal{F}) := \mathcal{F}(V) \in \mathcal{A}$ are called the **global sections** of $\mathcal{F}$. Moreover, for $U' \subseteq U \subseteq V$ open there is a **restriction morphism**

$$\rho_{U'}^{U} = \rho_{U',\mathcal{F}}^{U} := \mathcal{F}(\iota_{U'}^{U})\colon \mathcal{F}(U) \to \mathcal{F}(U')\colon f \mapsto f|_{U'},$$

such that for $U'' \subseteq U' \subseteq U \subseteq V$ open we have $\rho_{U''}^{U'} \circ \rho_{U'}^{U} = \rho_{U''}^{U}$, and $\rho_{U}^{U} = \mathrm{id}_{\mathcal{F}(U)}$.

**b)** Letting $\mathcal{F}$ and $\mathcal{G}$ be presheaves on $V$ with values in $\mathcal{A}$, a **morphism of presheaves** is a natural transformation $\Phi\colon \mathcal{F} \Rightarrow \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are called **isomorphic** if they are naturally isomorphic; in this case we again write $\mathcal{F} \cong \mathcal{G}$.

Thus a morphism of presheaves $\Phi$ assigns a morphism $\Phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ to all open subsets $U \subseteq V$, such that for all $U' \subseteq U \subseteq V$ open we have

$$\rho_{U',\mathcal{F}}^{U} \cdot \Phi_{U'} = \Phi_U \cdot \rho_{U',\mathcal{G}}^{U}\colon \mathcal{F}(U) \to \mathcal{G}(U').$$

We get the **functor category** $\mathbf{PSh}(V) = \mathbf{PSh}(V, \mathcal{A})$, whose objects are the presheaves on $V$ with values in $\mathcal{A}$, having the morphism of presheaves as morphisms; it is immediately checked that it is a category indeed.

**(6.2) Sheaves.** A presheaf $\mathcal{F}$ on $V$ with values in $\mathcal{A}$ is called a **sheaf**, if the following additional condition holds: Whenever $U \subseteq V$ is open, and $\{U_i; i \in \mathcal{I}\}$ is an open covering of $U$, where $\mathcal{I}$ is an index set, and $\{f_i \in \mathcal{F}(U_i); i \in \mathcal{I}\}$ are **compatible** sections, that is $(f_i)|_{U_{ij}} = (f_j)|_{U_{ij}}$, where $U_{ij} := U_i \cap U_j$, for all $i, j \in \mathcal{I}$, then there is a unique section $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$, for $i \in \mathcal{I}$.

Since for any section $f \in \mathcal{F}(U)$ we have $(f|_{U_i})|_{U_{ij}} = f|_{U_{ij}} = (f|_{U_j})|_{U_{ij}}$ the compatibility condition is necessary. Thus the sheaf condition says that the compatibility condition is also sufficient for the existence of such a section. Moreover, it also says that any section is uniquely defined by its restrictions to the given open 'patches'.

In particular, for $\emptyset \subseteq V$, in view of the empty covering given by $\mathcal{I} = \emptyset$, the sheaf condition says that $\mathcal{F}(\emptyset)$ is a singleton set, $\{0\}$ say; thus if $\mathcal{A} \subseteq \mathbf{Ab}$ being the trivial group. Generally speaking, $\{0\}$ is a terminal object in $\mathcal{A}$. Hence for any $U \subseteq V$ open the restriction map $\rho_{\emptyset}^{U}$ is the unique morphism $\mathcal{F}(U) \to \{0\}$. In turn, if $\mathcal{F}$ is a presheaf such that $\mathcal{F}(\emptyset) = \{0\}$, then the compatibility condition is vacuous whenever $U_i \cap U_j = \emptyset$.

The **full** subcategory $\mathbf{Sh}(V) = \mathbf{Sh}(V, \mathcal{A}) \subseteq \mathbf{PSh}(V, \mathcal{A})$ has the sheaves on $V$ with values in $\mathcal{A}$ as its objects, and all morphisms of presheaves as morphisms.

**(6.3) Example: Function spaces. a)** Let $V$ and $W$ be topological spaces. Letting $\mathcal{C}(U)$ be the set of continuous maps from $U$ to $W$, for $U \subseteq V$ open, defines a presheaf $\mathcal{C}$ on $V$, restriction maps being given by restriction of functions.

Actually, $\mathcal{C}$ is a sheaf: For any open subset $U \subseteq V$, and any open covering $\{U_i; i \in \mathcal{I}\}$ of $U$, where $\mathcal{I}$ is an index set, any map $\varphi\colon U \to W$ is uniquely defined by its restrictions $\varphi|_{U_i}$. Conversely, since $\varphi$ is continuous if and only if for any $v \in U$ there is an open neighborhood $v \in U' \subseteq U$ such that $\varphi|_{U'}$ is continuous, prescribing compatible continuous maps $\varphi_i \colon U_i \to W$ defines a continuous map $\varphi \colon U \to T$ such that $\varphi|_{U_i} = \varphi_i$. ♯

**b)** Letting $U \subseteq \mathbb{C}$ be open, with respect to the metric topology, let $\mathcal{H}(U)$ be the set of holomorphic $\mathbb{C}$-valued functions on $U$, which is a $\mathbb{C}$-algebra with respect to pointwise addition and multiplication. This defines a presheaf $\mathcal{H}$ of $\mathbb{C}$-algebras on $\mathbb{C}$, whose restriction maps are given by restriction of functions.

Actually, $\mathcal{H}$ is a sheaf: For any open subset $U \subseteq \mathbb{C}$, and any open covering $\{U_i; i \in \mathcal{I}\}$ of $U$, where $\mathcal{I}$ is an index set, any function $f \colon U \to \mathbb{C}$ is uniquely defined by its restrictions $f|_{U_i}$. Conversely, since $f$ is holomorphic if and only if for any $z \in U$ there is an open disc $D_\epsilon(z) \subseteq U$, for some radius $\epsilon > 0$, around $z$ such that $f|_{D_\epsilon(z)}$ is complex differentiable, where the open discs are basis of the metric topology, prescribing compatible holomorphic functions $f_i \colon U_i \to \mathbb{C}$ defines a holomorphic function $f \colon U \to \mathbb{C}$ such that $f|_{U_i} = f_i$. ♯

**(6.4) Restricted sheaves. a)** Let $V$ be a topological space, and let $\mathcal{O}_V$ be a presheaf of functions on $V$, whose restriction maps are given by restriction of functions. For $W \subseteq V$ the **restricted** presheaf $\mathcal{O}_V|_W$ is defined as follows:

For $U \subseteq W$ open let $\mathcal{O}_V|_W(U)$ be the set of all functions $f \colon U \to L$, such that for all $v \in U$ there exist an open neighborhood $v \in U_v \subseteq V$ and a section $f_v \in \mathcal{O}_V(U_v)$, such that $U_v \cap W \subseteq U$ and $f_v|_{U_v \cap W} = f|_{U_v \cap W}$.

Then the natural inclusion map $\iota_W^V \colon W \to V$, being continuous, induces a comorphism $(\iota_W^V)^* \colon \mathcal{O}_V \Rightarrow \mathcal{O}_V|_W$:

Let $U \subseteq V$ be open and $f \in \mathcal{O}_V(U)$. Then we have $U' := (\iota_W^V)^{-1}(U) = U \cap W$. Letting $U_v = U$ and $f_v = f$, for all $v \in U'$, we get $(\iota_W^V)_U^*(f) = f|_{U \cap W} = f|_{U'}$, showing that $f|_{U'} \in \mathcal{O}_V|_W(U')$. ♯

Actually, $\mathcal{O}_V|_W$ is a sheaf (even if $\mathcal{O}_V$ is not a sheaf):

Let $U \subseteq W$ be open, let $\{U_i; i \in \mathcal{I}\}$ be an open covering of $U$, where $\mathcal{I}$ is an index set, let $f_i \in \mathcal{O}_V|_W(U_i)$ be a compatible set of functions, for $i \in \mathcal{I}$. Letting $f \colon U \to L$ be defined by $f|_{U_i} = f_i$, we show $f \in \mathcal{O}_V|_W(U)$: For $v \in U$, let $i \in \mathcal{I}$ such that $v \in U_i$, hence there are $v \in U_v \subseteq V$ open and $f_v \in \mathcal{O}_V(U_v)$, such that $U_v \cap W \subseteq U_i \subseteq U$ and $f_v|_{U_v \cap W} = f_i|_{U_v \cap W} = f|_{U_v \cap W}$. ♯

**b)** The case of $W \subseteq V$ being open is of particular interest: In this case, any open subset $U \subseteq W$ is open in $V$ as well, and $(\iota_W^V)_U^* \colon \mathcal{O}_V(U) \to \mathcal{O}_V|_W(U) \colon f \mapsto f|_U = f$ is the identity map, hence is injective; thus the morphism of presheaves $(\iota_W^V)^* \colon \mathcal{O}_V \Rightarrow \mathcal{O}_V|_W$ is called **injective**.

We define the presheaf $\mathcal{O}_W$ on $W$ by letting $\mathcal{O}_W(U) := \mathcal{O}_V(U)$, for any $U \subseteq W$ open, where the restriction maps of $\mathcal{O}_W$ are inherited from $\mathcal{O}_V$. Thus we get an injective morphism of presheaves $\sigma_{V|W} \colon \mathcal{O}_W \Rightarrow \mathcal{O}_V|_W$. In particular, $\sigma_{V|V} \colon \mathcal{O}_V \Rightarrow \mathcal{O}_V|_V$ is injective, saying that $\mathcal{O}_V$ is a **subpresheaf** of $\mathcal{O}_V|_V$, where the sheaf $\mathcal{O}_V|_V$ is called the **sheafification** of $\mathcal{O}_V$. (The sheafification fulfills a universal property, which we do not discuss here; see Exercise (13.5).)

If $\mathcal{O}_V$ already is a sheaf, we show that $\sigma_{V|W}\colon \mathcal{O}_W \Rightarrow \mathcal{O}_V|_W$ is **surjective** as well, that is $(\iota_W^V)_U^*\colon \mathcal{O}_V(U) \to \mathcal{O}_V|_W(U)$ is surjective, for all $U \subseteq W$ open:

Let $f \in \mathcal{O}_V|_W(U)$. Then for all $v \in U$ let $v \in U_v \subseteq V$ be open, and let $f_v \in \mathcal{O}_V(U_v)$, such that $U_v \cap W \subseteq U$ and $f_v|_{U_v \cap W} = f|_{U_v \cap W}$. Since $W$ is open, we may assume that $U_v \subseteq W$, or equivalently $U_v \subseteq U$, and thus $f_v = f|_{U_v}$. Since $\{U_v; v \in U\}$ is an open covering of $U$, the sheaf properties imply $f \in \mathcal{O}_V(U)$. ♯

Thus in this case we have $\sigma_{V|W}(U) = \mathrm{id}_{\mathcal{O}_W(U)}$, for all $U \subseteq W$ open. In particular, sheafification does not change a given sheaf.

**Example: Constant sheaves. i)** Let $V$ be topological space, let $A \neq \emptyset$ be a set, and let $0 \in A$. Letting $U \mapsto A$ for $\emptyset \neq U \subseteq V$ open, and $\emptyset \mapsto \{0\}$, and restriction maps being given as $\rho_{U'}^U = \mathrm{id}_A$, for all $\emptyset \neq U' \subseteq U \subseteq V$ open, defines a presheaf $\mathcal{C}$ on $V$, called the **constant presheaf** with values in $A$. In other words, $\mathcal{C}(U)$ can be considered as consisting of the constant maps from $U$ to $A$.

Then $\mathcal{C}$ is not necessarily a sheaf: Assume that $|A| \geq 2$, and that $V$ is **disconnected**, that is there are $\emptyset \neq U, W \subseteq V$ open such that $V = U \,\dot\cup\, W$. Then letting $a \in \mathcal{C}(U) = A$ and $b \in \mathcal{C}(W) = A$, where $a \neq b$, be sections on $U$ and $W$, respectively, the compatibility condition is trivially fulfilled. But for any global section $c \in \mathcal{C}(V) = A$ we get $\rho_U^V(c) = \mathrm{id}_A(c) = c$ and $\rho_W^V(c) = \mathrm{id}_A(c) = c$, thus the prescribed sections cannot be obtained by restriction from a global one.

**ii)** Let $A$ be equipped with the discrete topology. For $U \subseteq V$ open let $\mathcal{C}^0(U)$ be the set of continuous maps from $U$ to $A$. Together with restriction maps being given by restriction of functions this defines a presheaf $\mathcal{C}^0$ on $V$.

Note that, due to $A$ carrying the discrete topology, a map $\varphi\colon U \to A$ is continuous if and only if $\varphi^{-1}(\{a\}) \subseteq U$ is open, for all $a \in A$, which in turn is equivalent to $\varphi$ being **locally constant**, that is any point in $U$ has an open neighborhood on which $\varphi$ is constant. Hence $\mathcal{C}^0$ coincides with the sheafification of $\mathcal{C}$, thus $\mathcal{C}^0$ actually is a sheaf, being called the **(locally) constant sheaf** on $V$ with values in $A$. Alternatively, it can be shown directly that $\mathcal{C}^0$ is a sheaf:

For $U \subseteq V$ open, and any open covering $\{U_i; i \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set, any map $\varphi\colon U \to A$ is uniquely defined by its restrictions $\varphi|_{U_i}$. Conversely, $\varphi$ is continuous if and only if $U_a := \varphi^{-1}(\{a\}) \subseteq U$ is open, for $a \in A$. Now $U_a \subseteq U$ is open if and only if $U_a \cap U_i \subseteq U_i$ is open, for $i \in \mathcal{I}$. Hence giving compatible continuous maps $\varphi_i\colon U_i \to A$ defines a continuous map $\varphi$ such that $\varphi|_{U_i} = \varphi_i$.

## 7 Localization

**(7.1) Localization.** Let $R$ be a ring, and let $\mathcal{S} \subseteq R$ be a **multiplicatively closed** subset, that is $1 \in \mathcal{S}$ and for any $f, g \in \mathcal{S}$ we have $fg \in \mathcal{S}$ as well.

**a)** Let $M$ be an $R$-module, and for $f \in R$ let $\rho_M(f)\colon M \to M\colon m \to mf$. An $R$-module $M_{\mathcal{S}}$ together with a '**natural**' $R$-module homomorphism $\sigma\colon M \to M_{\mathcal{S}}$ is called the **localization** of $M$ at $\mathcal{S}$, or the **module of fractions** of $M$ with respect to $\mathcal{S}$, if it fulfills the following **universal property** in **Mod**-$R$: **i)** The map $\rho_{M_{\mathcal{S}}}(f)$ is bijective for all $f \in \mathcal{S}$, and **ii)** for any $R$-module $N$ such that $\rho_N(f)$ is bijective for all $f \in \mathcal{S}$, and any $R$-module homomorphism $\alpha\colon M \to N$, there is a unique $R$-module homomorphism $\widehat{\alpha}\colon M_{\mathcal{S}} \to N$ such that $\alpha = \sigma \cdot \widehat{\alpha}$.

It is immediate that the localization of $M$ at $\mathcal{S}$ is unique up to unique isomorphism of $R$-modules, if it exists at all. We show that a localization exists:

We consider the set $M \times \mathcal{S}$, and the relation $\sim$ given by $[m, f] \sim [m', f']$ if there is $g \in \mathcal{S}$ such that $(mf' - m'f)g = 0$. Then $\sim$ is an equivalence relation indeed: Reflexivity and symmetry are immediate; to show transitivity let $[m, f] \sim [m', f']$ and $[m', f'] \sim [m'', f'']$, hence there are $g, h \in \mathcal{S}$ such that $(mf' - m'f)g = 0 = (m'f'' - m''f')h$, thus we get $(mf'' - m''f)f'gh = mf'f''gh - (mf' - m'f)f''gh - (m'f'' - m''f')fgh - m''ff'gh = 0 \in M$.

The set of equivalence classes in $M \times \mathcal{S}$ with respect to $\sim$ is denoted by $M/\mathcal{S}$, and the equivalence class of $[m, f]$ is denoted by $\frac{m}{f} \in M/\mathcal{S}$. Then $M/\mathcal{S}$ becomes an $R$-module by letting $\frac{m}{f} + \frac{m'}{f'} := \frac{mf'+m'f}{ff'}$ and $\frac{m}{f} \cdot g := \frac{mg}{f}$, for $g \in R$; independence of the choice of representatives is immediately checked. Then $\sigma\colon M \to M/\mathcal{S}\colon m \mapsto \frac{m}{1}$ is an $R$-module homomorphism. We show that $M \times \mathcal{S}$ together with $\sigma$ fulfills the required universal property:

Firstly, $\rho_{M/\mathcal{S}}(g)\colon M/\mathcal{S} \to M/\mathcal{S}\colon \frac{m}{f} \mapsto \frac{mg}{f}$ is bijective, for any $g \in \mathcal{S}$, with inverse $M/\mathcal{S} \to M/\mathcal{S}\colon \frac{m}{f} \mapsto \frac{m}{fg}$.

Secondly, let $\alpha\colon M \to N$ be an $R$-module homomorphism, where $\rho_N(f)$ is bijective for all $f \in \mathcal{S}$. Then, for any $R$-module homomorphism $\widehat{\alpha}\colon M/\mathcal{S} \to N$ such that $\alpha = \sigma \cdot \widehat{\alpha}$ we have $\widehat{\alpha}(\frac{m}{f}) \cdot f = \widehat{\alpha}(\frac{m}{f} \cdot f) = \widehat{\alpha}(\frac{mf}{f}) = \widehat{\alpha}(\frac{m}{1}) = \alpha(m) \in N$, so that $\widehat{\alpha}(\frac{m}{f}) = \rho_N(f)^{-1}(\alpha(m)) \in N$, for $f \in \mathcal{S}$. Thus $\widehat{\alpha}$ is unique, if it exists at all. Now it is immediately checked that the latter formula indeed defines an $R$-module homomorphism $\widehat{\alpha}$ as desired; see also Exercise (13.8). ♯

**b)** In particular, for the regular $R$-module we get the **localization** $R_{\mathcal{S}}$ of $R$ at $\mathcal{S}$, or the **ring of fractions** of $R$ with respect to $\mathcal{S}$. Indeed $R_{\mathcal{S}}$ becomes a ring, by letting $\frac{g}{f} \cdot \frac{g'}{f'} := \frac{gg'}{ff'}$, as is immediately checked.

Then $R_{\mathcal{S}}$ has the following universal property in the category of (commutative unital) rings: **i)** $\frac{f}{1} \in R_{\mathcal{S}}$ is a unit, for all $f \in \mathcal{S}$, and **ii)** for any ring homomorphism $\alpha\colon R \to T$, such that $\alpha(f) \in T$ is a unit for $f \in \mathcal{S}$, there is a unique ring homomorphism $\widehat{\alpha}\colon R_{\mathcal{S}} \to T$ such that $\alpha = \sigma \cdot \widehat{\alpha}$.

Indeed, firstly, $\rho_{R_{\mathcal{S}}}(f)$ being bijective, thus having inverse $\rho_{R_{\mathcal{S}}}(\frac{1}{f})$, is equivalent to saying that $\sigma(f) = \frac{f}{1} \in R_{\mathcal{S}}$ is a unit. Secondly, considering $T$ as an $R$-module via $\alpha$, the universal property of $R_{\mathcal{S}}$ as an $R$-module implies the existence of a unique $R$-module homomorphism $\widehat{\alpha}$ as desired; then it immediately checked that $\widehat{\alpha}$ even is a ring homomorphism; see also Exercise (13.8). ♯

In combination, since $\frac{f}{1} \in R_{\mathcal{S}}$ is a unit, for any $R_{\mathcal{S}}$-module $N$ the map $\rho_N(\frac{f}{1})$ is bijective, having inverse $\rho_N(\frac{1}{f})$. Conversely, if $\rho_M(f)$ is bijective for all $f \in \mathcal{S}$, then $M$ becomes an $R_{\mathcal{S}}$-module by letting $m \cdot \frac{g}{f} := \rho_M(f)^{-1}(mg)$, as is immediately checked. In particular, the localization $M_{\mathcal{S}}$ becomes an $R_{\mathcal{S}}$-module by letting $\frac{m}{h} \cdot \frac{g}{f} := \frac{mg}{fh}$; thus $M_{\mathcal{S}}$ is generated by $\sigma(M)$ as an $R_{\mathcal{S}}$-module.

**(7.2) Properties of localizations. a)** The natural map $\sigma\colon M \to M_{\mathcal{S}}$ is an isomorphism of $R$-modules if and only if $\rho_M(f)$ is bijective, for all $f \in \mathcal{S}$: Indeed, the latter condition implies that $M$ together with the natural map $\mathrm{id}_M$ is a localization of $M$ at $\mathcal{S}$.

Similarly, $\sigma\colon R \to R_{\mathcal{S}}$ is a ring isomorphism if and only if $\mathcal{S} \subseteq R$ consists of units: Indeed, the latter condition implies that $R$ together with the natural map $\mathrm{id}_R$ is a localization of $R$ at $\mathcal{S}$.

**b)** An element $f \in R$ is called a **zero-divisor** on $M$, if $\rho_M(f)$ is not injective, that is there is $0 \neq m \in M$ such that $mf = 0$. Let $\mathcal{Z} = \mathcal{Z}_R(M) \subseteq R$ be the set of zero-divisors on $M$. In particular, $R \setminus \mathcal{Z}$ is multiplicatively closed; note that $0 \in \mathcal{Z}$ if and only if $R \neq \{0\}$.

For $m \in M$ let the **annihilator** be defined as $\mathrm{ann}_R(m) := \{f \in R; mf = 0\} \trianglelefteq R$; then we have $\mathcal{Z} = \bigcup_{0 \neq m \in M} \mathrm{ann}_R(m) \subseteq R$. For the natural map $\sigma\colon M \to M_{\mathcal{S}}$ we have $\ker(\sigma) = \{m \in M; \mathrm{ann}_R(m) \cap \mathcal{S} \neq \emptyset\}$: We have $m \in \ker(\sigma)$ if and only if $\frac{m}{1} = 0 \in M_{\mathcal{S}}$, that is $mf = 0$ for some $f \in \mathcal{S}$, which is equivalent to $\mathrm{ann}_R(m) \cap \mathcal{S} \neq \emptyset$. In particular, $\sigma$ is injective if and only if $\mathcal{S} \cap \mathcal{Z}_R(M) = \emptyset$. Moreover, we have $M_{\mathcal{S}} = \{0\}$ if and only if $\ker(\sigma) = M$, which is equivalent to $\mathrm{ann}_R(m) \cap \mathcal{S} \neq \emptyset$ for all $m \in M$ (or for an $R$-module generating set of $M$).

Similarly, the natural map $\sigma\colon R \to R_{\mathcal{S}}$ is injective if and only if $\mathcal{S} \cap \mathcal{Z}_R = \emptyset$, where $\mathcal{Z}_R = \mathcal{Z}_R(R)$ is the set of zero-divisors of $R$ (including 0 whenever $R \neq \{0\}$). We have $R_{\mathcal{S}} = \{0\}$ if and only if $1 \in \ker(\sigma) \trianglelefteq R$, which since $\mathrm{ann}_R(1) = \{0\}$ is equivalent to $0 \in \mathcal{S}$; in this case we have $M_{\mathcal{S}} = \{0\}$, for any $R$-module $M$.

**Example.** The ring $\mathrm{Q}(R) := R_{R \setminus \mathcal{Z}_R}$ is called the **(full) ring of fractions** of $R$, and we have a natural embedding $R \to \mathrm{Q}(R)$. In particular, if $R$ is a domain, then $\mathcal{Z}_R = \{0\}$, and $\mathrm{Q}(R) := R_{R \setminus \{0\}}$ is a field, called the **field of fractions** of $R$, which again comes with a natural embedding $R \to \mathrm{Q}(R)$.

**c)** An element $f \in R$ is called **nilpotent** on $M$, if $\rho_M(f)$ is nilpotent. Let $\mathcal{N}_R(M) \trianglelefteq R$ be the set of nilpotent elements on $M$. In particular, for $M = R$ we recover the nilradical $\mathcal{N}_R(R) = \mathrm{nil}(R) \trianglelefteq R$; note that $\mathrm{nil}(R) \subseteq \mathcal{N}_R(M)$, and that $0 \in \mathcal{S}$ if and only if $\mathcal{S} \cap \mathrm{nil}(R) \neq \emptyset$.

Now, for $f \in R$ let $\mathcal{S} := \{f^k \in R; k \in \mathbb{N}\}$. Then $R_f := R_{\mathcal{S}}$ is called the **localization of** $R$ **at** $f$. We have $R_f = \{0\}$ if and only if $f \in \mathrm{nil}(R)$.

**i)** If $R$ is reduced, then so is $R_f$: Since $\frac{1}{f} \in R_f$ is a unit, we consider $\frac{g}{1} \in \mathrm{nil}(R_f)$. Thus $\frac{g^k}{1} = 0 \in R_f$, for some $k \in \mathbb{N}$, hence $(gf^r)^k = g^k f^{rk} = 0 \in R$, for some

$r \in \mathbb{N}_0$; from $\mathrm{nil}(R) = \{0\}$ we conclude $gf^r = 0 \in R$, thus $\frac{g}{1} = 0 \in R_f$. $\sharp$

**ii)** For $f, g \in R$ we have a natural isomorphism rings $(R_f)_g \cong R_{fg}$: The image of $f$ and $g$, with respect to the natural maps, is a unit in both $(R_f)_g$ and $R_{fg}$. Hence the natural map $R \to R_{fg}$ factors through a natural map $\alpha\colon (R_f)_g \to R_{fg}$, and the natural map $R \to R_f \to (R_f)_g$ factors through a natural map $\beta\colon R_{fg} \to (R_f)_g$. By naturality we have $\alpha\beta = \mathrm{id}_{(R_f)_g}$ and $\beta\alpha = \mathrm{id}_{R_{fg}}$. $\sharp$

**iii)** Letting $T$ be an indeterminate, we have $R_f \cong R[T]/\langle fT - 1\rangle$: Since $f = \frac{1}{T} \in R[T]/\langle fT-1\rangle =: S$, the ring homomorphism $R \to R[T]$ extends to a homomorphism of $R$-algebras given by $\varphi\colon R_f \to S\colon \frac{1}{f} \mapsto T$. Conversely, the homomorphism of $R$-algebras given by $\widehat{\psi}\colon R[T] \to R_f\colon T \mapsto \frac{1}{f}$ factors through $\langle fT - 1\rangle \trianglelefteq R[T]$, giving rise to $\psi\colon S \to R_f$. Then $\varphi\psi = \mathrm{id}_{R_f}$ and $\psi\varphi = \mathrm{id}_S$. $\sharp$

We recover the **radical membership test**, see (2.9): We have $f \in \mathrm{nil}(R)$, that is $f$ is nilpotent, if and only if $R_f = \{0\}$, which is equivalent to $\langle fT-1\rangle = R[T]$.

**(7.3) Graded localization.** Let $R$ be a graded $K$-algebra, let $\mathcal{Z} \subseteq R$ be the set of zero-divisors, let $\mathcal{S} \subseteq R$ be multiplicatively closed such that $\mathcal{S} \cap \mathcal{Z} = \emptyset$, and let $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$ be its (multiplicatively closed) subset of homogeneous elements. Then $R_{\widetilde{\mathcal{S}}}$ is $\mathbb{Z}$-graded by $\deg(\frac{f}{g}) := \deg f - \deg(g)$, whenever $0 \neq f \in R$ is homogeneous. Recall that the natural map $R \to R_{\widetilde{\mathcal{S}}} \to R_{\mathcal{S}}$ is an embedding.

The homogeneous component $R^\sharp_{\mathcal{S}} := (R_{\widetilde{\mathcal{S}}})_0$ of $R_{\widetilde{\mathcal{S}}}$ of degree 0 is called the **graded localization** of $R$ at $\mathcal{S}$; it is a $K$-algebra again. In particular, $\mathrm{Q}^\sharp(R) := R^\sharp_{R\setminus\mathcal{Z}} \subseteq R_{R\setminus\mathcal{Z}} = \mathrm{Q}(R)$ is called the **(full) graded ring of fractions** of $R$. If moreover $R$ is a domain this yields the **graded field of fractions**

$$\mathrm{Q}^\sharp(R) = \{0\} \,\dot\cup\, \{\frac{f}{g} \in \mathrm{Q}(R); 0 \neq f, g \in R \text{ homogeneous}, \deg(f) = \deg(g)\}.$$

## 8 Spaces with functions

**(8.1) Principal open subsets.** Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed, let $n \in \mathbb{N}_0$, let $\mathcal{X} := \{X_1, \ldots, X_n\}$ and $\mathcal{X}^\sharp := \{X_0\} \,\dot\cup\, \mathcal{X}$ be indeterminates, and let $A := K[\mathcal{X}]$ and $A^\sharp := K[\mathcal{X}^\sharp]$.

**a)** Let $\mathbf{V} \subseteq L^n$ be closed, and let $f \in K[\mathbf{V}]$. The associated **principal open subset** is defined as

$$D_f := \mathbf{V} \setminus \mathbf{V}_{\mathbf{V}}(f) = \{v \in \mathbf{V}; f(v) \neq 0\} \subseteq \mathbf{V}.$$

Then we have $D(f^k) = D(f)$ for $k \in \mathbb{N}$, and $D_{fg} = D_f \cap D_g$ for $g \in K[\mathbf{V}]$, and $D_0 = \emptyset$ and $D_1 = \mathbf{V}$. Moreover, the set $\{D_f \subseteq \mathbf{V}; f \in K[\mathbf{V}]\}$ is a **basis** of the Zariski topology on $\mathbf{V}$, where actually any open subset $U \subseteq \mathbf{V}$ is a finite union of principal open subsets:

Let $U = \mathbf{V} \setminus \mathbf{V}_{\mathbf{V}}(I)$, where $I = \langle f_1, \ldots, f_r \rangle \trianglelefteq K[\mathbf{V}]$, for some $r \in \mathbb{N}_0$. Then for all $f \in I$ we have $\mathbf{V}_{\mathbf{V}}(I) \subseteq \mathbf{V}_{\mathbf{V}}(f)$, that is $D_f \subseteq U$. Moreover, for any $u \in U = \mathbf{V} \setminus \mathbf{V}_{\mathbf{V}}(I)$ there is $i \in \{1, \ldots, r\}$ such that $f_i(u) \neq 0$, that is $u \in D_{f_i}$. This implies that $U = \bigcup_{f \in I} D_f = \bigcup_{i=1}^r D_{f_i}$. ♯

Letting $U \subseteq \mathbf{V}$ be open, the topology induced by the Zariski topology on $\mathbf{V}$ is called the **($K$-)Zariski topology** on $U$. Then $\{D_f \subseteq U; f \in \mathbf{I}_{\mathbf{V}}(\mathbf{V} \setminus U)\}$ is a basis of the Zariski topology on $U$. Moreover, any subset $M \subseteq \mathbf{V}$ is quasi-compact, in a very explicit sense: (We have seen this in (3.3) for $\mathbf{V}$ itself, and it actually holds for any Noetherian topological space, see Exercise (12.11).)

We may assume that the covering is given as $M = \bigcup_{i \in \mathcal{I}} (D_{f_i} \cap M) = M \cap \bigcup_{i \in \mathcal{I}} D_{f_i}$, where $\mathcal{I}$ is an index set. Let $I := \langle f_i; i \in \mathcal{I} \rangle = \langle f_j; j \in \mathcal{J} \rangle \trianglelefteq K[\mathbf{V}]$, for some $\mathcal{J} \subseteq \mathcal{I}$ finite. Then we get $\mathbf{V} \setminus \bigcup_{i \in \mathcal{I}} D_{f_i} = \bigcap_{i \in \mathcal{I}} (\mathbf{V} \setminus D_{f_i}) = \bigcap_{i \in \mathcal{I}} \mathbf{V}_{\mathbf{V}}(f_i) = \mathbf{V}_{\mathbf{V}}(I) = \bigcap_{j \in \mathcal{J}} \mathbf{V}_{\mathbf{V}}(f_j) = \bigcap_{j \in \mathcal{J}} (\mathbf{V} \setminus D_{f_i}) = \mathbf{V} \setminus \bigcup_{j \in \mathcal{J}} D_{f_j}$. Thus we have $\bigcup_{i \in \mathcal{I}} D_{f_i} = \bigcup_{j \in \mathcal{J}} D_{f_j}$, hence $M = M \cap \bigcup_{j \in \mathcal{J}} D_{f_j} = \bigcup_{j \in \mathcal{J}} (D_{f_j} \cap M)$. ♯

**b)** Let $\mathbf{V} \subseteq \mathbf{P}$ be closed. For $f \in K[\mathbf{V}]$ homogeneous let $D_f := \mathbf{V} \setminus \mathbf{V}_L^\sharp(f) \subseteq \mathbf{V}$, being open in $\mathbf{V}$. Letting $D_i = D_{X_i}$, for $i \in \{0, \ldots, n\}$, we have $D_0 = \mathbf{V} \cap L^n \subseteq L^n$, and similarly $D_i$ can be identified with a subset of $L^n$, for all $i$.

Silently identifying $f \in K[\mathbf{V}]$ homogeneous with a homogeneous representative $f \in A$, we have $D_f \cap D_0 = D_{fX_0} = D_{f'} \subseteq L^n$, where conversely for $g \in A^\sharp$ from $(g^\sharp)' = g$ we get $D_g = D_{g^\sharp X_0} = D_{g^\sharp} \cap D_0 \subseteq L^n$; and similar for all $i$.

Hence $\mathbf{V} = \bigcup_{i=0}^n D_i$ implies that $\{D_f \subseteq \mathbf{V}; f \in K[\mathbf{V}] \text{ homogeneous}\}$ is a basis of the Zariski topology on $\mathbf{V}$. Moreover, any subset of $\mathbf{V}$ is a quasi-compact topological space, with respect to the induced topology.

Letting $U \subseteq \mathbf{V}$ be open, the topology induced by the Zariski topology on $\mathbf{V}$ is called the **($K$-)Zariski topology** on $U$, which has the set $\{D_f \subseteq U; f \in \mathbf{I}_K^\sharp(\mathbf{V} \setminus U) \text{ homogeneous}\}$ as a basis; note that $\mathbf{I}_K^\sharp(\mathbf{V}) \subseteq \mathbf{I}_K^\sharp(\mathbf{V} \setminus U)$.

**(8.2) Regular functions on affine varieties. a)** Let $\mathbf{V} \subseteq L^n$ be closed, and let $U \subseteq \mathbf{V}$ be open. A function $\varphi \colon U \to L$ is called **regular** at a point $v \in U$, if there are $f, g \in K[\mathbf{V}]$, such that $v \in D_g \subseteq U$ and $\varphi(u) = \frac{f(u)}{g(u)}$, for all $u \in D_g$; note that we may assume $D_g \subseteq U$. Moreover, $\varphi$ is called **regular** on $U$, if it is regular at any point of $U$. (It will follow from (9.2) below that any regular function is continuous. Hence a regular function is already uniquely determined on any dense subset.)

If $K[\mathbf{V}]$ is a domain, that is $\mathbf{V}$ is irreducible, then we may write $\frac{f}{g} \in K(\mathbf{V}) := \mathrm{Q}(K[\mathbf{V}])$, where $K(\mathbf{V})$ is called the field of **rational functions** on $\mathbf{V}$. Note that $K(\mathbf{V})$ induces regular functions which are only defined on certain (dense) open subsets of $\mathbf{V}$. (But by continuity this suffices to identify $K(\mathbf{V})$ with this set of 'globally rational' functions.)

**b)** It is immediate that the set $\mathcal{O}_{\mathbf{V}}(U)$ of regular functions on $U$ is a $K$-algebra; recall that $D_g \cap D_{g'} = D_{gg'}$ for $g, g' \in K[\mathbf{V}]$. Then it is immediate that associating the $K$-algebra $\mathcal{O}_{\mathbf{V}}(U)$ to any open subset $U \subseteq \mathbf{V}$, together with restriction of functions $\rho^U_{U'} : \mathcal{O}_{\mathbf{V}}(U) \to \mathcal{O}_{\mathbf{V}}(U') : f \mapsto f|_{U'}$ for any pair $U' \subseteq U \subseteq \mathbf{V}$ of open subsets, and letting $\mathcal{O}_{\mathbf{V}}(\emptyset) := \{0\}$, defines a presheaf $\mathcal{O}_{\mathbf{V}} : \mathcal{T}_{\mathbf{V}} \to K\text{-}\mathbf{Alg}$, where $K$-**Alg** is the category of $K$-algebras together with all homomorphisms of $K$-algebras as morphisms. Actually, $\mathcal{O}_{\mathbf{V}}$ is a sheaf:

For any open subset $U \subseteq \mathbf{V}$, and any open covering $\{U_i; i \in \mathcal{I}\}$ of $U$, where $\mathcal{I}$ is an index set, any function $f : U \to L$ is uniquely defined by its restrictions $f|_{U_i}$. Conversely, $f : U \to L$ is regular if and only if for any $v \in U$ there is a principal open subset $v \in D_g \subseteq U$ such that $f|_{D_g} = \frac{h}{g}$, for some $g, h \in K[\mathbf{V}]$; hence, since the principal open subsets are a basis of the Zariski topology, prescribing compatible regular functions $f_i : U_i \to L$ defines a regular function on $U$. ♯

Then $\mathbf{V}$, together with the **sheaf of regular functions** or **structure sheaf** $\mathcal{O}_{\mathbf{V}}$, is called an **affine ($K$-)variety**. We will show in (9.1) below that $\Gamma(\mathcal{O}_{\mathbf{V}}) = K[\mathbf{V}]$, and that $\mathcal{O}_{\mathbf{V}}$ is determined by $K[\mathbf{V}]$, so that the present definition of an affine variety coincides with the one given in (3.5). Note that the Zariski topology is built into the structure sheaf anyway.

Moreover, we get the restricted sheaf $\mathcal{O}_U := \mathcal{O}_{\mathbf{V}}|_U : \mathcal{T}_U \to K\text{-}\mathbf{Alg}$, that is we have $\mathcal{O}_U(U') := \mathcal{O}_{\mathbf{V}}(U')$ for any $U' \subseteq U \subseteq \mathbf{V}$ open. Then $U$, together with the sheaf $\mathcal{O}_U$, is called a **quasi-affine ($K$-)variety**.

**(8.3) Example: A quadric hypersurface.** We show that a regular function on a quasi-affine variety is not necessarily globally rational:

**i)** Let $K = L = \mathbb{C}$, and $f := WX - YZ \in A := \mathbb{C}[W, X, Y, Z]$. Then, since $f$ is homogeneous, it is immediate that $f$ is irreducible. We consider the irreducible affine hypersurface $\mathbf{V} := \mathbf{V}(f) \subseteq \mathbb{C}^4$, where $\mathbb{C}[\mathbf{V}] = A/\langle f \rangle$ is a graded domain.

Since $f$ has degree 2, we conclude that $\{W, X, Y, Z\} \subseteq \mathbb{C}[\mathbf{V}]_1$ is a $\mathbb{C}$-basis. In particular, the elements $W, X, Y, Z$ are irreducible and pairwise non-associated. This implies that none of these elements is prime: It suffices to consider $X$; we have $X \mid YZ \in \mathbb{C}[\mathbf{V}]$, but $X \nmid Y$ and $X \nmid Z$. Thus $\mathbb{C}[\mathbf{V}]$ is not factorial.

**ii)** We proceed to present an open subset $U \subseteq \mathbf{V}$ and a regular function $\varphi \in \mathcal{O}_{\mathbf{V}}(U)$ which is not given by an element of $\mathbb{C}(\mathbf{V})$. We need a few preparations:

Firstly, $A$ is $\mathbb{Z}$-graded by letting $\delta(X) = 1 = \delta(Y)$ and $\delta(W) = -1 = \delta(Z)$; note that none of the associated homogeneous components is finite-dimensional. Then, since $f$ is homogeneous of degree $\delta(f) = 0$, the coordinate algebra $\mathbb{C}[\mathbf{V}]$ is $\mathbb{Z}$-graded as well, with respect to the inherited grading. Comparing with the standard grading, for $k \in \mathbb{N}_0$ and $W^a X^b Y^c Z^d \in A_k$ we get $\delta(W^a X^b Y^c Z^d) = (b - a) + (c - d) = k - 2(a + d) \in \{-k, -k + 2, \ldots, k - 2, k\}$.

Next, since $f$ is binomial, we infer that the following set is a $\mathbb{C}$-basis of $\mathbb{C}[\mathbf{V}]$:

$$\mathcal{B} := \left(\{1\} \mathbin{\dot\cup} \{X^i; i \geq 1\} \mathbin{\dot\cup} \{W^i; i \geq 1\}\right) \cdot \{Y^i; i \geq 0\} \cdot \{Z^i; i \geq 0\} \subseteq \mathbb{C}[\mathbf{V}].$$

We show that the divisors of $Y^m \in \mathbb{C}[\mathbf{V}]$ are $\{1, Y, \ldots, Y^m\}$, up to associates:

We proceed by induction on $m \geq 0$, the cases $m \leq 1$ being trivial; let $m \geq 2$. Since the divisors of $Y^m$ are homogeneous with respect to both gradings, let $Y^m = pq$, where $p$ and $q$ are homogeneous of degree $k \geq 1$ and $l \geq 1$, respectively, such that $k + l = m$. Then, since $\delta(Y^m) = m$, we conclude that $p$ and $q$ are homogeneous such that $\delta(p) = k$ and $\delta(q) = l$. This entails $p = \sum_{i=0}^{k} a_i X^i Y^{k-i}$ and $q = \sum_{j=0}^{l} b_j X^j Y^{l-j}$, yielding $Y^m = pq = \sum_{i=0}^{k} \sum_{j=0}^{l} a_i b_j X^{i+j} Y^{m-(i+j)}$. Comparing coefficients, for $i = k$ and $j = l$ we get $a_k b_l = 0$, so that we may assume that $a_k = 0$. Hence writing $p = p' \cdot Y$ we get $p'q = Y^{m-1}$. ♯

**iii)** Let $U := D_W \cup D_Y \subseteq \mathbf{V}$ open, and let $\varphi \in \mathcal{O}_\mathbf{V}(U)$ such that $\varphi|_{D_W} = \frac{Z}{W}$ and $\varphi|_{D_Y} = \frac{X}{Y}$, where since $\frac{Z}{W}|_{D_{WY}} = \frac{X}{Y}|_{D_{WY}}$ the function $\varphi$ is well-defined.

Assume that $\varphi = \frac{h}{g}$ on $U$, where $0 \neq g, h \in \mathbb{C}[\mathbf{V}]$; hence we have $\mathbf{V}(g) \subseteq \mathbf{V} \setminus U = \mathbf{V}_\mathbf{V}(W, Y)$. Since $f \in \langle W, Y\rangle_A \trianglelefteq A$, we have $\mathbb{C}[\mathbf{V}]/\langle W, Y\rangle \cong A/\langle W, Y\rangle_A \cong \mathbb{C}[X, Z]$, hence $\langle W, Y\rangle \trianglelefteq \mathbb{C}[\mathbf{V}]$ is prime. Thus we get $\langle W, Y\rangle \subseteq \sqrt{\langle g\rangle} \trianglelefteq \mathbb{C}[\mathbf{V}]$, hence there is $l \in \mathbb{N}$ such that $g \mid Y^l \in \mathbb{C}[\mathbf{V}]$.

Thus we may assume that $g = Y^k$, for some $k \in \mathbb{N}_0$. Then considering $v := [1, 0, 0, 1] \in D_W \setminus D_Y \subseteq U \subseteq \mathbf{V}$ yields $0^k = Y^k(v) = g(v) \neq 0$, hence $k = 0$, that is $g = 1$. By continuity, which follows from (9.2) below, this entails $h = \frac{X}{Y} \in \mathbb{C}(\mathbf{V})$. Thus we get $h \cdot Y = X \in \mathbb{C}[\mathbf{V}]$, a contradiction. ♯

**(8.4) Regular functions on projective varieties. a)** Let again $K \subseteq L$ be a field extension, where $L$ is algebraically closed, let $\mathbf{V} \subseteq \mathbf{P}$ be closed, and let $U \subseteq \mathbf{V}$ be open. Recall that the homogeneous coordinate algebra $K[\mathbf{V}]$ does not induce functions on $\mathbf{P}$, due to the use of homogeneous coordinates. But $K[\mathbf{V}]$ is a graded $K$-algebra:

Let $0 \neq f, g \in K[\mathbf{V}]$ such that $d := \deg(f) = \deg(g)$. Then for $[x_0 \colon \ldots \colon x_n] \in D_g \subseteq \mathbf{V}$ we have $\frac{f(\lambda\cdot[x_0,\ldots,x_n])}{g(\lambda\cdot[x_0,\ldots,x_n])} = \frac{\lambda^d \cdot f(x_0,\ldots,x_n)}{\lambda^d \cdot g(x_0,\ldots,x_n)} = \frac{f(x_0,\ldots,x_n)}{g(x_0,\ldots,x_n)}$, for all $0 \neq \lambda \in L$. Hence this indeed defines a function on $D_g$. Having observed this:

A function $\varphi \colon U \to L$ is called **regular** at a point $v \in U$, if there are $f, g \in K[\mathbf{V}]$, where $g \neq 0$, and $f = 0$ or $\deg(f) = \deg(g)$, such that $v \in D_g \subseteq U$ and $\varphi(u) = \frac{f(u)}{g(u)}$, or all $u \in D_g$; note that we may assume that $D_g \subseteq U$. Moreover, $\varphi$ is called **regular** on $U$, if it is regular at any point of $U$. (We will see in (9.4) below that any regular function is continuous.)

If $K[\mathbf{V}]$ is a domain, that is $\mathbf{V}$ is irreducible, then we may write $\frac{f}{g} \in K(\mathbf{V}) := \mathrm{Q}^\sharp(K[\mathbf{V}])$, where $K(\mathbf{V})$ is called the field of **rational functions** on $\mathbf{V}$. Note that $K(\mathbf{V})$ induces regular functions which are only defined on certain (dense) open subsets of $\mathbf{V}$. (Again, it follows from continuity that this suffices to identify $K(\mathbf{V})$ with this set of 'globally rational' functions.)

**b)** It is immediate that the set $\mathcal{O}_\mathbf{V}(U)$ of regular functions on $U$ is a $K$-algebra. Moreover, associating the $K$-algebra $\mathcal{O}_\mathbf{V}(U)$ to any open subset $U \subseteq \mathbf{V}$, together with restriction of functions between open subsets, and letting $\mathcal{O}_\mathbf{V}(\emptyset) := \{0\}$, defines a sheaf $\mathcal{O}_\mathbf{V} \colon \mathcal{T}_\mathbf{V} \to K$-**Alg** on $\mathbf{V}$, and by restriction a sheaf $\mathcal{O}_U :=$

$\mathcal{O}_{\mathbf{V}}|_U : \mathcal{T}_U \to K$-**Alg** on any open subset $U \subseteq \mathbf{V}$. Then $\mathcal{O}_U$ is called the **sheaf of regular functions** on $U$, or the **structure sheaf** of $U$.

Then $\mathbf{V}$, together with the sheaf $\mathcal{O}_{\mathbf{V}}$, is called a **projective ($K$-)variety**; and $U$, together with the sheaf $\mathcal{O}_U$, is called a **quasi-projective ($K$-)variety**. Note that the Zariski topology is built into the structure sheaf anyway. Unfortunately, if $\mathbf{V}$ is irreducible, we will show in (9.4) below that $\Gamma(\mathcal{O}_{\mathbf{V}}) \cong K$, so that the global sections are not too useful, and that the homogeneous coordinate algebra $K[\mathbf{V}]$ cannot be recovered from $\mathcal{O}_{\mathbf{V}}$.

**(8.5) Varieties as spaces with functions. a)** Any affine, quasi-affine, projective or quasi-projective variety is briefly called a **($K$-)variety (over $L$)**.

Then, generally speaking, a variety $V$ is a topological space together with a sheaf $\mathcal{O}_V$ of $K$-algebras of $L$-valued functions, where $K$ is identified with the constant (thus $K$-valued) functions, and whose restriction maps are given by restriction of functions. Then the pair $(V, \mathcal{O}_V)$ is called a **space with functions**. (Actually, this is a special case of a **ringed space**.)

This yields the definition of morphisms of varieties, and thus the category of ($K$-)varieties (over $L$). We will show in (9.1) below that for affine varieties the new definition of morphisms and the earlier notion of regular maps coincide.

**b)** Let $(U, \mathcal{O}_U)$ and $(V, \mathcal{O}_V)$ be spaces with functions. Then a continuous map $\varphi\colon U \to V$ is called a **morphism** (of spaces with functions), if for any $W \subseteq V$ open and any $f \in \mathcal{O}_V(W)$ we have $\varphi^*_W(f) := (f \circ \varphi)|_{\varphi^{-1}(W)} \in \mathcal{O}_U(\varphi^{-1}(W))$.

Thus the **comorphism** $\varphi^*\colon \mathcal{O}_V \Rightarrow \mathcal{O}_U$ induces homomorphisms of $K$-algebras $\varphi^*_W\colon \mathcal{O}_V(W) \to \mathcal{O}_U(\varphi^{-1}(W))$, which commute with the restriction of functions, that is for $W' \subseteq W \subseteq V$ open we have $\varphi^*_{W'} \circ \rho^W_{W' \subseteq V} = \rho^{\varphi^{-1}(W)}_{\varphi^{-1}(W') \subseteq U} \circ \varphi^*_W$. (The pair $(\varphi, \varphi^*)$ is a special case of a **morphism of ringed spaces**.)

In other words, the assignment $\varphi^*\colon \mathcal{O}_V \Rightarrow \mathcal{O}_U$ behaves like a (contravariant) natural transformation (that is a morphism of sheaves, where we consider sheaves as functors) from the sheaf $\mathcal{O}_V\colon \mathcal{T}_V \to K$-**Alg** to the sheaf $\mathcal{O}_U\colon \mathcal{T}_U \to K$-**Alg**; alone the sheaves in question are based on different topological spaces (that is the functors are defined on different categories). Anyway:

In particular, $\mathrm{id}_U$ is a morphism, whose comorphism $(\mathrm{id}_U)^*\colon \mathcal{O}_U \Rightarrow \mathcal{O}_U$ induces the identity isomorphism on all sections of $\mathcal{O}_U$. Moreover, if $(W, \mathcal{O}_W)$ is a space with functions, and $\psi\colon V \to W$ is a morphism, then $\varphi\psi\colon U \to W$ is a morphism again, where the associated comorphisms fulfill $(\varphi\psi)^* = \psi^*\varphi^*$. This gives rise to the category of spaces with ($K$-algebras of $L$-valued) functions.

A morphism $\varphi\colon U \to V$ is called an **isomorphism** (of spaces with functions), if there is a morphism $\psi\colon V \to U$ such that $\varphi\psi = \mathrm{id}_U$ and $\psi\varphi = \mathrm{id}_V$. This is equivalent to saying that $\varphi\colon U \to V$ is a homeomorphism, such that the associated comorphisms fulfill $\psi^*\varphi^* = (\mathrm{id}_U)^*$ and $\varphi^*\psi^* = (\mathrm{id}_V)^*$, where the latter in turn is equivalent to saying that $\varphi^*_W\colon \mathcal{O}_V(W) \to \mathcal{O}_U(\varphi^{-1}(W))$ is an isomorphism with inverse $\psi^*_{\psi(W)}\colon \mathcal{O}_U(\psi(W)) \to \mathcal{O}_V(W)$, for any $W \subseteq V$ open.

# 9 Varieties

**(9.1) Affine varieties. a)** Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed, and let $\mathbf{V} \subseteq L^n$ be an affine variety.

**Theorem.** Let $f \in K[\mathbf{V}]$. Then we have $\mathcal{O}_{\mathbf{V}}(D_f) = K[\mathbf{V}]_f$.

**Proof.** We may assume that $f \neq 0$, and let $\varphi \in \mathcal{O}_{\mathbf{V}}(D_f)$. Then there are $f_i, g_i \in K[\mathbf{V}]$ for $i \in \{1, \ldots, r\}$, such that $\varphi|_{D_{f_i}} = \frac{g_i}{f_i}|_{D_{f_i}}$, where $D_f = \bigcup_{i=1}^r D_{f_i}$. Then we have $(f_i g_j - f_j g_i)|_{D_{f_i f_j}} = 0$, for all $i, j$, hence $f_i f_j (f_i g_j - f_j g_i) = 0 \in K[\mathbf{V}]$. Rewriting this as $f_i^2 (f_j g_j) - f_j^2 (f_i g_i) = 0$, and observing $\varphi|_{D_{f_i}} = \frac{f_i g_i}{f_i^2}|_{D_{f_i}}$, we may assume that $\varphi|_{D_{f_i}} = \frac{g_i}{f_i}|_{D_{f_i}}$, where $f_i g_j - f_j g_i = 0 \in K[\mathbf{V}]$.

From $\mathbf{V}_{\mathbf{V}}(f) = \mathbf{V} \setminus D_f = \bigcap_{i=1}^r (\mathbf{V} \setminus D_{f_i}) = \bigcap_{i=1}^r \mathbf{V}_{\mathbf{V}}(f_i) = \mathbf{V}_{\mathbf{V}}(f_1, \ldots, f_r)$ we get $f \in \mathbf{I}_{\mathbf{V}}(\mathbf{V}_{\mathbf{V}}(f_1, \ldots, f_r)) = \sqrt{\langle f_1, \ldots, f_r \rangle}$, hence there are $k \in \mathbb{N}$ and $h_i \in K[\mathbf{V}]$ such that $f^k = \sum_{i=1}^r h_i f_i$. This yields $f^k g_j = \sum_{i=1}^r h_i f_i g_j = \sum_{i=1}^r h_i g_i f_j = (\sum_{i=1}^r h_i g_i) f_j \in K[\mathbf{V}]$. Thus letting $h := \sum_{i=1}^r h_i g_i \in K[\mathbf{V}]$ we get $\frac{g_j}{f_j}|_{D_{f_j}} = \frac{h}{f^k}|_{D_{f_j}}$. This implies that $\varphi = \frac{h}{f^k}|_{D_f}$.

Since $f \in \mathcal{O}_{\mathbf{V}}(D_f)$ is a unit, there is a $K$-algebra epimorphism $\alpha \colon K[\mathbf{V}]_f \to \mathcal{O}_{\mathbf{V}}(D_f)$. In order to show that $\alpha$ in injective, let $\frac{g}{f^k}|_{D_f} = \frac{h}{f^l}|_{D_f}$, where $g, h \in K[\mathbf{V}]$ and $k, l \in \mathbb{N}_0$. Then we have $(gf^l - hf^k)|_{D_f} = 0$, which implies that $(gf^l - hf^k) f = 0 \in K[\mathbf{V}]$. Thus we have $\frac{g}{f^k} = \frac{h}{f^l} \in K[\mathbf{V}]_f$. ♯

**Corollary.** We have $\Gamma(\mathcal{O}_{\mathbf{V}}) = K[\mathbf{V}]$.

Recall that the Zariski topology on $\mathbf{V}$ can be recovered from $K[\mathbf{V}]$. Moreover, a map $\varphi \colon \mathbf{V} \to L$ is regular in the sense of (3.5), that is $\varphi \in K[\mathbf{V}]$, if and only if it is a regular function on $\mathbf{V}$, that is $\varphi \in \mathcal{O}_{\mathbf{V}}(\mathbf{V}) = \Gamma(\mathcal{O}_{\mathbf{V}})$.

We observe that the $K$-algebra of regular functions on any principal open subset of $\mathbf{V}$ can be recovered from $K[\mathbf{V}]$ as well, thus by the sheaf properties this holds for any open subset of $\mathbf{V}$. In other words, the structure sheaf of any quasi-affine variety is determined by the coordinate algebra of the affine variety it is open in. In particular, this holds for affine varieties themselves, so that the present definition of affine varieties coincides with the earlier one given in (3.5).

**b)** Let $\mathbf{W} \subseteq L^m$ be an affine variety. Then a map $\varphi \colon \mathbf{V} \to \mathbf{W}$ is regular in the sense of (3.6) if and only if $\varphi^*(K[\mathbf{W}]) \subseteq K[\mathbf{V}]$, see (3.7); in this case $\varphi^* \colon K[\mathbf{W}] \to K[\mathbf{V}]$ is a homomorphism of $K$-algebras. Actually, the earlier notion of regularity indeed coincides with the present definition of morphisms:

**Corollary.** The map $\varphi$ is regular if and only if it is a morphism of varieties.

**Proof.** If $\varphi$ is a morphism of varieties, that is $\varphi$ is continuous and for any open subset $U \subseteq \mathbf{W}$ we have $\varphi_U^*(\mathcal{O}_\mathbf{W}(U)) \subseteq \mathcal{O}_\mathbf{V}(\varphi^{-1}(U))$, then for $U = \mathbf{W}$ we get $\varphi^*(K[\mathbf{W}]) = \varphi_\mathbf{W}^*(\mathcal{O}_\mathbf{W}(\mathbf{W})) \subseteq \mathcal{O}_\mathbf{V}(\varphi^{-1}(\mathbf{W})) = \mathcal{O}_\mathbf{V}(\mathbf{V}) = K[\mathbf{V}]$.

Conversely, if $\varphi$ is regular, then it is continuous. It remains to show that $\varphi^*(\mathcal{O}_\mathbf{W}(U)) \subseteq \mathcal{O}_\mathbf{V}(\varphi^{-1}(U))$ for any open subset $U \subseteq \mathbf{W}$. Since the principal open subsets are a basis of the Zariski topology on $U$, by the sheaf properties it suffices to consider the principal open subsets $D_f \subseteq \mathbf{W}$, where $f \in K[\mathbf{W}]$:

We have $\varphi^{-1}(D_f) = \{v \in \mathbf{V}; f(\varphi(v)) \neq 0\} = D_{\varphi^*(f)} \subseteq \mathbf{V}$. Moreover, the homomorphism $\varphi^* \colon K[\mathbf{W}] \to K[\mathbf{V}]$ extends naturally to $\varphi_f^* \colon K[\mathbf{W}]_f \to K[\mathbf{V}]_{\varphi^*(f)}$. Thus we have $\varphi_{D_f}^*(\mathcal{O}_\mathbf{W}(D_f)) = \varphi_f^*(K[\mathbf{W}]_f) \subseteq K[\mathbf{V}]_{\varphi^*(f)} = \mathcal{O}_\mathbf{V}(D_{\varphi^*(f)})$. ♯

In particular, for $\mathbf{W} = L$, we conclude that the set $\Gamma(\mathcal{O}_\mathbf{V}) = K[\mathbf{V}]$ of regular functions on $\mathbf{V}$ consists precisely of the morphisms of affine varieties $\mathbf{V} \to L$.

**(9.2) Principal open subsets of affine varieties.** Let $\mathbf{V} \subseteq L^n$ be closed. We show that any principal open subset of $\mathbf{V}$ is isomorphic to an affine variety:

Let $I := \mathbf{I}_K(\mathbf{V}) \trianglelefteq A$, and let $T$ be an indeterminate. Then for $f \in K[\mathbf{V}] = A/I$ let $\widehat{I}_f := \langle I, fT-1\rangle \trianglelefteq A[T]$ (which is independent of the chosen coset representative for $f$). We consider the affine closed set

$$\widehat{\mathbf{V}}_f := \mathbf{V}_L(\widehat{I}_f) = \{[v,t] \in \mathbf{V} \times L; f(v)\cdot t = 1\} \subseteq L^n \times L = L^{n+1}.$$

In order to show $K[\widehat{\mathbf{V}}_f] = A[T]/\mathbf{I}_K(\widehat{\mathbf{V}}_f) = A[T]/\widehat{I}_f$, we show that $A[T]/\widehat{I}_f$ is reduced: From $A[T]/\langle I\rangle \cong (A/I)[T]$ we get $A[T]/\widehat{I}_f \cong (A/I)[T]/\langle fT-1\rangle = K[\mathbf{V}][T]/\langle fT-1\rangle \cong K[\mathbf{V}]_f$. Now, since $K[\mathbf{V}]$ is reduced, so is $K[\mathbf{V}]_f$. ♯

**Theorem.** We have $\widehat{\mathbf{V}}_f \cong D_f$ as varieties.

**Proof.** Let $\widehat{\pi} \colon L^{n+1} \to L^n$ be the projection onto the first $n$ coordinates; since $\widehat{\pi}$ is a regular map (in the earlier sense), it is continuous. Thus $\widehat{\pi}$ restricts to a continuous map $\pi \colon \widehat{\mathbf{V}}_f \to D_f$, which is bijective having inverse $\pi^{-1} \colon D_f \to \widehat{\mathbf{V}}_f \colon v \mapsto [v, \frac{1}{f(v)}]$. We proceed to show that $\pi$ is a homeomorphism:

A basis of the Zariski topology on $\widehat{\mathbf{V}}_f$ is given by the principal open subsets. Letting $\frac{g}{f^k} \in K[\mathbf{V}]_f$, for some $g \in K[\mathbf{V}]$ and $k \in \mathbb{N}_0$, we get $\widehat{D}_{\frac{g}{f^k}} = \widehat{D}_g \subseteq \widehat{\mathbf{V}}_f$, where $\mathcal{O}_{\widehat{\mathbf{V}}_f}(\widehat{D}_g) = (K[\mathbf{V}]_f)_g$. For $\widehat{D}_h \subseteq \widehat{D}_g$, for $h \in K[\mathbf{V}]$, restriction is given by the natural map $(K[\mathbf{V}]_f)_g \to (K[\mathbf{V}]_f)_h$; note that $g \in (K[\mathbf{V}]_f)_h$ is a unit.

Similarly, a basis of the Zariski topology on $D_f$ is given by the principal open subsets $D_{fg} = D_g \cap D_f \subseteq D_f$, for some $g \in K[\mathbf{V}]$, where $\mathcal{O}_{D_f}(D_{fg}) = \mathcal{O}_\mathbf{V}(D_{fg}) = K[\mathbf{V}]_{fg}$. For $D_{fh} \subseteq D_{fg}$, where $h \in K[\mathbf{V}]$, restriction is given by the natural map $K[\mathbf{V}]_{fg} \to K[\mathbf{V}]_{fh}$; note that $fg \in K[\mathbf{V}]_{fh}$ is a unit.

From $g([v,t]) = g(v)$, for all $[v,t] \in \widehat{\mathbf{V}}_f$, we infer that $\pi$ induces a bijection $\widehat{D}_g \to D_{fg}$, for any $g \in K[\mathbf{V}]$. Thus $\pi$ is an open map, hence is a homeomorphism.

(Note that this also reproves that $\pi$ is continuous.) It remains to be shown that $\pi^*\colon \mathcal{O}_{D_f} \Rightarrow \mathcal{O}_{\widehat{\mathbf{V}}_f}$ induces isomorphisms on the level of $K$-algebras of functions:

By the sheaf properties it suffices to consider principal open subsets $D_{fg} \subseteq D_f$, where $g \in K[\mathbf{V}]$. Then we have $\pi^{-1}(D_{fg}) = \widehat{D}_g$, and

$$\pi^*_{D_{fg}}\colon \mathcal{O}_{D_f}(D_{fg}) = K[\mathbf{V}]_{fg} \to (K[\mathbf{V}]_f)_g = \mathcal{O}_{\widehat{\mathbf{V}}_f}(\widehat{D}_g)$$

boils down to the natural isomorphism $\pi^*_{fg}\colon K[\mathbf{V}]_{fg} \cong (K[\mathbf{V}]_f)_g$. $\sharp$

In particular, we infer that the set $\Gamma(\mathcal{O}_{D_f}) = K[\mathbf{V}]_f$ of regular functions on $D_f$ consists precisely of the morphisms of affine varieties $D_f \to L$. This entails that any quasi-affine variety $U \subseteq \mathbf{V}$ has an open covering consisting of affine open subsets. Thus by the sheaf properties we conclude that the set $\Gamma(\mathcal{O}_U)$ of regular functions on $U$ consists precisely of the morphisms of varieties $U \to L$.

**(9.3) Example.** We show that a quasi-affine variety is not necessarily affine:

Letting $K = L = \mathbb{C}$, we consider the affine space $\mathbf{V} := \mathbb{C}^2$, having coordinate algebra $A := \mathbb{C}[X, Y]$, and the open subset $U := D_X \cup D_Y = \mathbf{V} \setminus \{[0, 0]\} \subseteq \mathbf{V}$, having structure sheaf $\mathcal{O}_U = \mathcal{O}_{\mathbf{V}}|_U$. We determine $\Gamma(\mathcal{O}_U) = \mathcal{O}_{\mathbf{V}}(U)$:

Letting $\varphi \in \mathcal{O}_{\mathbf{V}}(U)$, there are $f, g \in A$ and $k, l \in \mathbb{N}_0$ such that $\varphi|_{D_X} = \frac{f}{X^k} \in A_X$ and $\varphi|_{D_Y} = \frac{g}{Y^l} \in A_Y$; recall that by continuity $\mathrm{Q}(A)$ can be identified with the set of regular functions it induces. Thus on $D_X \cap D_Y = D_{XY}$ we have $\frac{f}{X^k} = \frac{g}{Y^l} \in A_{XY}$, from which we get $fY^l = gX^k \in A$. Since $A$ is factorial, where $X$ and $Y$ are irreducible and non-associated, we infer that $X^k \mid f$ and $Y^l \mid g$, hence $\varphi \in A$. Conversely, $\rho^{\mathbf{V}}_{D_{XY}}\colon A \to A_{XY}$ is injective, hence so is $\rho^{\mathbf{V}}_U\colon A \to \mathcal{O}_{\mathbf{V}}(U)$. This implies $\mathcal{O}_{\mathbf{V}}(U) = A$ and $\rho^{\mathbf{V}}_U = \mathrm{id}_A$. $\sharp$

Let $\iota^{\mathbf{V}}_U$ be the natural inclusion map; recall $(\iota^{\mathbf{V}}_U)^* = \rho^{\mathbf{V}}_U$. Then $\iota^{\mathbf{V}}_U$ is a morphism of varieties; see also (6.4): It is continuous, and for any $W \subseteq \mathbf{V}$ open we have

$$(\iota^{\mathbf{V}}_U)^*_W(\mathcal{O}_{\mathbf{V}}(W)) = \rho^W_{W\cap U\subseteq \mathbf{V}}(\mathcal{O}_{\mathbf{V}}(W)) \subseteq \mathcal{O}_{\mathbf{V}}(W \cap U) = \mathcal{O}_U((\iota^{\mathbf{V}}_U)^{-1}(W)).$$

Now assume that $U$ is affine. Then $\iota^{\mathbf{V}}_U$ is a morphism of affine varieties, that is a regular map. The associated comorphism induces the identity $\mathrm{id}_A = \rho^{\mathbf{V}}_U\colon \Gamma(\mathcal{O}_{\mathbf{V}}) = A \to A = \Gamma(\mathcal{O}_U)$ on global sections. The latter is an isomorphism of $K$-algebras, so that $\iota^{\mathbf{V}}_U$ is an isomorphism of varieties, hence is bijective, a contradiction. This shows that $U$ together with $\mathcal{O}_U$ is not an affine variety. $\sharp$

**(9.4) Projective varieties.** We first consider $\mathbf{P} = \mathbf{P}^n$, having structure sheaf $\mathcal{O}_{\mathbf{P}}$. For $i \in \{0, \ldots, n\}$ the principal open subset $D_i := D_{X_i} \subseteq \mathbf{P}$ has structure sheaf $\mathcal{O}_{D_i} = \mathcal{O}_{\mathbf{P}}(D_i)$; recall that $\mathbf{P} = \bigcup_{i=0}^n D_i$. For notational simplicity we proceed to consider the case $i = 0$; the other open pieces are treated similarly.

Then homogenization $\sigma\colon L^n \to D_0\colon v = [x_1, \ldots, x_n] \mapsto [1\colon x_1\colon \ldots\colon x_n] = v^\sharp$ and dehomogenization $\tau\colon D_0 \to L^n\colon v = [x_0\colon \ldots\colon x_n] \mapsto [\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] = v'$

are mutually inverse homeomorphisms. Still, for $g \in A^\sharp$ let $g' \in A$ be its dehomogenization, and for $f \in A$ let $f^\sharp \in A^\sharp$ be its homogenization. Recall that $\mathbf{A} = \mathbf{A}^n = L^n$ is an affine variety, where $\Gamma(\mathcal{O}_\mathbf{A}) = A$.

**Theorem.** We have $\mathbf{A} \cong D_0$ as varieties.

**Proof.** Given $U \subseteq D_0$ open, we have to show $\sigma^*(\mathcal{O}_\mathbf{P}(U)) \subseteq \mathcal{O}_\mathbf{A}(\sigma^{-1}(U))$: Since the principal open subsets are a basis of the Zariski topology, by the sheaf properties it suffices to consider regular functions of shape $\varphi = \frac{f}{g}|_{D_g}$ on $D_g \subseteq D_0$, where $\frac{f}{g} \in \mathrm{Q}^\sharp(A^\sharp)$. For any $v \in L^n$ we have $\sigma^*(g)(v) = g(v^\sharp) = g'(v)$. This implies $\sigma^*(g) = g'$ and $\sigma^{-1}(D_g) = D_{g'}$, thus $\sigma^*(\varphi) = \frac{f'}{g'}|_{D_{g'}}$, where $\frac{f'}{g'} \in \mathrm{Q}(A)$.

Conversely, given $V \subseteq L^n$ open, we have to show $\tau^*(\mathcal{O}_\mathbf{A}(V)) \subseteq \mathcal{O}_\mathbf{P}(\tau^{-1}(V))$: Again, it suffices to consider regular functions of shape $\varphi = \frac{f}{g}|_{D_g}$, where $\frac{f}{g} \in \mathrm{Q}(A)$. For any $v \in D_0$ we have $\tau^*(g)(v) = g(v') = g(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0})(v)$. This implies $\tau^*(g) = \frac{g^\sharp}{X_0^{\deg(g)}}|_{D_0}$, and $\tau^{-1}(D_g) = D_0 \cap D_{g^\sharp} = D_{X_0 g^\sharp}$, thus we get $\tau^*(\varphi) = (X_0^{\deg(g)-\deg(f)} \cdot \frac{f^\sharp}{g^\sharp})|_{D_{X_0 g^\sharp}}$, where $X_0^{\deg(g)-\deg(f)} \cdot \frac{f^\sharp}{g^\sharp} \in \mathrm{Q}^\sharp(A^\sharp)$. ♯

**Corollary.** Let $\mathbf{V} \subseteq \mathbf{P}$ be closed and irreducible. Then we have $\Gamma(\mathcal{O}_\mathbf{V}) = K$.

**Proof.** Let $V_i := \mathbf{V} \cap D_i$, for $i \in \{0, \ldots, n\}$, so that $\mathbf{V} = \bigcup_{i=0}^n V_i$. If $V_n = \emptyset$, say, then we have $\mathbf{V} \subseteq \mathbf{P}^{n-1}$, and we may proceed with coordinates $X_0, \ldots, X_{n-1}$ instead. Thus we may assume that $V_i \neq \emptyset$ for all $i \in \{0, \ldots, n\}$. Moreover, if $n = 0$ then $\mathbf{V}$ is a singleton set; hence we may assume that $n \geq 1$.

Then $V_i \subseteq \mathbf{V}$ is open, hence dense, so that $\overline{V_i} = \mathbf{V}$. Since $D_i$ is affine and $V_i \subseteq D_i$ is closed, we conclude that $V_i$ is irreducible affine, such that $\mathcal{O}_\mathbf{V}(V_i) = \Gamma(\mathcal{O}_{V_i}) = K[V_i]$. Identifying $D_i \cong \mathbf{A}$ and $V_i \cong \mathbf{V}_i \subseteq \mathbf{A}$, we get $K[V_i] \cong K[\mathbf{V}_i] = K[\mathcal{X}^\sharp \setminus \{X_i\}]/\mathbf{I}_K(\mathbf{V}_i)$, where by (4.9) the vanishing ideals $\mathbf{I}_K^\sharp(\mathbf{V}) \trianglelefteq A^\sharp$ and $\mathbf{I}_K(\mathbf{V}_i)$ are related by (de)homogenization at position $i$.

Now let $0 \neq \varphi \in \Gamma(\mathcal{O}_\mathbf{V})$. Then we have $\varphi|_{V_i} \in K[V_i]$, which is regular, thus continuous on the affine variety $V_i$. Hence from $\mathbf{V} = \bigcup_{i=0}^n V_i$ we conclude that $\varphi$ is continuous, which since $V_i \subseteq \mathbf{V}$ is dense implies $\varphi|_{V_i} \neq 0$. Hence we have $\varphi|_{V_i} = f_i|_{V_i} = \frac{f_i}{X_i^{d_i}}|_{V_i}$, where $0 \neq \frac{f_i}{X_i^{d_i}} \in \mathrm{Q}^\sharp(K[\mathbf{V}])$ such that $f_i \in K[\mathbf{V}]$ is the homogenization at position $i$ of an element of $K[\mathbf{V}_i]$; thus we have $X_i \nmid f_i$ and $d_i = \deg(f_i)$. (Note that by the relation between the vanishing ideals involved this is well-defined indeed.)

For comparison, we consider $\emptyset \neq V_0 \cap V_i \subseteq V_0$, for $i \geq 1$, which is identified with the principal open subset $D_{X_i} \subseteq \mathbf{V}_0$. Dehomogenisation yields $(f_0)' = \frac{(f_i)'}{X_i^{d_i}} \in \mathcal{O}_{\mathbf{V}_0}(D_{X_i}) = K[\mathbf{V}_0]_{X_i} \subseteq K(\mathbf{V}_0)$. Thus we infer that $d_i = 0$, that is $f_i \in K$, for all $i \geq 1$. This implies $\varphi = f_0 = f_1 = \ldots = f_n \in K$, that is $\varphi$ is constant. ♯

From this, we get the following property of quasi-projective varieties: Let $\mathbf{V} \subseteq \mathbf{P}$ be closed, and let $U \subseteq \mathbf{V}$ be open. We have see above that $D_i \cap \mathbf{V}$ is affine as well, so that $D_i \cap \mathbf{V} \subseteq \mathbf{V}$ is affine open, for all $i \in \{0, \ldots, n\}$. Thus, since $U \cap D_i = U \cap (D_i \cap \mathbf{V}) \subseteq D_i \cap \mathbf{V}$ is open, we conclude that $U \cap D_i$ is quasi-affine, hence $U \cap D_i \subseteq U$ is quasi-affine open. Thus $U = \bigcup_{i=0}^n (U \cap D_i)$ has a quasi-affine open covering, so that $U$ also has an affine open covering.

In particular, by the sheaf properties we conclude that the set $\Gamma(\mathcal{O}_U)$ of regular functions on $U$ consists precisely of the morphisms of varieties $U \to L$.

**Example.** We show that the homogeneous coordinate algebra of a projective variety cannot be recovered from its structure sheaf, but strictly depends on its embedding into a projective space (contrary to the analogous fact that the coordinate algebra of an affine variety is intimately related to its structure sheaf):

Let $K = L = \mathbb{C}$, let $A^\sharp = \mathbb{C}[W, X, Y, Z]$, let $P^\sharp := \langle Y^2 - XZ, WZ - XY, WY - X^2 \rangle \trianglelefteq A^\sharp$, and let $\mathbf{V} := \mathbf{V}(P^\sharp) \subseteq \mathbf{P} := \mathbf{P}^3$ be the twisted cubic again, see (4.10). Let $\mathbf{P}^1$ be the projective line, having homogeneous coordinate algebra $K[T, S]$. Then we have the bijection $\varphi \colon \mathbf{P}^1 \to \mathbf{V} \colon [s \colon t] \mapsto [s^3 \colon s^2 t \colon s t^2 \colon t^3]$. Moreover, we observe that both restrictions $\varphi|_{D_S} \colon D_S \to \mathbf{V} \cap D_W \colon t \mapsto [1 \colon t \colon t^2 \colon t^3]$ and $\varphi|_{D_T} \colon D_T \to \mathbf{V} \cap D_Z \colon s \mapsto [s^3 \colon s^2 \colon s \colon 1]$ are isomorphisms of affine varieties. Hence it follows from (10.1) below that $\varphi$ is an isomorphism of projective varieties. But we have $\mathbb{C}[\mathbf{V}] \cong \mathbb{C}[S, T]_{3\mathbb{N}_0} \not\cong \mathbb{C}[S, T] \cong \mathbb{C}[\mathbf{P}^1]$. ♯

**(9.5) Algebraic-geometric correspondence again. a)** Let $K$**-Alg** be the category of all $K$-algebras together with all homomorphisms of $K$-algebras $\operatorname{Hom}(?, ??)$ as morphisms. Moreover, let $K$**-fgAlg** be its full subcategory of quotient algebras of finitely generated polynomial $K$-algebras, and let in turn $K$**-AffAlg** be its full subcategory of reduced algebras. (Note that the object classes of the latter are sets.)

Then, given a fixed $K$-algebra $A$, assigning $B \mapsto \operatorname{Hom}(A, B)$, and letting

$$\big(\alpha \colon B \to B'\big) \mapsto \big(\alpha_* \colon \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B') \colon \beta \mapsto \alpha \circ \beta\big)$$

defines a (covariant) functor $\operatorname{Hom}(A, ?) \colon K\textbf{-Alg} \to \textbf{Sets}$.

**b)** Let $K$**-Var** be the category of all $K$-varieties over $L$, together with all morphisms of varieties $\operatorname{Mor}_K(?, ??)$ as morphisms. Moreover, let $K\textbf{-AffVar} := \coprod_{n \in \mathbb{N}_0} \{\mathbf{V} \subseteq L^n \ K\text{-closed}\}$ be its full subcategory of affine $K$-varieties over $L$, where the morphisms of affine varieties coincide with the regular maps. (Note that the object class of the latter is a set.)

Then, given a fixed $K$-variety $V$, assigning $U \mapsto \operatorname{Mor}_K(U, V)$, and letting

$$\big(\varphi \colon U' \to U\big) \mapsto \big(\varphi^*_{U,V} \colon \operatorname{Mor}_K(U, V) \to \operatorname{Mor}_K(U', V) \colon \psi \mapsto \psi \circ \varphi\big)$$

defines a contravariant functor $\operatorname{Mor}_K(?, V) \colon K\textbf{-Var} \to \textbf{Sets}$.

**c)** Now, assigning $U \mapsto \Gamma(\mathcal{O}_U)$, where $U$ is a variety, and letting

$$\big(\varphi\colon U' \to U\big) \mapsto \big(\varphi_U^*\colon \Gamma(\mathcal{O}_U) = \mathcal{O}_U(U) \to \mathcal{O}_{U'}(\varphi^{-1}(U)) = \mathcal{O}_{U'}(U') = \Gamma(\mathcal{O}_{U'})\big)$$

defines the contravariant **global section** functor $\Gamma\colon K\text{-}\mathbf{Var} \to K\text{-}\mathbf{Alg}$.

Motivated by the close relationship between morphisms of affine varieties and the associated (co)morphisms of coordinate algebras, we observe the following:

**Theorem.** Let $\mathbf{V}$ be an affine $K$-variety. Then the contravariant functors $\mathrm{Mor}_K(?, \mathbf{V})$ and $\mathrm{Hom}(K[\mathbf{V}], \Gamma(?))$ are isomorphic as functors $K\text{-}\mathbf{Var} \to \mathbf{Sets}$.

**Proof.** We show that $\mathcal{N}\colon \mathrm{Mor}_K(?, \mathbf{V}) \Rightarrow \mathrm{Hom}(K[\mathbf{V}], \Gamma(?))$ given by

$$\mathcal{N}_U\colon \mathrm{Mor}_K(U, \mathbf{V}) \to \mathrm{Hom}(K[\mathbf{V}], \Gamma(U))\colon \psi \mapsto \psi_{\mathbf{V}}^*$$

is a natural isomorphism:

For $\psi \in \mathrm{Mor}_K(U, \mathbf{V})$ we have $\psi_{\mathbf{V}}^* \in \mathrm{Hom}(\Gamma(\mathcal{O}_{\mathbf{V}}), \Gamma(\mathcal{O}_U)) = \mathrm{Hom}(K[\mathbf{V}], \Gamma(U))$; thus $\mathcal{N}$ is well-defined. For any morphism $\varphi\colon U' \to U$ and any morphism $\psi\colon U \to \mathbf{V}$ we have $(\psi \circ \varphi)_{\mathbf{V}}^* = \varphi_U^* \circ \psi_{\mathbf{V}}^*\colon K[\mathbf{V}] \to \Gamma(U')$, that is

$$\varphi_{U,\mathbf{V}}^* \cdot \mathcal{N}_{U'} = \mathcal{N}_U \cdot (\varphi_U^*)_*\colon \mathrm{Mor}_K(U, \mathbf{V}) \to \mathrm{Hom}(K[\mathbf{V}], \Gamma(U'));$$

this shows that $\mathcal{N}$ is a natural transformation. We show that $\mathcal{N}_U$ is bijective:

Let $\mathbf{V} \subseteq L^n$, and let $K[\mathbf{V}] = A/\mathbf{I}_K(\mathbf{V})$. Given a morphism $\psi\colon U \to \mathbf{V}$, for $i \in \{1, \ldots, n\}$ and $X_i \in K[\mathbf{V}]$ let $f_i := \pi_{\mathbf{V}}^*(X_i) \in \Gamma(U)$. Then we have $\psi(v) = [f_1(v), \ldots, f_n(v)]$ for all $v \in U$. Thus $\psi$ can be recovered from $\psi_{\mathbf{V}}^*$, implying that $\mathcal{N}_U$ is injective.

In order to show that $\mathcal{N}_U$ is surjective, let $\pi \in \mathrm{Hom}(K[\mathbf{V}], \Gamma(U))$, let $p_i := \pi(X_i) \in \Gamma(U)$, for $i \in \{1, \ldots, n\}$, and let $\pi^\vee\colon U \to L^n\colon v \mapsto [p_1(v), \ldots, p_n(v)]$. We show that $\pi^\vee(U) \subseteq \mathbf{V}$: For $f \in \mathbf{I}_K(\mathbf{V})$ we have $f \circ \pi^\vee = f(p_1, \ldots, p_n) = f(\pi(X_1), \ldots, \pi(X_n)) = \pi(f) = 0$ on $U$, thus $\pi^\vee(U) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})) = \mathbf{V}$. This entails $(\pi^\vee)_{\mathbf{V}}^*(X_i) = X_i \circ \pi^\vee = p_i$, so that $(\pi^\vee)_{\mathbf{V}}^* = \pi$.

It remains to be shown that $\pi^\vee$ actually is a morphism of varieties: To this end, since $U$ is covered by affine open subsets, by the sheaf properties we may assume that $U$ is affine; then we have $(\pi^\vee)_{\mathbf{V}}^* = \pi \in \mathrm{Hom}(K[\mathbf{V}], K[U])$, which is equivalent to $\pi^\vee\colon U \to \mathbf{V}$ being a morphism of affine varieties. ♯

**d)** Having this in place, the case of affine varieties reads as follows (where actually the affine case has been used in the proof of the general case):

If $\varphi\colon \mathbf{V} \to \mathbf{W}$ is a morphism of affine varieties, then $\Gamma(\varphi) = \varphi_{\mathbf{W}}^*\colon K[\mathbf{W}] = \Gamma(\mathbf{W}) \to \Gamma(\mathbf{V}) = K[\mathbf{V}]$ coincides with the comorphism $\varphi^*$ associated with the regular map $\varphi$. We get a contravariant functor $\Gamma\colon K\text{-}\mathbf{AffVar} \to K\text{-}\mathbf{AffAlg}$ by

$$\mathbf{V} \mapsto K[\mathbf{V}] \quad \text{and} \quad \big(\varphi\colon \mathbf{V} \to \mathbf{W}\big) \mapsto \big(\varphi^*\colon K[\mathbf{W}] \to K[\mathbf{V}]\big).$$

Moreover, since any affine variety $\mathbf{V}$ can be recovered from its coordinate algebra $K[\mathbf{V}]$ (which does *not* generalize to arbitrary varieties and their global sections), we get a contravariant functor $\mathcal{V}\colon K\textbf{-AffAlg} \to K\textbf{-AffVar}$ by letting

$$K[\mathcal{X}]/I \mapsto \mathbf{V}_L(I) \quad \text{and} \quad \big(\alpha\colon K[\mathcal{X}]/I \to K[\mathcal{Y}]/J\big) \mapsto \big(\alpha^\vee\colon \mathbf{V}_L(J) \to \mathbf{V}_L(I)\big),$$

where $\alpha^\vee$ is given by $v \mapsto [\alpha(X_1)(v), \ldots, \alpha(X_n)(v)]$.

Then we have $\mathcal{V} \circ \Gamma = \mathrm{Id}_{K\textbf{-AffAlg}}$ and $\Gamma \circ \mathcal{V} = \mathrm{Id}_{K\textbf{-AffVar}}$, so that $K$**-AffVar** and $K$**-AffAlg** are contravariantly isomorphic. In particular, both functors are **surjective** on objects; **fully faithful**, that is surjective and injective on morphisms, respectively (which is a special case of the above theorem); and they respect and reflect isomorphisms.

## 10 Prevarieties

**(10.1) Prevarieties. a)** Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed. A space with ($K$-algebras of $L$-valued) functions $(V, \mathcal{O}_V)$ is called a **prevariety**, if **i)** $V$ is connected, and **ii)** there is an open covering $\{V_i; i \in \mathcal{I}\}$ of $V$, where $\mathcal{I}$ is a finite index set, such that the space with functions $(V_i, \mathcal{O}_V|_{V_i})$ is (isomorphic to) an irreducible affine variety, for $i \in \mathcal{I}$.

In particular, any irreducible affine, quasi-affine, projective or quasi-projective variety is a prevariety.

**Proposition. i)** Any prevariety $V$ is Noetherian, hence $V$ is quasi-compact.
**ii)** If $V \neq \emptyset$, then $V$ is irreducible.

**Proof. i)** Let $V \supseteq W_1 \supseteq W_2 \supseteq \cdots$ be a descending chain of closed subsets. Then for $i \in \mathcal{I}$ we have $V_i \supseteq (V_i \cap W_1) \supseteq (V_i \cap W_2) \supseteq \cdots$. Since $V_i$ is Noetherian, each of the latter chains stabilizes after finitely many steps. Thus, since $\mathcal{I}$ is finite and $V = \bigcup_{i \in \mathcal{I}} V_i$, so does the given chain.

**ii)** Let $\emptyset \neq V = \bigcup_{j=1}^{r} W_j$ be the irreducible components of $V$, where $r \in \mathbb{N}$, and assume that $r \geq 2$. Since $W_1 \subseteq V$ and $\bigcup_{i \geq 2} W_i \subseteq V$ are closed, and $V$ is connected, we may assume that there is $v \in W_1 \cap W_2 \neq \emptyset$. Let $v \in V_i$, for some $i \in \mathcal{I}$. Then $\emptyset \neq V_i \cap W_j \subseteq W_j$ is open, hence dense, for both $j \in \{1, 2\}$. Thus we have $W_1 \cup W_2 \subseteq \overline{V}_i \subseteq V$. Since $V_i$ is irreducible, so is its closure $\overline{V}_i$. Thus $W_1 \not\subseteq W_2 \not\subseteq W_1$ cannot possibly be irreducible components, a contradiction. ♯

An open subset $\emptyset \neq U \subseteq V$, such that $(U, \mathcal{O}_V|_U)$ is an (irreducible) affine variety, is called an **affine open** subset of $V$. Since the (irreducible) affine open subsets form a basis of the Zariski topology on an irreducible affine variety, we conclude that the affine open subsets form a basis of the topology on $V$.

**b)** Morphisms of prevarieties are their morphisms as spaces with functions. This gives rise to the category of prevarieties, being a full subcategory of the category of spaces with functions. Here is a useful criterion ensuring that a map between prevarieties actually is a morphism:

**Proposition.** Let $(V, \mathcal{O}_V)$ and $(W, \mathcal{O}_W)$ be prevarieties, let $\{V_i; i \in \mathcal{I}\}$ be an open covering of $V$, let $\{W_i; i \in \mathcal{I}\}$ be an affine open covering of $W$, where $\mathcal{I}$ is an index set, and let $\varphi\colon V \to W$ be a map such that $\varphi(V_i) \subseteq W_i$ and $(\varphi|_{V_i})^*(\mathcal{O}_W(W_i)) \subseteq \mathcal{O}_V(V_i)$, for all $i \in \mathcal{I}$. Then $\varphi$ is a morphism of prevarieties.

**Proof.** If $U \subseteq V_i$ is open, then from $(\varphi|_U)^*(f) = (\varphi|_{V_i})^*(f)|_U$, for any $f\colon W_i \to L$, we get $(\varphi|_U)^*(\mathcal{O}_W(W_i)) \subseteq \mathcal{O}_V(U)$. Hence, since the affine open subsets form a basis of the topology on $V$, we may assume that the $V_i$ are affine open.

Then, abbreviating $\varphi_i := \varphi|_{V_i}$, since $\mathcal{O}_V(V_i) = \Gamma(\mathcal{O}_{V_i})$ and $\mathcal{O}_W(W_i) = \Gamma(\mathcal{O}_{W_i})$, we have $(\varphi_i)^*(\Gamma(\mathcal{O}_{W_i})) \subseteq \Gamma(\mathcal{O}_{V_i})$. Thus, $V_i$ and $W_i$ being affine, we conclude that $\varphi_i$ is a morphism. In particular, $\varphi_i$ is continuous, thus $\varphi$ is so as well.

For $U \subseteq W$ open, the set $\{U \cap W_i; i \in \mathcal{I}\}$ is an open covering of $U$, and we have $\varphi_i^{-1}(U \cap W_i) = \varphi^{-1}(U \cap W_i) \cap V_i = \varphi^{-1}(U) \cap V_i$. Then for $f \in \mathcal{O}_W(U)$ we have $f|_{U \cap W_i} \in \mathcal{O}_W(U \cap W_i) = \mathcal{O}_{W_i}(U \cap W_i)$, hence we get

$$\varphi^*(f)|_{\varphi^{-1}(U)\cap V_i} = (\varphi_i)^*(f|_{U\cap W_i}) \in \mathcal{O}_{V_i}(\varphi_i^{-1}(U \cap W_i)) = \mathcal{O}_V(\varphi^{-1}(U) \cap V_i).$$

Since $\{\varphi^{-1}(U) \cap V_i; i \in \mathcal{I}\}$ is an open covering of $\varphi^{-1}(U)$, by the sheaf properties from this we infer that $\varphi^*(f) \in \mathcal{O}_V(\varphi^{-1}(U))$. ♯

**(10.2) Remark: Comparison with manifolds.** Recall that a topological space $V$ is called **Hausdorff**, if for any $x \neq y \in V$ there are open neighborhoods $U_x$ and $U_y$ of $x$ and $y$, respectively, such that $U_x \cap U_y = \emptyset$.

Now, for $n \in \mathbb{N}$, we may define an $n$-dimensional **complex manifold** $\mathcal{M}$ along the above lines, as being a connected Hausdorff space, together with a sheaf of $\mathbb{C}$-valued functions $\mathcal{H}_\mathcal{M}$, such that there is an open covering $\{\mathcal{M}_i; i \in \mathcal{I}\}$ of $\mathcal{M}$, where $\mathcal{I}$ is an index set, such that $(\mathcal{M}_i, \mathcal{H}_\mathcal{M}|_{\mathcal{M}_i})$ is isomorphic to $(\mathbb{C}^n, \mathcal{H}_{\mathbb{C}^n})$, for $i \in \mathcal{I}$, where $\mathcal{H}_{\mathbb{C}^n}$ is the sheaf of holomorphic functions on $\mathbb{C}^n$.

In particular, for $n = 1$ we recover **Riemann surfaces**. Similarly, we may define an $n$-dimensional **differentiable manifold** by being connected Hausdorff, and prescribing $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ on an open covering, where $\mathcal{C}^\infty_{\mathbb{R}^n}$ is the sheaf of smooth $\mathbb{R}$-valued functions on $\mathbb{R}^n$. Note that in the algebraic setting we cannot proceed with 'smooth transition maps' between 'charts', as is usually done for manifolds.

**(10.3) Subprevarieties. a)** Let $(V, \mathcal{O}_V)$ be a prevariety. We show that open or irreducible closed subsets of $V$, carry the structure of a prevariety again:

**Proposition.** Let $U \subseteq V$ be open. Then $(U, \mathcal{O}_V|_U)$ is a prevariety again, being called an **open subprevariety**.

**Proof.** We may assume that $U \neq \emptyset$. Since $V$ is irreducible, $U$ is dense, hence is irreducible as well, thus is connected. Moreover, since $V$ is quasi-compact, so is $U$. Finally, since the affine open subsets form a basis of the topology on $V$, we conclude that $U$ is covered by affine open subsets. ♯

**b)** To deal with the case of closed subsets, let first $\mathbf{V}$ be irreducible affine with structure sheaf $\mathcal{O}_{\mathbf{V}}$, and let $\mathbf{W} \subseteq \mathbf{V}$ be closed and irreducible. Then $\mathbf{W}$ carries the structure of an irreducible affine variety with structure sheaf $\mathcal{O}_{\mathbf{W}}$, and the structure of a space with functions with respect to the sheaf $\mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}$. The natural inclusion map $\iota_{\mathbf{W}}^{\mathbf{V}} \colon \mathbf{W} \to \mathbf{V}$, which is regular, induces a comorphism $(\iota_{\mathbf{W}}^{\mathbf{V}})^* \colon \mathcal{O}_{\mathbf{V}} \Rightarrow \mathcal{O}_{\mathbf{W}}$, as well as a comorphism $(\iota_{\mathbf{W}}^{\mathbf{V}})^* \colon \mathcal{O}_{\mathbf{V}} \Rightarrow \mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}$ anyway.

**Proposition.** We have $\mathcal{O}_{\mathbf{V}}|_{\mathbf{W}} = \mathcal{O}_{\mathbf{W}}$.

**Proof.** Letting $U \subseteq \mathbf{W}$ open, we show that $\mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}(U) = \mathcal{O}_{\mathbf{W}}(U)$:

Let first $f \in \mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}(U)$, for $v \in U$ let $v \in U_v \subseteq \mathbf{V}$ be open, and let $f_v \in \mathcal{O}_{\mathbf{V}}(U_v)$, such that $U_v \cap \mathbf{W} \subseteq U$ and $f_v|_{U_v \cap \mathbf{W}} = f|_{U_v \cap \mathbf{W}}$. Thus we have $f|_{U_v \cap \mathbf{W}} \in \mathcal{O}_{\mathbf{W}}(U_v \cap \mathbf{W})$, by the sheaf properties implying $f \in \mathcal{O}_{\mathbf{W}}(U)$.

Conversely, again by the sheaf properties, it suffices to consider principal open subsets $D_g \subseteq \mathbf{W}$, where $0 \neq g \in K[\mathbf{W}]$. Letting $I := \mathbf{I}_{\mathbf{V}}(\mathbf{W}) \trianglelefteq K[\mathbf{V}]$, we have the natural epimorphism $(\iota_{\mathbf{W}}^{\mathbf{V}})^*_{\mathbf{V}} \colon K[\mathbf{V}] \to K[\mathbf{V}]/I = K[\mathbf{W}]$. Thus we get the natural epimorphism $K[\mathbf{V}]_{K[\mathbf{V}] \setminus I} \to \mathrm{Q}(K[\mathbf{W}]) = K(\mathbf{W})$; note that $I$ being prime, $K[\mathbf{V}] \setminus I$ is multiplicatively closed. Thus any $f = \frac{h}{g} \in \mathcal{O}_{\mathbf{W}}(D_g) = K[\mathbf{W}]_g \subseteq K(\mathbf{W})$ lifts to an element $\widehat{f} = \frac{\widehat{h}}{\widehat{g}} \in K[\mathbf{V}]_{\widehat{g}} \subseteq K[\mathbf{V}]_{K[\mathbf{V}] \setminus I}$. Then we have $D(\widehat{g}) \cap \mathbf{W} = D(g)$ and $\widehat{f}|_{D(\widehat{g}) \cap \mathbf{W}} = f$, thus $f \in \mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}(D_g)$. ♯

**Corollary.** Let $W \subseteq V$ be closed and irreducible. Then $(W, \mathcal{O}_V|_W)$ is a prevariety again, being called a **closed subprevariety**.

**Proof.** Since $W$ is connected, it only remains to be shown that $(W, \mathcal{O}_V|_W)$ has a finite affine open covering: To this end, let $\{V_i; i \in \mathcal{I}\}$ be an affine open covering of $V$, where $\mathcal{I}$ is a finite index set. Then $\{V_i \cap W; i \in \mathcal{I}\}$ is a finite open covering of $W$; note that $V_i \cap W$ is empty or irreducible. Then since $V_i \cap W \subseteq W$ and $V_i \subseteq V$ are open, we have $(\mathcal{O}_V|_W)|_{V_i \cap W} = \mathcal{O}_V|_{V_i \cap W} = (\mathcal{O}_V|_{V_i})|_{V_i \cap W}$. Since $\mathcal{O}_V|_{V_i} = \mathcal{O}_{V_i}$ is affine, and $V_i \cap W \subseteq V_i$ is closed, we conclude that $(\mathcal{O}_V|_{V_i})|_{V_i \cap W} = \mathcal{O}_{V_i}|_{V_i \cap W} = \mathcal{O}_{V_i \cap W}$ is affine. ♯

**(10.4) Example: Affine line with an additional point.** Let $U_1 \cong L \cong U_2$ be two copies of the affine line, with coordinate algebras $K[U_1] = K[S]$ and $K[U_2] = K[T]$. Let $U_0 := L \setminus \{0\}$, with coordinate algebra $K[X]_X = K[X^{\pm 1}]$, let $U_1' := (U_1)_S$ and $U_2' := (U_2)_T$, with coordinate algebras $K[U_1'] = K[S^{\pm 1}]$ and $K[U_2'] = K[T^{\pm 1}]$, respectively, and let $\psi_i \colon U_0 \to U_i'$ be isomorphisms such that $\psi_1^* \colon K[S^{\pm 1}] \to K[X^{\pm 1}] \colon S \mapsto X$ and $\psi_2^* \colon K[T^{\pm 1}] \to K[X^{\pm 1}] \colon T \mapsto X$,

We 'glue' $U_1$ and $U_2$ along an identification of $U_1'$ and $U_2'$ as affine varieties. To do so, let $\varphi_0^*$ be a $K$-algebra automorphism of $K[X^{\pm 1}]$. Hence we get an associated automorphism $\varphi_0$ of $U_0$, and an isomorphism $\varphi := \psi_1^{-1} \varphi_0 \psi_2 \colon U_1' \to U_2'$.

Let $V := U_1 \sqcup_\varphi U_2$ be the **$\varphi$-twisted** fibre sum of $U_1$ and $U_2$, that is with respect to the inclusions $\psi_1 \colon U_0 \to U_1$ and $\varphi_0 \psi_2 \colon U_0 \to U_2$. Thus we have embeddings

$\iota_i\colon U_i \to V$, for $i \in \{1,2\}$, such that letting $V' := \iota_1(U_1') = \iota_2(U_2')$ we have $V = \{0_1\} \,\dot\cup\, V' \,\dot\cup\, \{0_2\}$, where $V_i := \iota_i(U_i) = \{0_i\} \,\dot\cup\, V'$, and $(\iota_1|_{U_1'}) = \varphi \cdot (\iota_2|_{U_2'})$.

The topology on $V$ is defined as follows: A subset $W \subseteq V$ is open if both $\iota_i^{-1}(W) \subseteq U_i$ are open; this is the coarsest topology on $V$ such that both maps $\iota_i$ are continuous. Moreover, both maps $\iota_i$ are open: It suffices to consider $\iota_1$; let $U \subseteq U_1$ be open, then we have $\iota_1^{-1}(\iota_1(U)) = U \subseteq U_1$ open, and $\iota_2^{-1}(\iota_1(U)) = \iota_2^{-1}(\iota_1(U) \cap V') = \iota_2^{-1}(\iota_1(U \cap U_1') \cap U_2') = \varphi(U \cap U_1') \subseteq U_2' \subseteq U_2$ open. Hence both maps $\iota_i$ are homeomorphisms, allowing us to identify $U_i$ and $V_i$.

Next, $V$ is connected: Let $V = W_1 \,\dot\cup\, W_2$, where $W_j \subseteq V$ are open; since $W_j \cap V_i \subseteq V_i$ is open, and $V_i$ is irreducible, we may assume that $W_2 \cap V_1 = \emptyset$ and $W_1 \cap V_2 = \emptyset$, thus $W_1 \subseteq \{0_1\}$ and $W_2 \subseteq \{0_2\}$, a contradiction.

We proceed to define a structure sheaf $\mathcal{O}_V$ on $V$: Firstly, let $\mathcal{O}_{V_i}$ be the sheaf of functions on $V_i$ obtained by pre-composition with the homeomorphism $\iota_i^{-1}\colon V_i \to U_i$. Thus $\iota_i^*\colon \mathcal{O}_{V_i} \Rightarrow \mathcal{O}_{U_i}$ is an isomorphism of sheaves, so that $V_i$ carries the structure of an affine variety. Moreover, we get the isomorphism

$$\iota_2^* \cdot \varphi^* \cdot (\iota_1^{-1})^*\colon \mathcal{O}_{V_2}|_{V'} \Rightarrow \mathcal{O}_{U_2}|_{U_2'} = \mathcal{O}_{U_2'} \Rightarrow \mathcal{O}_{U_1'} = \mathcal{O}_{U_1}|_{U_1'} \Rightarrow \mathcal{O}_{V_1}|_{V'},$$

where on $V'$ we indeed have $\iota_2^* \cdot \varphi^* \cdot (\iota_1^{-1})^* = (\iota_1^{-1} \cdot \varphi \cdot \iota_2)^* = (\mathrm{id}_{V'})^*$.

Now, for $W \subseteq V$ open, let $\mathcal{O}_V(W)$ be the set of all functions $f\colon W \to L$ such that both $f|_{V_i} \in \mathcal{O}_{V_i}(W \cap V_i)$. It is immediate that this defines a presheaf on $V$. We show that $\mathcal{O}_V$ is a sheaf:

Let $\{W_j; j \in \mathcal{J}\}$ be an open covering of $W$, and let $f\colon W \to L$ be a function such that $f|_{W_j} \in \mathcal{O}_V(W_j)$, for $j \in \mathcal{J}$. Then $\{W_j \cap V_i; j \in \mathcal{J}\}$ is an open covering of $W \cap V_i$, and the functions $f|_{W_j \cap V_i} = (f|_{W_j})|_{W_j \cap V_i} \in \mathcal{O}_V(W_j \cap V_i) = \mathcal{O}_{V_i}(W_j \cap V_i)$ are compatible. Thus there is $f_i \in \mathcal{O}_{V_i}(W \cap V_i)$ such that $f_i|_{W_j \cap V_i} = f|_{W_j \cap V_i}$, for $j \in \mathcal{J}$. This entails both $f|_{V_i} = f_i \in \mathcal{O}_{V_i}(W \cap V_i)$, thus $f \in \mathcal{O}_V(W)$. ♯

Hence $\{V_1, V_2\}$ is an affine open covering, where we identify $K[V_i] = K[U_i]$. Thus $(V, \mathcal{O}_V)$ is a prevariety. (Actually, generally speaking, $\mathcal{O}_V$ is obtained by 'gluing' the sheaves $\mathcal{O}_{U_1}$ and $\mathcal{O}_{U_2}$ along $\varphi^*\colon \mathcal{O}_{U_2'} \Rightarrow \mathcal{O}_{U_1'}$, see Exercise (13.6).)

**i)** Let $\varphi^*\colon K[T^{\pm 1}] \to K[S^{\pm 1}]\colon T \mapsto S^{-1}$, hence $\varphi\colon U_1' \to U_2'\colon s \mapsto s^{-1}$.

Let $\mathbf{P}^1$ be the projective line, having homogeneous coordinate algebra $K[T,S]$, and let $\psi\colon \mathbf{P}^1 \to V$ be the bijection given by

$$\psi|_{D_T}\colon D_T \to V_1\colon [t\colon s] \mapsto \iota_1(\frac{s}{t}) \quad \text{and} \quad \psi|_{D_S}\colon D_S \to V_2\colon [t\colon s] \mapsto \iota_2(\frac{t}{s});$$

note that on $D_S \cap D_T = D_{ST}$ we have $\iota_2(\frac{t}{s}) = \iota_2(\varphi(\frac{s}{t})) = \iota_1(\frac{s}{t}) \in V'$, so that $\psi$ is well-defined indeed. Recall that $D_T \subseteq \mathbf{P}^1$ and $D_S \subseteq \mathbf{P}^1$ are affine open such that $K[D_T] = K[S]$ and $K[D_S] = K[T]$, respectively. Thus $(\psi|_{D_T})^* = \mathrm{id}_{K[S]}$ and $(\psi|_{D_S})^* = \mathrm{id}_{K[T]}$ shows that both $\psi|_{D_T}$ and $\psi|_{D_S}$ are isomorphisms of affine varieties. Hence we conclude that $\psi$ is an isomorphism of prevarieties, where $\mathbf{P}^1$ actually is a projective variety.

**ii)** Let $\varphi^*\colon K[T^{\pm 1}] \to K[S^{\pm 1}]\colon T \mapsto S$, hence $\varphi\colon U'_1 \to U'_2\colon s \mapsto s$.

Writing $V' = L \setminus \{0\}$, we have $V = \{0_1\} \mathbin{\dot\cup} (L \setminus \{0\}) \mathbin{\dot\cup} \{0_2\}$, which is called the **affine line with one point doubled**. (This prevariety will be shown not to be an abstract variety, see (11.6) below; in particular it is not isomorphic as prevarieties to any of the varieties we have seen before.)

## 11 Abstract varieties

**(11.1) Categorical products of prevarieties.** Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed. We aim at showing that the category of prevarieties has (finite direct) products.

As a motivation, we would certainly like to have $L^n \times L^m \cong L^{n+m}$ as affine varieties, for $n, m \in \mathbb{N}_0$, but the Zariski topology on $L^{n+m}$ is strictly finer than the product topology of the Zariski topologies on $L^n$ and $L^m$, respectively, whenever $n, m \geq 1$, see Exercise (12.13). Thus we have to define products in the categorical sense. As an indication we observe the following:

**Lemma.** Let $K = L$, let $V$ and $W$ be prevarieties, and let $P$ be a product of $V$ and $W$. Then $P$ can be identified with the Cartesian product $V \times W$ of sets.

**Proof.** For $v \in V$ there is an affine open subset $v \in U \subseteq V$, implying that $U \setminus \{v\} \subseteq U \subseteq V$ is open, thus $\{v\} \subseteq V$ is closed; recall that any singleton subset of an affine $L$-variety is closed. Hence $\{v\}$ is a closed subprevariety, being affine, isomorphic to $\{0\} \cong L^0$. Thus there is a unique morphism $\{0\} \to \{v\}$, entailing a bijection $V \to \mathrm{Mor}(\{0\}, V)$. By the universal property of products we have bijections $P \cong \mathrm{Mor}(\{0\}, P) \cong \mathrm{Mor}(\{0\}, V) \times \mathrm{Mor}(\{0\}, W) \cong V \times W$, the second one being induced by post-composition with $\pi_V\colon P \to V$ and $\pi_W\colon P \to W$. ♯

**(11.2) Theorem: Products of affine varieties.** Let $\mathbf{V} \subseteq L^n$ and $\mathbf{W} \subseteq L^m$, where $n, m \in \mathbb{N}_0$, be affine varieties having coordinate algebras $K[\mathbf{V}]$ and $K[\mathbf{W}]$.

**a)** Then $\mathbf{U} := \mathbf{V} \times \mathbf{W} \subseteq L^n \times L^m \cong L^{n+m}$ is closed, hence becomes an affine variety, which has coordinate algebra $K[\mathbf{U}] \cong K[\mathbf{V}] \otimes_K K[\mathbf{W}]$.

**b)** If both $\mathbf{V}$ and $\mathbf{W}$ are irreducible, then so is $\mathbf{U}$, and together with the projections $\pi_{\mathbf{V}}$ and $\pi_{\mathbf{W}}$ it is a product of $\mathbf{V}$ and $\mathbf{W}$ in the category of prevarieties.

**Proof. a)** Let $\mathcal{X} := \{X_1, \ldots, X_n\}$ and $\mathcal{Y} := \{Y_1, \ldots, Y_m\}$ be indeterminates, let $A := K[\mathcal{X}]$ and $B := K[\mathcal{Y}]$, and let $I := \mathbf{I}_K(\mathbf{V}) \trianglelefteq A$ and $J := \mathbf{I}_K(\mathbf{W}) \trianglelefteq B$, so that $K[\mathbf{V}] = A/I$ and $K[\mathbf{W}] = B/J$.

We may identify $L^n \times L^m \cong L^{n+m}$ as sets, where $L^{n+m}$ is an affine variety having coordinate algebra $C := K[\mathcal{X}, \mathcal{Y}] \cong K[\mathcal{X}] \otimes_K K[\mathcal{Y}]$, where the tensor product of $K$-vector spaces naturally becomes a $K$-algebra again. (This coincides with the fibre sum of $K[\mathcal{X}]$ and $K[\mathcal{Y}]$ in the category of $K$-algebras, with respect to the structural homomorphisms $K \to K[\mathcal{X}]$ and $K \to K[\mathcal{Y}]$.) Thus $f(\mathcal{X})g(\mathcal{Y}) \in C$ induces the regular function $(f \otimes g)^\bullet \colon L^n \times L^m \to L \colon [v,w] \mapsto f(v)g(w)$.

Then for $[v,w] \in L^n \times L^m$ we have $[v,w] \in \mathbf{U}$ if and only if $v \in \mathbf{V}_L(I)$ and $w \in \mathbf{V}_L(J)$. In other words, $\mathbf{U} = \mathbf{V}_L(\langle I \otimes 1, 1 \otimes J\rangle) \subseteq L^{n+m}$ is closed, such that $\langle I \otimes 1, 1 \otimes J\rangle \subseteq \mathbf{I}_K(\mathbf{U}) \trianglelefteq C$.

We have a $K$-bilinear map $(A/I) \times (B/J) \to C/\langle I \otimes 1, 1 \otimes J\rangle \colon [f,g] \mapsto f(\mathcal{X})g(\mathcal{Y})$, which hence gives rise to an epimorphism of $K$-algebras

$$\alpha \colon (A/I) \otimes_K (B/J) \to C/\langle I \otimes 1, 1 \otimes J\rangle \colon X_i \otimes 1 \mapsto X_i,\ 1 \otimes Y_j \mapsto Y_j.$$

Concatenating $\alpha$ with the natural map $C/\langle I \otimes 1, 1 \otimes J\rangle \to C/\mathbf{I}_K(\mathbf{U})$, we get an epimorphism $\widetilde{\alpha} \colon (A/I) \otimes_K (B/J) \to C/\mathbf{I}_K(\mathbf{U})$. We show that $\widetilde{\alpha}$ is injective:

Let $\mathcal{A} := \{f_k; k \in \mathcal{K}\} \subseteq (A/I)$ and $\mathcal{B} := \{g_l; l \in \mathcal{L}\} \subseteq (B/J)$ be $K$-bases, where $\mathcal{K}$ and $\mathcal{L}$ are index sets. Then $\mathcal{A} \otimes \mathcal{B} \subseteq (A/I) \otimes_K (B/J)$ is a $K$-basis, so that $\widetilde{\alpha}(\mathcal{A} \otimes \mathcal{B})$ spans $C/\mathbf{I}_K(\mathbf{U})$. We show that $\widetilde{\alpha}(\mathcal{A} \otimes \mathcal{B})$ is $K$-linearly independent:

Let $\lambda_{kl} \in K$ such that $\sum_{k\in\mathcal{K}} \sum_{l\in\mathcal{L}} \lambda_{kl} f_k(\mathcal{X}) g_l(\mathcal{Y}) = 0 \in C/\mathbf{I}_K(\mathbf{U}) = K[\mathbf{U}]$. Hence for all $v \in \mathbf{V}$ and $w \in \mathbf{W}$ we have $\sum_{k\in\mathcal{K}} f_k(v) \cdot \left(\sum_{l\in\mathcal{L}} \lambda_{kl} g_l(w)\right) = 0$. Keeping $w$ fixed, we infer that $\sum_{k\in\mathcal{K}} f_k(\mathcal{X}) \cdot \left(\sum_{l\in\mathcal{L}} \lambda_{kl} g_l(w)\right) = 0 \in K[\mathbf{V}] = A/I$. Since $\mathcal{A} \subseteq (A/I)$ is $K$-linearly independent, we get $\sum_{l\in\mathcal{L}} \lambda_{kl} g_l(w) = 0$, for all $k \in \mathcal{K}$. This entails $\sum_{l\in\mathcal{L}} \lambda_{kl} g_l(\mathcal{Y}) = 0 \in K[\mathbf{W}] = B/J$. Since $\mathcal{B} \subseteq (B/J)$ is $K$-linearly independent, we finally get $\lambda_{kl} = 0$, for all $k \in \mathcal{K}$ and $l \in \mathcal{L}$. ♯

Hence both $\alpha$ and the natural map $C/\langle I \otimes 1, 1 \otimes J\rangle \to C/\mathbf{I}_K(\mathbf{U})$ are injective, and thus are isomorphisms. Hence we have $\langle I \otimes 1, 1 \otimes J\rangle = \mathbf{I}_K(\mathbf{U}) \trianglelefteq C$, and

$$K[\mathbf{U}] = C/\mathbf{I}_K(\mathbf{U}) \cong C/\langle I \otimes 1, 1 \otimes J\rangle \cong (A/I) \otimes_K (B/J) = K[\mathbf{V}] \otimes_K K[\mathbf{W}].$$

**b)** Assume that $I \trianglelefteq A$ and $J \trianglelefteq B$ are prime. We show that $\mathbf{I}_K(\mathbf{U}) \trianglelefteq C$ is prime:

Let $h_1, h_2 \in C$ such that $h_1 h_2 \in \mathbf{I}_K(\mathbf{U})$. For $w \in \mathbf{W}$ fixed let $\mathbf{V}_i(w) := \{v \in \mathbf{V}; h_i(v,w) = 0\} = \mathbf{V}_{\mathbf{V}}(h_i(\mathcal{X}, w)) \subseteq \mathbf{V}$ closed, for $i \in \{1,2\}$. Since $h_1 h_2 \in \mathbf{I}_K(\mathbf{U})$, for any $v \in \mathbf{V}$ we have $h_1(v,w) = 0$ or $h_2(v,w) = 0$, that is $v \in \mathbf{V}_1(w)$ or $v \in \mathbf{V}_2(w)$. Thus we get $\mathbf{V} = \mathbf{V}_1(w) \cup \mathbf{V}_2(w)$. Hence, since $\mathbf{V}$ is irreducible, we have $\mathbf{V}_1(w) = \mathbf{V}$ or $\mathbf{V}_2(w) = \mathbf{V}$.

Let $\mathbf{W}_i := \{w \in \mathbf{W}; \mathbf{V}_i(w) = \mathbf{V}\} = \{w \in \mathbf{W}; h_i(v,w) = 0 \text{ for all } v \in \mathbf{V}\} = \mathbf{V}_{\mathbf{W}}(\{h_i(v, \mathcal{Y}); v \in \mathbf{V}\}) \subseteq \mathbf{W}$ closed. Then by the above we have $\mathbf{W} = \mathbf{W}_1 \cup \mathbf{W}_2$. Since $\mathbf{W}$ is irreducible, we get $\mathbf{W}_1 = \mathbf{W}$ or $\mathbf{W}_2 = \mathbf{W}$. Finally, if $\mathbf{W}_i = \mathbf{W}$ then we have $h_i(v,w) = 0$ for all $v \in \mathbf{V}$ and $w \in \mathbf{W}$, thus $h_i(v,w) \in \mathbf{I}_K(\mathbf{U})$. ♯

We show that $\mathbf{U}$ is a product in the category of prevarieties: Firstly, since $\pi_{\mathbf{V}}^* \colon K[\mathbf{V}] \to K[\mathbf{U}] \colon X_i \mapsto X_i \otimes 1$ and $\pi_{\mathbf{W}}^* \colon K[\mathbf{W}] \to K[\mathbf{U}] \colon Y_j \mapsto 1 \otimes Y_j$ are homomorphisms of $K$-algebras, $\pi_{\mathbf{V}}$ and $\pi_{\mathbf{W}}$ are morphisms of affine varieties.

Now let $Z$ be a prevariety, and let $\varphi' \colon Z \to \mathbf{V}$ and $\varphi'' \colon Z \to \mathbf{W}$ be morphisms. Since $\mathbf{U} = \mathbf{V} \times \mathbf{W}$ as sets (that is a product in the category of sets), there is a unique product map $\varphi = \varphi' \times \varphi'' \colon Z \to \mathbf{U}$ such that $\varphi' = \varphi\pi_{\mathbf{V}}$ and $\varphi'' = \varphi\pi_{\mathbf{W}}$. Hence we have to show that $\varphi$ is a morphism of prevarieties:

Since $\mathbf{U}$ is affine, it suffices to show that $\varphi^* \colon K[\mathbf{U}] = \Gamma(\mathcal{O}_{\mathbf{U}}) \to \Gamma(\mathcal{O}_Z)$ is a homomorphism of $K$-algebras, which just amounts to showing that it is well-defined: Recall that $K[\mathbf{U}]$ is generated by the $X_i \otimes 1$ and $1 \otimes Y_j$. Since $\varphi'$ and $\varphi''$ are morphisms, we get $\varphi^*(X_i \otimes 1) = \varphi^*(\pi_{\mathbf{V}}^*(X_i)) = (\varphi\pi_{\mathbf{V}})^*(X_i) = (\varphi')^*(X_i) \in \Gamma(\mathcal{O}_Z)$ and $\varphi^*(1 \otimes Y_j) = \varphi^*(\pi_{\mathbf{W}}^*(Y_j)) = (\varphi\pi_{\mathbf{W}})^*(Y_j) = (\varphi'')^*(Y_j) \in \Gamma(\mathcal{O}_Z)$. ♯

**(11.3) Theorem: Products of prevarieties.** Let $V$ and $W$ be prevarieties. Then the product $V \times W$ as sets carries the structure of a prevariety, as which it is a product of $V$ and $W$ in the category of prevarieties.

**Proof.** We may assume that $V \neq \emptyset \neq W$. We let the topology on $P := V \times W$ be defined by having the basis consisting of all subsets $V' \times W' \subseteq P$ such that both $V' \subseteq V$ and $W' \subseteq W$ are affine open, where $V' \times W'$ carries the structure of an affine variety: Noting that $(V' \times W') \cap (V'' \times W'') = (V' \cap V'') \times (W' \cap W'')$, where $V' \cap V'' \subseteq V$ and $W' \cap W'' \subseteq W$ are open, and recalling that $V$ and $W$ have bases consisting of affine open subsets, we infer that this indeed is a basis.

Since $V$ is irreducible, any two non-empty open subsets of $V$ intersect non-trivially; similarly for $W$. Thus any two non-empty open subsets of $P$ belonging to the above basis intersect non-trivially, entailing that $P$ is irreducible. Since both $V$ and $W$ have a finite affine open covering, this holds for $P$ as well.

Next, as is immediately seen, the following defines a sheaf $\mathcal{O}_P$ on $P$: For $U \subseteq P$ open let $\mathcal{O}_P(U)$ be the $K$-algebra of all $f \colon U \to L$ such that $f|_{U \cap (V' \times W')} \in \mathcal{O}_{V' \times W'}(U \cap (V' \times W'))$, for $V' \subseteq V$ and $W' \subseteq W$ affine open. Thus $P$ together with $\mathcal{O}_P$ becomes a prevariety, so that $V' \times W'$ together with $\mathcal{O}_P|_{V' \times W'} = \mathcal{O}_{V' \times W'}$ is an affine open subprevariety, for all $V' \times W'$ as above.

We show that $P$, together with the projections $\pi_V$ and $\pi_W$, is a product of $V$ and $W$ in the category of prevarieties: Firstly, $\pi_V$ and $\pi_W$ are morphisms: We consider the affine open covering by subsets $V' \times W' \subseteq P$ as above. Then $\pi_V|_{V' \times W'} \colon V' \times W' \to V'$ and $\pi_W|_{V' \times W'} \colon V' \times W' \to W'$ are morphisms, hence $\pi_V$ and $\pi_W$ are continuous, and by the sheaf properties are morphisms.

Now let $Z$ be a prevariety, and let $\varphi_V \colon Z \to V$ and $\varphi_W \colon Z \to W$ be morphisms. Then there is a unique product map $\varphi = \varphi_V \times \varphi_W \colon Z \to P$ such that $\varphi_V = \varphi\pi_V$ and $\varphi_W = \varphi\pi_W$. We show that $\varphi$ is a morphism of prevarieties:

We again consider the affine open covering by subsets $V' \times W' \subseteq P$ as above. Let $U := \varphi_V^{-1}(V') \cap \varphi_W^{-1}(W') \subseteq Z$ open. Then $\varphi|_U \colon U \to V' \times W'$ is the product map associated with the morphisms $\varphi_V|_U \colon U \to V'$ and $\varphi_W|_U \colon U \to W'$. Since $V' \times W'$ is affine, $\varphi|_U$ is a morphism. Again we infer that $\varphi$ is a morphism. ♯

**Corollary.** If $U \subseteq V$ is an open subprevariety, then so is $U \times W \subseteq P$.

**Proof.** Picking affine open coverings of $U$ and $W$, respectively, shows that $U \times W \subseteq P$ is open, and that $\mathcal{O}_P|_{U \times W} = \mathcal{O}_{U \times W}$. ♯

In particular, the product of two irreducible quasi-affine varieties is irreducible quasi-affine again, and by (11.5) below the product of two irreducible quasi-projective varieties is irreducible quasi-projective again.

**(11.4) Proposition.** Let $V$ and $W$ be prevarieties, and let $Z \subseteq V$ be a closed subprevariety. Then $Z \times W \subseteq V \times W$ is a closed subprevariety as well.

**Proof.** Since $(V \setminus Z) \times W \subseteq V \times W =: P$ is open, $Z \times W \subseteq P$ is closed. Now $P$ is covered by affine open subsets $V' \times W' \subseteq P$, where $V' \subseteq V$ and $W' \subseteq W$ are affine open. Then $Z' := Z \cap V' \subseteq V'$ is closed, where we may assume that $Z' \neq \emptyset$. Since $Z' \subseteq Z$ is open and $Z$ is irreducible, $Z'$ is irreducible as well.

Then $Z' \times W' \subseteq V' \times W'$ is closed, hence is affine with respect to $\mathcal{O}_{V' \times W'}|_{Z' \times W'}$, where we have $\Gamma(\mathcal{O}_{V' \times W'}|_{Z' \times W'}) = K[V' \times W']/\mathbf{I}_K(Z' \times W') \cong (K[V'] \otimes_K K[W'])/\langle \mathbf{I}_K(Z') \otimes 1, 1 \otimes \mathbf{I}_K(W') \rangle \cong (K[V']/\langle \mathbf{I}_K(Z') \rangle) \otimes_K K[W']$.

Similarly, $Z' \subseteq V'$ is affine with respect to $\mathcal{O}_{V'}|_{Z'} = \mathcal{O}_{Z'}$, where $\Gamma(\mathcal{O}_{Z'}) = K[Z'] = K[V']/\mathbf{I}_K(Z')$. Then $Z' \times W'$ is affine with respect to $\mathcal{O}_{Z' \times W'}$, where $\Gamma(\mathcal{O}_{Z' \times W'}) = K[Z' \times W'] \cong K[Z'] \otimes_K K[W'] = (K[V']/\langle \mathbf{I}_K(Z') \rangle) \otimes_K K[W']$. Thus we infer that $\mathcal{O}_{V' \times W'}|_{Z' \times W'} = \mathcal{O}_{Z' \times W'}$.

Then from this we get $(\mathcal{O}_P|_{Z \times W})|_{Z' \times W'} = \mathcal{O}_P|_{Z' \times W'} = (\mathcal{O}_P|_{V' \times W'})|_{Z' \times W'} = \mathcal{O}_{V' \times W'}|_{Z' \times W'} = \mathcal{O}_{Z' \times W'} = \mathcal{O}_{Z \times W}|_{Z' \times W'}$. Since $Z \times W$ is covered by affine open subsets of the above form, we conclude that $\mathcal{O}_P|_{Z \times W} = \mathcal{O}_{Z \times W}$. ♯

**(11.5) Theorem: Products of projective varieties.** Let $\mathbf{V}$ and $\mathbf{W}$ be projective varieties. Then $\mathbf{V} \times \mathbf{W}$ is a projective variety again.

**Proof.** Let $\mathbf{V} \subseteq \mathbf{P}^n$ and $\mathbf{W} \subseteq \mathbf{P}^m$ be closed, where $n, m \in \mathbb{N}_0$. Then $\mathbf{V} \times \mathbf{W} \subseteq \mathbf{P}^n \times \mathbf{P}^m$ is closed as well. Since a closed subprevariety of a projective variety is projective again again, it suffices to show that $\mathbf{P}^n \times \mathbf{P}^m$ is projective. We show that the latter can be identified with a closed subprevariety of $\mathbf{P}^s$, where $s := (n+1)(m+1) - 1 = n + m + nm$:

Letting $\mathcal{X}^\sharp = \{X_0, \ldots, X_n\}$, $\mathcal{Y}^\sharp = \{Y_0, \ldots, Y_m\}$, and $\mathcal{Z}^\sharp = \{Z_{00}, Z_{01}, \ldots, Z_{nm}\}$ be indeterminates associated with $\mathbf{P}^n$, $\mathbf{P}^m$, and $\mathbf{P}^s$, respectively, we consider the **Segre map** (which is immediately seen to be well-defined)

$$\sigma \colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^s \colon [[x_0 \colon \ldots \colon x_n], [y_0 \colon \ldots \colon y_m]] \mapsto [\ldots \colon x_i y_j \colon \ldots].$$

We show that $\sigma$ is injective: Let $p = [x_0 \colon \ldots \colon x_n]$, $p' = [x'_0 \colon \ldots \colon x'_n]$, and $q = [y_0 \colon \ldots \colon y_m]$, $q' = [y'_0 \colon \ldots \colon y'_m]$, such that $\sigma(p, q) = \sigma(p', q')$. We may assume that $x_0 \neq 0 \neq y_0$. Then $x_0 y_0 \neq 0$ implies $x'_0 y'_0 \neq 0$, thus $x'_0 \neq 0 \neq y'_0$ as well. Hence we have $\frac{x_i y_j}{x_0 y_0} = \frac{x'_i y'_j}{x'_0 y'_0}$, entailing $\frac{x_i}{x_0} = \frac{x_i y_0}{x_0 y_0} = \frac{x'_i y'_0}{x'_0 y'_0} = \frac{x'_i}{x'_0}$ and $\frac{y_j}{y_0} = \frac{x_0 y_j}{x_0 y_0} = \frac{x'_0 y'_j}{x'_0 y'_0} = \frac{y'_j}{y'_0}$, for all $i$ and $j$, thus $p = p' \in \mathbf{P}^n$ and $q = q' \in \mathbf{P}^m$.

Now, writing $D_i := D_{X_i} \subseteq \mathbf{P}^n$, $D_j := D_{Y_j} \subseteq \mathbf{P}^m$, and $D_{ij} := D_{Z_{ij}} \subseteq \mathbf{P}^s$, we have $\sigma^{-1}(D_{ij}) = D_i \times D_j \subseteq \mathbf{P}^n \times \mathbf{P}^m$. Thus this gives rise to maps $\sigma_{ij} = \sigma|_{D_i \times D_j} \colon D_i \times D_j \to D_{ij}$, where we let $V_{ij} := \sigma_{ij}(D_i \times D_j) \subseteq D_{ij} \subseteq \mathbf{P}^s$.

For convenience letting $i = 0 = j$, and identifying $L^n \cong D_0 \subseteq \mathbf{P}^n$ and $L^m \cong D_0 \subseteq \mathbf{P}^m$ and $L^s \cong D_{00} \subseteq \mathbf{P}^s$ as affine varieties, we get the regular map $\sigma_{00} \colon L^n \times L^m \to L^s \colon [[x_1, \ldots, x_n], [y_1, \ldots, y_m]] \mapsto [z_{ij}]_{ij}$, given by $z_{ij} = x_i y_j$ and $z_{i,0} = x_i$ and $z_{0,j} = y_j$, for $i, j \geq 1$. Moreover, the comorphism $\sigma_{00}^* \colon K[\mathcal{Z}] \to K[\mathcal{X}] \otimes_K K[\mathcal{Y}]$, where $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are the respective dehomogenizations, is given by $Z_{ij} \mapsto X_i \otimes Y_j$ and $Z_{i,0} \mapsto X_i \otimes 1$ and $Z_{0,j} \mapsto 1 \otimes Y_j$, for $i, j \geq 1$.

Hence we conclude that the $\sigma_{ij}^*$ are epimorphisms of $K$-algebras, implying that $\sigma_{ij}$ is a closed embedding, for all $i$ and $j$, that is $V_{ij} \subseteq D_{ij}$ is closed such that $\sigma_{ij} \colon D_i \times D_j \to V_{ij}$ is an isomorphism of affine varieties.

Let $\mathbf{V} := \sigma(\mathbf{P}^n \times \mathbf{P}^m) = \bigcup_{i,j \geq 0} V_{ij} \subseteq \mathbf{P}^s$. Then we have $\mathbf{V} \cap D_{ij} = \sigma(D_i \times D_j) = V_{ij}$. Since $V_{ij} \subseteq D_{ij}$ is closed, for all $i$ and $j$, we conclude that $\mathbf{V} \subseteq \mathbf{P}^s$ is closed as well, so that $\mathbf{V}$ is a projective variety. Moreover, by the sheaf properties, $\sigma \colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{V}$ is an isomorphism of prevarieties. (For a description of the homogeneous vanishing ideal $\mathbf{I}_K^\sharp(\mathbf{V}) \trianglelefteq K[\mathcal{Z}^\sharp]$, see Exercise (12.33).) $\sharp$

**(11.6) Abstract varieties. a)** A prevariety $V$ is called an **(abstract) variety** if the **Hausdorff axiom** holds, that is for any prevariety $U$ and any morphisms $\varphi, \psi \in \mathrm{Mor}(U, V)$ the associated **(difference) kernel**

$$\ker(\varphi, \psi) := \{u \in U; \varphi(u) = \psi(u)\} \subseteq U$$

is closed. (This naming is reminiscent of the Hausdorff property of topological spaces, where product spaces carry the product topology.)

Morphisms of varieties are their morphisms as prevarieties. This gives rise to the category of varieties, being a full subcategory of the category of prevarieties. There are prevarieties which are not varieties, as the example below shows.

**b)** We present a characterization of varieties amongst prevarieties:

Let $U$ and $V$ be prevarieties, and let $\varphi \in \mathrm{Mor}(U, V)$. Then let the **graph morphism** $\gamma_\varphi = \mathrm{id}_U \times \varphi \colon U \to U \times V \colon u \mapsto [u, \varphi(u)]$ be the product morphism arising from $\mathrm{id}_U$ and $\varphi$, and let $\Gamma_\varphi := \gamma_\varphi(U) \subseteq U \times V$ be the **graph** of $\varphi$.

In particular, for the identity morphism $\mathrm{id}_V$ let $\delta_V := \gamma_{\mathrm{id}_V} = \mathrm{id}_V \times \mathrm{id}_V \colon V \to V \times V \colon v \mapsto [v, v]$ be the **diagonal morphism**, and let $\Delta_V := \delta_V(V) \subseteq V \times V$ be the **diagonal** of $V \times V$.

**Proposition. a)** Let $V$ be a prevariety. Then $V$ is a variety if and only if $V$ is **separated**, that is the diagonal $\Delta_V \subseteq V \times V$ is closed.

**b)** Let $V$ be a variety, let $U$ be a prevariety, and let $\varphi \in \mathrm{Mor}(U, V)$. Then $\gamma_\varphi \colon U \to U \times V$ is a closed embedding. In particular, so is $\delta_V \colon V \to V \times V$.

**Proof. a)** Let $V$ be a variety. Since $\Delta_V = \ker(\pi_1, \pi_2) \subseteq V \times V$, where the $\pi_i \colon V \times V \to V$ are the projections on the first and second factors, respectively, we conclude that $\Delta_V \subseteq V \times V$ is closed.

Now let $V$ be separated, let $U$ be a prevariety, let $\varphi_1, \varphi_2 \in \mathrm{Mor}(U, V)$, and let $\varphi := \varphi_1 \times \varphi_2 \colon U \to V \times V$ be the associated product morphism. Then we have $\ker(\varphi_1, \varphi_2) = \ker(\varphi\pi_1, \varphi\pi_2) = \{u \in U; \varphi(u) \in \Delta_V\} = \varphi^{-1}(\Delta_V) \subseteq U$. Since $\Delta_V \subseteq V \times V$ is closed, we infer that $\varphi^{-1}(\Delta_V) \subseteq U$ is closed as well.

**b)** Letting $P := U \times V$, we consider the morphisms $\pi_V \colon P \to V$ and $\pi_U \varphi \colon P \to V$. Thus, since $V$ is a variety, the kernel $\ker(\pi_V, \pi_U\varphi) = \{[u, v] \in P; \pi_V(u, v) = \varphi(\pi_U(u, v))\} = \{[u, v] \in P; v = \varphi(u)\} = \Gamma_\varphi \subseteq P$ is closed. In particular, $\Gamma_\varphi$ together with the structure sheaf $\mathcal{O}_P|_{\Gamma_\varphi}$ is a prevariety.

Moreover, the map $\gamma'_\varphi \colon U \to \Gamma_\varphi \colon u \mapsto \gamma_\varphi(u)$, obtained from $\gamma_\varphi$ by restricting its range, is a morphism as well: Let $W \subseteq \Gamma_\varphi$ be open, and let $f \in \mathcal{O}_P|_{\Gamma_\varphi}(W)$; then by the definition of a restricted sheaf we may assume that there is $\widehat{W} \subseteq P$ open and $\widehat{f} \in \mathcal{O}_P(\widehat{W})$ such that $\widehat{W} \cap \Gamma_\varphi = W$ and $\widehat{f}|_W = f$; then we have $(\gamma'_\varphi)^*(f) = \gamma^*_\varphi(\widehat{f}) \in \mathcal{O}_U(\gamma_\varphi^{-1}(\widehat{W})) = \mathcal{O}_U((\gamma'_\varphi)^{-1}(W))$.

Conversely, we have the morphism $\pi_U|_{\Gamma_\varphi} = \iota^P_{\Gamma_\varphi} \cdot \pi_U \colon \Gamma_\varphi \to U$; recall that the inclusion map $\iota^P_{\Gamma_\varphi} \colon \Gamma_\varphi \to P$ is a morphism with respect to the restricted sheaf. Then we get $\gamma'_\varphi \cdot (\pi_U|_{\Gamma_\varphi}) = \mathrm{id}_U$ and $(\pi_U|_{\Gamma_\varphi}) \cdot \gamma'_\varphi = \mathrm{id}_{\Gamma_\varphi}$. ♯

**Example: The affine line with one point doubled.** Following (10.4), we consider the fibre sum $V := L \sqcup_U L$ of two disjoint copies of the affine line, with respect to the inclusion $U := L \setminus \{0\} \subseteq L$. Thus we have injective morphisms $\iota_i \colon L \to V$, such that $\iota_1|_U = \iota_2|_U$. Letting $V' := \iota_i(U) \subseteq V$ we have $V = \{0_1\} \,\dot\cup\, V' \,\dot\cup\, \{0_2\}$, where $V_i := \iota_i(L) = \{0_i\} \,\dot\cup\, V'$. Then $V$ becomes a prevariety, having $\{V_1, V_2\}$ as an affine open covering. But $\ker(\iota_1, \iota_2) = \{a \in L; \iota_1(a) = \iota_2(a)\} = \{a \in L; \iota_i(a) \in V'\} = U \subseteq L$ is not closed; hence $V$ is not a variety. ♯

**(11.7) Proposition.** Any irreducible affine variety $\mathbf{V}$ is a variety.

**Proof.** Let $U$ be any prevariety, and let $\varphi, \psi \in \mathrm{Mor}(U, \mathbf{V})$. In order to show that $\ker(\varphi, \psi) \subseteq U$ is closed, by taking an affine open covering of $U$, we may assume that $U$ is an affine variety. Then, given $u \in U$, we have $\varphi(u) = \psi(u) \in \mathbf{V}$ if and only if $f(\varphi(u)) = f(\psi(u))$ for all $f \in L[\mathbf{V}] = L \otimes_K K[\mathbf{V}]$. Comparing the associated maximal ideals of $L[\mathbf{V}]$ and $K[\mathbf{V}]$, respectively, it follows that the latter property is fulfilled if and only if it holds for all $f \in K[\mathbf{V}]$. Thus we have $\varphi(u) = \psi(u)$ if and only if $(\varphi^*(f))(u) = f(\varphi(u)) = f(\psi(u)) = (\psi^*(f))(u)$ for all $f \in K[\mathbf{V}]$, that is $\ker(\varphi, \psi) = \mathbf{V}_L((\varphi^* - \psi^*)(K[\mathbf{V}])) \subseteq U$ is closed. ♯

From this we get the following consequence on difference kernels in general:

Recall first that a subset of a topological space $V$ is called **locally closed**, if it is the intersection of an open and a closed subset of $V$. Then $W \subseteq V$ is locally closed if and only if it is open in its closure: If $W = U \cap Z$, where $U \subseteq V$ is open and $Z \subseteq V$ is closed, then $W \subseteq Z$ is open, hence $W \subseteq \overline{W}$ is open; conversely, if $W \subseteq \overline{W}$ is open, then there is $U \subseteq V$ open such that $W = U \cap \overline{W}$.

**Corollary. a)** Let $U$ and $V$ be prevarieties, and let $\varphi, \psi \in \mathrm{Mor}(U, V)$. Then the kernel $\ker(\varphi, \psi) \subseteq U$ is locally closed (rather than closed, if $V$ is a variety).

**b)** Assume that for any $v, w \in V$ there is an affine open subset of $V$ containing $v$ and $w$. Then $V$ is a variety.

**Proof. a)** Let $u \in \ker(\varphi, \psi)$, let $\varphi(u) = \psi(u) \in V' \subseteq V$ be affine open, and let $U' := \varphi^{-1}(V') \cap \psi^{-1}(V') \subseteq U$ open, hence $u \in U'$. Since $V'$ is affine and thus is a variety, for the morphisms $\varphi|_{U'} \colon U' \to V'$ and $\psi|_{U'} \colon U' \to V'$ we infer that $\ker(\varphi, \psi) \cap U' = \ker(\varphi|_{U'}, \psi|_{U'}) \subseteq U'$ is closed. Taking the union of the subsets $U' \subseteq U$ as above, we conclude that $\ker(\varphi, \psi)$ is closed in an open subset of $U$. Thus $\ker(\varphi, \psi)$ is the intersection of a closed and an open subset of $U$.

**b)** Let $U$ be a prevariety, and let $\varphi, \psi \in \mathrm{Mor}(U, V)$. Given $u \in U$, there is an affine open subset $V' \subseteq V$ such that $\varphi(u), \psi(u) \in V'$. Let $U' := \varphi^{-1}(V') \cap \psi^{-1}(V') \subseteq U$ open, hence $u \in U'$. Then $\ker(\varphi, \psi) \cap U' = \ker(\varphi|_{U'}, \psi|_{U'}) \subseteq U'$ is closed. Now $U$ is covered by subsets as above, thus $\ker(\varphi, \psi) \subseteq U$ is closed. $\sharp$

**(11.8) Proposition.** The (irreducible) projective space $\mathbf{P}$ is a variety.

**Proof.** We consider the natural $L$-linear (right) action of the general linear group $G := \mathrm{GL}_{n+1}(K)$ on $L^{n+1}$, which induces an action of $G$ on $\mathbf{P}$. We show that $G$ acts on $\mathbf{P}$ by automorphisms, where it suffices to show that the map induced by any $\alpha = [\alpha_{ij}] \in G$ is a morphism:

Let $\mathbf{P}$ have homogeneous coordinate algebra $A^\sharp = K[X_0, \ldots, X_n]$. We consider the affine open subset $D_i := D_{X_i} \subseteq \mathbf{P}$, for $i \in \{0, \ldots, n\}$, where $\Gamma(\mathcal{O}_{D_i}) = K[\frac{X_0}{X_i}, \ldots, \widehat{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}]$. Then $\alpha^*(X_i) = \sum_{j=0}^n \alpha_{ji} X_j$ is homogeneous of degree 1, entailing a bijection $D_{\alpha^*(X_i)} = \alpha^{-1}(D_i) \to D_i$. We have $\alpha^*(\frac{X_j}{X_i}) = \frac{\alpha^*(X_j)}{\alpha^*(X_i)}$, being homogeneous of degree 0, thus being regular on $D_{\alpha^*(X_i)}$. Hence $\alpha|_{D_{\alpha^*(X_i)}}$ is a morphism. Since $\mathbf{P} = \bigcup_{i=0}^n D_{\alpha^*(X_i)}$, so is the map induced by $\alpha$. $\sharp$

This shows that $D_{\alpha^*(X_i)} \cong L^n$ is an affine variety. From this we infer that any hyperplane complement $D_f \subseteq \mathbf{P}$, where $f \in A^\sharp$ is homogeneous of degree 1, is an affine variety isomorphic to $L^n$. Finally, it is immediate that for any $v, w \in \mathbf{P}$ there is a hyperplane complement $D_f$, such that both $v, w \in D_f$. (It suffices to consider the case $v \in D_i \setminus D_j$ and $w \in D_j \setminus D_i$, for $i \neq j$.) $\sharp$

**(11.9) Persistence properties.** We show that the full subcategory of varieties within the category of prevarieties is closed with respect to certain constructions:

**Proposition. i)** The product of two varieties is a variety again.

**ii)** Any subprevariety of a variety is a variety again. In particular, any irreducible quasi-affine, projective or quasi-projective variety is a variety.

**Proof. i)** Let $V$ and $W$ be varieties. Then $\Delta_V \subseteq V \times V$ and $\Delta_W \subseteq W \times W$ are closed, thus so is $\Delta_{V \times W} = \Delta_V \times \Delta_W \subseteq (V \times V) \times (W \times W) \cong (V \times W) \times (V \times W)$.

**ii)** Let $V$ be a variety, and let $W \subseteq V$ be a subprevariety. Moreover, let $U$ be a prevariety, and let $\varphi, \psi \colon U \to W$ be morphisms of prevarieties. Then, since the injective inclusion map $\iota_W^V \colon W \to V$ is a morphism, we conclude that $\ker(\varphi, \psi) = \ker(\varphi \iota_W^V, \psi \iota_W^V) \subseteq U$ is closed. ♯

**Proposition.** Let $U$ be a variety, and let $V, W \subseteq U$ be affine open subsets, having coordinate algebras $K[V]$ and $K[W]$, respectively. Then $V \cap W \subseteq U$ is affine open again, having coordinate algebra $K[V \cap W] = K[V] \cdot K[W]$.

**Proof.** We may assume that $V \neq \emptyset \neq W$. The product $V \times W \subseteq U \times U$ is an affine open subset. Since $\delta_U \colon U \to U \times U$ is a closed embedding, we infer that the morphism $\epsilon := \delta_U|_{V \cap W} \colon V \cap W \to V \times W$ induces an isomorphism onto its image $\{[x, x] \in U \times U; x \in V \cap W\} = \Delta_U \cap (V \times W)$, where $\Delta_U \cap (V \times W) \subseteq V \times W$ is closed. Noting that $V \cap W \subseteq U$ is irreducible, we conclude that $\epsilon \colon V \cap W \to V \times W$ is a closed embedding, thus $V \cap W$ is affine.

Finally, as comorphism we get the surjective homomorphism of $K$-algebras $\epsilon^* \colon K[V] \otimes_K K[W] \to K[V \cap W] \colon f \otimes g \mapsto f|_{V \cap W} \cdot g|_{V \cap W}$, where since $V \cap W$ is dense in both $V$ and $W$, both restriction maps to $V \cap W$ are injective. ♯

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# III Exercises and references

## 12 Exercises for Part I

**(12.1) Exercise: Noetherian rings and modules.**
Let $R$ be a (commutative, associative, unital) ring, and let $M$ be a finitely generated $R$-module. Recall that the set of $R$-submodules of $M$ is partially ordered by set-theoretic inclusion.

**a)** Let $M \neq \{0\}$. Show that $M$ has a maximal (proper) $R$-submodule.

**b)** Show that the following properties are equivalent:
**i)** $M$ is **Noetherian**, that is any $R$-submodule of $M$ is finitely generated.
**ii)** Any strictly ascending chain of $R$-submodule of $M$ terminates.
**iii)** Any non-empty set of $R$-submodules of $M$ contains a maximal element.

**c)** Let $N \leq M$ is an $R$-submodule. Show that $M$ is Noetherian if and only if both $N$ and $M/N$ are Noetherian.

**d)** Let $M$ be Noetherian. If $\mathcal{S} \subseteq M$ is any (possibly infinite) generating set, show that there is a finite subset $\mathcal{S}' \subseteq \mathcal{S}$ generating $M$. Moreover, show that any surjective $R$-endomorphism of $M$ is injective.

**e)** Let $R$ be **Noetherian**, that is $R$ is Noetherian as an $R$-module; in other words, any ideal of $R$ is finitely generated. Show that $M$ is Noetherian as well.

**(12.2) Exercise: Non-Noetherian rings.**
Let $K$ be a field, and let $\mathcal{X} := \{X_1, X_2, \ldots\}$ be an infinite set of indeterminates. Show that $K[\mathcal{X}]$ is not Noetherian.

**(12.3) Exercise: Polynomial functions.**
Let $K$ be a field, and let $A$ be a polynomial $K$-algebra in $n \in \mathbb{N}_0$ indeterminates. For $f \in A$ let $f^\bullet \colon L^n \to L \colon v \mapsto f(v)$ be the associated **polynomial function**. Show that the set $\mathcal{A}$ of all such polynomial functions carries the structure of a $K$-algebra, with respect to which the map $f \mapsto f^\bullet \colon A \to \mathcal{A}$ is a homomorphism of $K$-algebras, which moreover is injective if and only if $K$ is infinite.

**(12.4) Exercise: Finite fields.**
Let $\mathbb{F}$ be a finite field, and let $n \in \mathbb{N}_0$. Show that any function $\mathbb{F}^n \to \mathbb{F}$ is polynomial. Conclude that any subset of $\mathbb{F}^n$ is algebraic.

**(12.5) Exercise: Algebraically closed fields.**
Show that any algebraically closed field is infinite.

**(12.6) Exercise: Complements of algebraic sets.**
Let $K \subseteq L$ be a field extension such that $L$ is infinite, let $n \geq 1$, and let $\mathbf{V} \subset L^n$ be a proper algebraic subset. Show that $L^n \setminus \mathbf{V}$ is infinite.

**(12.7) Exercise: Scalar extensions.**
Let $K \subseteq L$ be a field extension, let $A_K := K[\mathcal{X}]$ be a polynomial $K$-algebra in $n \in \mathbb{N}_0$ indeterminates, and let $A_L := L[\mathcal{X}]$ be its **scalar extension** to $L$.

**a)** Let $V \subseteq L^n$ be $L$-algebraic. Show that $U := V \cap K^n$ is $K$-algebraic, such that $\mathbf{I}_K(U) = \mathbf{I}_L(V) \cap A_K \trianglelefteq A_K$.

**b)** Let $U \subseteq K^n$ be $K$-algebraic, and let $V \subseteq L^n$ be the smallest $L$-algebraic subset containing $U$. Show that $V \cap K^n = U$, and that $\mathbf{I}_L(V) = \mathbf{I}_K(U) \cdot A_L \trianglelefteq A_L$.

**c)** Let $K \subseteq L \subseteq M$, where $M$ is algebraically closed, and let $W \subseteq M^n$ be $K$-algebraic. Show that $W$ is $L$-algebraic, and that $\mathbf{I}_L(W) = \sqrt{\mathbf{I}_K(W) \cdot A_L} \trianglelefteq A_L$.

**(12.8) Exercise: Hilbert's Nullstellensatz.**
Let $K$ be a field, let $A := K[\mathcal{X}]$ be a finitely generated polynomial $K$-algebra, and let $P \lhd A$ be a maximal ideal.
**a)** Show that $K \subseteq (A/P)$ is a finite field extension.
**b)** Let $K \subseteq L$ be a field extension. Show that $\mathbf{V}_L(P)$ is finite.

**(12.9) Exercise: A generalized Nullstellensatz.**
Let $K$ be a field which is *not* algebraically closed, let $A$ be a finitely generated polynomial $K$-algebra, and let $I \trianglelefteq A$.
**a)** Show that there is $g \in A$ such that $\mathbf{V}_K(I) = \mathbf{V}_K(g)$.
**b)** Show that $\mathbf{V}_K(I) \neq \emptyset$ if and only if $\mathbf{V}_K(f) \neq \emptyset$ for all $f \in I$. (This holds for algebraically closed fields as well, by Hilbert's Nullstellensatz.)

**Hint.** Show first that there is $h \in A$ such that $\mathbf{V}_K(h) = \{0\}$.

**(12.10) Exercise: Topological spaces.**
Let $V$ be an arbitrary topological space; recall that a maximal (closed) irreducible subset of $V$ is called an irreducible component. Show that any irreducible subset of $V$ is contained in an irreducible component. Deduce that $V$ is the irredundant union of its irreducible components.

**(12.11) Exercise: Noetherian spaces.**
**a)** A topological space is called **quasi-compact**, if any open cover has a finite subcover. Show that a topological space is Noetherian if and only if any open subspace is quasi-compact.

**b)** A topological space $V$ is called **Hausdorff**, if for any $x \neq y \in V$ there are open neighborhoods $U_x$ and $U_y$ of $x$ and $y$, respectively, such that $U_x \cap U_y = \emptyset$. Show that $V$ is Hausdorff Noetherian if and only if it is finite and discrete.

**c)** The space $V$ is called **T1**, if for any $x \neq y \in V$ there is an open neighborhood $U_x$ of $x$, such that $y \notin U_x$. Show that any quasi-projective variety is a T1 space.

**(12.12) Exercise: Hausdorff spaces.**
Let $V$ be a topological space. Show that the following are equivalent:

**i)** $V$ is a Hausdorff space.
**ii)** The set $\{[v, v] \in V \times V; v \in V\} \subseteq V \times V$ is closed in the product topology.
**iii)** For any topological space $U$ and any continuous map $\varphi\colon U \to V$, the graph $\{[u, \varphi(u)] \in U \times V; u \in U\} \subseteq U \times V$ is closed in the product topology.
**iv)** For any topological space $U$ and continuous maps $\varphi, \psi\colon U \to V$, the difference kernel $\{u \in U; \varphi(u) = \psi(u)\} \subseteq U$ is closed.

**(12.13) Exercise: Zariski topology.**
**a)** Identifying $\mathbb{C}\times\mathbb{C}$ with $\mathbb{C}^2$, show that the the Zariski topology on $\mathbb{C}^2$ is strictly finer than the product topology afforded by the Zariski topology on $\mathbb{C}$.

**b)** Show that the Zariski topology on $\mathbb{C}$ is strictly coarser than its metric topology, and conclude that the exponential function $\mathbb{C} \to \mathbb{C}\colon z \mapsto \exp(z)$ (for example) is not continuous with respect to the Zariski topology.

**(12.14) Exercise: Radicals.**
**a)** Let $R$ be a ring. For $I, J \trianglelefteq R$ show that $\sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J} = \sqrt{IJ}$. Moreover, for $I \lhd R$ show that $\sqrt{I} = \bigcap\{P \lhd A \text{ prime}; I \subseteq P\}$.

**b)** Let $A$ be a finitely-generated $K$-algebra, where $K$ is a field. For $I \lhd R$ show that $\sqrt{I} = \bigcap\{P \lhd A \text{ maximal}; I \subseteq P\}$. Does this also hold for arbitrary rings? (The above intersection of ideals is also called the **Jacobson radical** of $I$.)

**(12.15) Exercise: Maximal ideals.**
Let $A$ be a finitely generated $K$-algebra, where $K$ is a field, let $P \lhd A$ be a maximal ideal, and let $B \subseteq A$ be a $K$-subalgebra. Show that $P \cap B \lhd B$ is a maximal ideal again.

**(12.16) Exercise: Maximal ideals of coordinate algebras.**
**a)** Let $K \subseteq L$ be a field extension, let $\mathbf{V} \subseteq L^n$ be closed, let $\text{Hom}(K[\mathbf{V}], L)$ be the set of all $K$-algebra homomorphisms from $K[\mathbf{V}]$ to $L$, and for $v \in \mathbf{V}$ let $\epsilon_v\colon K[\mathbf{V}] \to L\colon f \mapsto f(v)$ be the associated **evaluation map**. Show that the map $\mathbf{V} \to \text{Hom}(K[\mathbf{V}], L)\colon v \mapsto \epsilon_v$ is a bijection.

**b)** Now let $K = L$ be algebraically closed. Show that for $v \in \mathbf{V}$ the inclusion map $\iota_v\colon \{v\} \to \mathbf{V}$ is an embedding such that $(\iota_v)^* = \epsilon_v$. Conclude that $\text{Hom}(K[\mathbf{V}], K)$ is in bijective correspondence with the maximal ideals of $K[\mathbf{V}]$.

**(12.17) Exercise: Hypersurfaces.**
Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed, let $A$ be a polynomial $K$-algebra in $n \in \mathbb{N}$ indeterminates, let $f = \prod_{i=1}^r f_i^{a_i} \in A$, where $r \in \mathbb{N}$ and $a_i \in \mathbb{N}$, and the $f_i \in A$ are pairwise non-associated and irreducible.

**a)** If $n \geq 2$, show that the hypersurface $\mathbf{V}_L(f) \subseteq L^n$ is infinite.
**b)** Show that $\mathbf{I}_K(\mathbf{V}_L(f)) = \prod_{i=1}^r \langle f_i \rangle \lhd A$.
**c)** Determine the irreducible components of $\mathbf{V}_L(f)$.

**(12.18) Exercise: Linear subspaces.**
Let $K$ be an infinite field, and let $A := K[X_1, \ldots, X_n]$ for some $n \in \mathbb{N}_0$.
**a)** Let $V \leq K^n$ be a $K$-subspace. Show that $V = \mathbf{V}_K(f_1, \ldots, f_m)$ for some $m \leq n$, where $f_j = \sum_{i=1}^n a_{ji} X_i$ for some $a_{ji} \in K$. How is $m$ related to $\dim_K(V)$?
**b)** Let $m$ be chosen minimal. Show that $\mathbf{I}_K(V) = \langle f_1, \ldots, f_m \rangle \lhd A$, that $V$ is irreducible, and that $K[V]$ is a polynomial algebra in $n - m$ indeterminates.

**(12.19) Exercise: Closed sets in the affine plane.**
Let $K$ be an algebraically closed field, let $A := K[X, Y]$, and let $f, g \in A$ be coprime. Show that $\mathbf{V}_K(f, g)$ is finite. Use this to give a description of all closed subsets of the affine plane $K^2$. Moreover, show that a prime ideal of $A$ is either maximal or principal.

**(12.20) Exercise: Irreducible components.**
Let $K$ be an algebraically closed field, and let $A := K[X, Y, Z]$. Determine the irreducible components of $\mathbf{V} := \mathbf{V}_K(X^2 - YZ, XZ - X)$. Moreover, compute the coordinate algebras of $\mathbf{V}$ and its irreducible components.

**(12.21) Exercise: Parametrisation of curves.**
**a)** Let $L \subseteq \mathbb{C}$, and let $\mathbf{V} := \{[t^2 - 1, t(t^2 - 1)] \in L^2; t \in L\}$. Show that $\mathbf{V} \subseteq L^2$ is closed, irreducible and defined over $\mathbb{Q}$, and determine $\mathbf{I}(\mathbf{V})$. Is the given parametrisation an isomorphism? Try to depict $\mathbf{V}$ for $L = \mathbb{R}^2$.

**b)** Let $\mathbf{W} := \{[t^3, t^4, t^5] \in L^3; t \in L\}$. Show that $\mathbf{W} \subseteq L^3$ is closed, irreducible and defined over $\mathbb{Q}$, and determine $\mathbf{I}(\mathbf{W})$. Is the given parametrisation an isomorphism? Can you show that $\mathbf{I}(\mathbf{W})$ cannot be generated by two elements?

**(12.22) Exercise: Parametrisation of surfaces.**
**a)** Let $L \subseteq \mathbb{C}$, let $W := \{[uv, v, u^2] \in L^3; u, v \in L\}$, and let $\mathbf{W} := \overline{W} \subseteq L^3$ be the **Whitney umbrella** surface over $L$. Show that $\mathbf{W}$ is irreducible and defined over $\mathbb{Q}$, and determine $\mathbf{I}(\mathbf{W})$. For $L = \mathbb{C}$ show that the given parametrisation is bijective. Is it an isomorphism? For $L = \mathbb{R}$ determine $\mathbf{W} \setminus W$.

**b)** Let $Z := \{[uv, uv^2, u^2] \in L^3; u, v \in L\}$, and let $\mathbf{Z} := \overline{Z} \subseteq L^3$. Show that $\mathbf{Z}$ is irreducible and defined over $\mathbb{Q}$, and determine $\mathbf{I}(\mathbf{W})$. Show that the given parametrisation is injective. For $L \in \{\mathbb{R}, \mathbb{C}\}$ determine $\mathbf{W} \setminus W$.

**(12.23) Example: Frobenius morphisms.**
Let $q$ be a prime power, let $\mathbf{F}$ be an algebraic closure of $\mathbb{F}_q$, let $\mathbf{V} \subseteq \mathbf{F}^n$ be $\mathbf{F}$-closed, for some $n \in \mathbb{N}_0$, and let $\Phi = \Phi_q \colon \mathbf{F}^n \to \mathbf{F}^n \colon [x_1, \ldots, x_n] \mapsto [x_1^q, \ldots, x_n^q]$ be the associated **(geometric) Frobenius morphism**.

**a)** Show that $\Phi$ is regular and bijective, but is not an isomorphism.

**b)** Show the equivalence of the following assertions:
**i)** We have $\Phi(\mathbf{V}) = \mathbf{V}$. **ii)** The set $\mathbf{V}$ is $\mathbb{F}_q$-closed.
**iii)** There are $f_1, \ldots, f_s \in \mathbb{F}_q[X_1, \ldots, X_n]$ such that $\mathbf{V} = \mathbf{V}_{\mathbf{F}}(f_1, \ldots, f_s) \subseteq \mathbf{F}^n$.

**iv)** There are $f_1, \ldots, f_r \in \mathbb{F}_q[X_1, \ldots, X_n]$ such that $\mathbf{I}_{\mathbf{F}}(\mathbf{V}) = \langle f_1, \ldots, f_r \rangle \trianglelefteq \mathbf{F}[\mathbf{V}]$.
**v)** There is a finitely generated $\mathbb{F}_q$-algebra $A$ such that $\mathbf{F}[\mathbf{V}] \cong A \otimes_{\mathbb{F}_q} \mathbf{F}$.

**c)** Let $\mathbf{V}$ be $\mathbb{F}_q$-closed. Determine the $\mathbb{F}_q$-closed points of $\mathbf{V}$.

**(12.24) Example: Automorphisms.**
Let $K$ be an algebraically closed field. An isomorphism of an affine variety $\mathbf{V} \subseteq K^n$ onto itself, for some $n \in \mathbb{N}_0$, is called an **automorphism** of $\mathbf{V}$. Let $\mathrm{Aut}(\mathbf{V})$ be the group of automorphisms of $\mathbf{V}$.

**a)** Determine $\mathrm{Aut}(K)$.

**b)** For $\varphi\colon K^n \to K^n \colon v \mapsto [f_1(v), \ldots, f_n(v)]$ regular, where $f_i \in K[\mathcal{X}]$, let

$$J(\varphi) := \det\big([\frac{\partial f_i}{\partial X_j}]_{ij}\big) \in K[X_1, \ldots, X_n]$$

be the **Jacobian** of $\varphi$, where $\frac{\partial}{\partial X_j}$ denotes the partial derivative with respect to $X_j$. Show that $J\colon \mathrm{Aut}(K^n) \to K \setminus \{0\} \colon \varphi \mapsto J(\varphi)$ is a group homomorphism.

**(12.25) Exercise: Computing the image of a regular map.**
Let $K \subseteq L$ be a field extension, let $A := K[X_1, \ldots, X_n]$ and $B := K[Y_1, \ldots, Y_m]$, for $n, m \in \mathbb{N}_0$, and $C := A \otimes_K B \cong K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$.

**a)** Show that $\pi\colon L^{n+m} \to L^m \colon [x_1, \ldots, x_n, y_1, \ldots, y_m] \mapsto [y_1, \ldots, y_m]$ is a morphism. Determine the associated comorphism $\pi^*$.

**b)** Let now $\varphi\colon L^n \to L^m \colon v \mapsto [f_1(v), \ldots, f_m(v)]$ be regular, where $f_j \in A$, and let $J := \langle Y_1 - f_1, \ldots, Y_m - f_m \rangle \trianglelefteq C$. Show that

$$\pi(\mathbf{V}_L(J)) = \varphi(L^n) \quad \text{and} \quad \overline{\varphi(L^n)} = \mathbf{V}_L(J \cap B).$$

(Then any Gröbner basis of $J$ with respect to some lexicographical order on monomials fulfilling $\mathcal{X} > \mathcal{Y}$ contains a Gröbner basis of $J \cap B$.)

**c)** Letting $K = \mathbb{Q}$ and $L = \mathbb{C}$, apply this to the twisted cubic $\mathbf{C} := \{[t, t^2, t^3] \in \mathbb{C}^3; t \in \mathbb{C}\}$, and compute $\mathbf{I}_{\mathbb{Q}}(\mathbf{C})$ from the given parametrisation.

**(12.26) Exercise: Graded rings.**
Let $R$ be a **(non-negatively) graded ring**, that is $R = \bigoplus_{d \geq 0} R_d$ as $\mathbb{Z}$-modules, such that $R_i R_j \subseteq R_{i,j}$ for $i, j \geq 0$. Then the following are equivalent:
**i)** $R$ is Noetherian.
**ii)** $R_0$ is Noetherian and $R$ is a finitely generated $R_0$-algebra.
**iii)** $R_0$ is Noetherian and $R_+ := \bigoplus_{d \geq 1} R_d \trianglelefteq R$ is a finitely generated ideal.
**iv)** The subring $R^{(n)} := \bigoplus_{n \mid d} R_d \subseteq R$ is Noetherian for all $n \in \mathbb{N}$.

**(12.27) Exercise: Euler Identity.**
Let $K$ be a field, let $\mathcal{X} := \{X_1, \ldots, X_n\}$, for some $n \in \mathbb{N}_0$, and let $A := K[\mathcal{X}]$.

**a)** Let $0 \neq f \in A$ be homogeneous. Show that $f$ fulfills the **Euler Identity**

$$\sum_{i=1}^{n} X_i \cdot \frac{\partial f}{\partial X_i} = \deg(f) \cdot f.$$

**b)** Assume that $\operatorname{char}(K) = 0$, and let conversely $0 \neq f \in A$ fulfill the Euler Identity. Show that $f$ is homogeneous.

**(12.28) Exercise: Affine cones.**
Let $K \subseteq L$ be a field extension, let $\mathbf{V} \subseteq L^n$ be an irreducible affine variety, for some $n \in \mathbb{N}_0$, and let $\widetilde{\mathbf{V}} := \bigcup_{v \in \mathbf{V}} \langle v \rangle_L$. Show that both $\widetilde{\mathbf{V}} \subseteq L^n$ and its closure $\overline{\widetilde{\mathbf{V}}} \subseteq L^n$ are irreducible cones, and determine the vanishing ideal $\mathbf{I}_K(\widetilde{\mathbf{V}}) = \mathbf{I}_K(\overline{\widetilde{\mathbf{V}}})$.

**Hint.** If $P \lhd A = \bigoplus_{d \geq 0} A_d$ is prime, then so is $\widetilde{P} := \bigoplus_{d \geq 0} (P \cap A_d) \subseteq P \lhd A$.

**(12.29) Exercise: Projective closure.**
Let $L \subseteq \mathbb{C}$, let $V := \mathbf{V}_L(Y - X(X^2 + 1)) \subseteq L^2$, and let $\mathbf{V} := \overline{V} \subseteq \mathbf{P}^2(L)$.

**a)** Show that $\mathbf{V}$ is irreducible, and determine $\mathbf{I}^\sharp(\mathbf{V})$ and the points at infinity of $V$. Determine the affine closed subsets $\mathbf{V} \cap D_X$ and $\mathbf{V} \cap D_Y$ in the '$(y,z)$-' and the '$(x,z)$-plane', respectively, where $Z$ denotes the homogenizing coordinate.

**b)** Now let $L = \mathbb{R}$. Depict $V$, and the affine sets $\mathbf{V} \cap D_X$ and $\mathbf{V} \cap D_Y$. Do the points at infinity of $V$ have distinguishing properties?

**(12.30) Exercise: Projective closure.**
Let $L \subseteq \mathbb{C}$, let $\mathcal{Q} := \mathbf{V}_L(Y - X^2) \subseteq L^2$ be a **parabola**, and let $\mathbf{Q} := \overline{\mathcal{Q}} \subseteq \mathbf{P}^2(L)$ be its projective closure.

**a)** Show that $\mathbf{Q}$ is irreducible, and determine $\mathbf{I}^\sharp(\mathbf{Q})$ and the points at infinity of $\mathcal{Q}$. Determine the affine closed subsets $\mathbf{Q} \cap D_X$ and $\mathbf{Q} \cap D_Y$ in the '$(y,z)$-' and the '$(x,z)$-plane', respectively, where $Z$ denotes the homogenizing coordinate.

**b)** Now let $L = \mathbb{R}$. Depict $\mathcal{Q}$, and the affine sets $\mathbf{Q} \cap D_X$ and $\mathbf{Q} \cap D_Y$. Show that $\mathbf{Q}$ is 'tangent' to the line at infinity $\mathbf{H}_Z := \mathbf{V}^\sharp(Z) \subseteq \mathbf{P}^2(\mathbb{R})$.

**(12.31) Exercise: Classification of binary quadrics.**
**a)** Let $L$ be an algebraically closed field, let $A := L[X, Y]$, let $\mathbf{V} := \mathbf{V}(Y - X^2)$ and $\mathbf{W} := \mathbf{V}(XY - 1)$. Show that $L[\mathbf{V}] \cong L[X]$ and $L[\mathbf{W}] \cong L[X]_X$.

**b)** Let $\operatorname{char}(L) \neq 2$, let $f \in A$ be irreducible of degree 2, and let $\mathbf{Q} := \mathbf{V}_L(f)$ be the associated **binary quadric**. Show that either $L[\mathbf{Q}] \cong L[\mathbf{V}]$ or $L[\mathbf{Q}] \cong L[\mathbf{W}]$, and find a criterion on the coefficients of $f$ to decide which case occurs. Conclude that $\mathbf{Q}$ is isomorphic to either $L$ or $L \setminus \{0\}$.

**c)** Let $A^\sharp := L[X, Y, Z]$. Show that the projective closures $\overline{\mathbf{V}} \subseteq \mathbf{P}^2$ and $\overline{\mathbf{W}} \subseteq \mathbf{P}^2$ are both isomorphic to $\mathbf{V}^\sharp(XZ - Y^2) \subseteq \mathbf{P}^2$.

**(12.32) Exercise: Veronese embeddings.**
Let $K \subseteq L$ be a field extension, where $L$ is infinite, and let $\mathbf{P}^n$ be the projective space over $L$, for some $n \in \mathbb{N}_0$. Let $\{M_0, \ldots, M_N\}$ be the set of monomials of degree $d \in \mathbb{N}_0$ in the indeterminates $\mathcal{X} := \{X_0, \ldots, X_n\}$, where $N := \binom{n+d}{n} - 1$. We consider the $d$-fold **Veronese embedding** or **$d$-uple embedding**

$$\varphi\colon \mathbf{P}^n \to \mathbf{P}^N \colon [x_0\colon \ldots \colon x_n] \mapsto [\ldots \colon M_i(x_0, \ldots, x_n)\colon \ldots].$$

**a)** Show that $\varphi$ is well-defined and injective.
**b)** Show that the image $\varphi(\mathbf{P}^n) \subseteq \mathbf{P}^N$ is closed and irreducible.
**c)** Show that $\varphi(\mathbf{P}^n)$ is homeomorphic via $\varphi$ to $\mathbf{P}^n$.

**Hint.** Consider the kernel of the homomorphism of $K$-algebras

$$K[Y_0, \ldots, Y_N] \to K[X_0, \ldots, X_n] \colon Y_i \mapsto M_i.$$

**(12.33) Exercise: Segre embedding.**
Let $K \subseteq L$ be a field extension, where $L$ is infinite, and let $\mathbf{P}^n$ and $\mathbf{P}^m$ be projective spaces over $L$, for some $n, m \in \mathbb{N}_0$. Letting $s := (n+1)(m+1) - 1$, we consider the **Segre embedding**

$$\sigma\colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^s \colon [[x_0\colon \ldots \colon x_n], [y_0\colon \ldots \colon y_m]] \mapsto [\ldots \colon x_i y_j \colon \ldots].$$

Moreover, letting $R := K[Z_{ij}; i \in \{0, \ldots, n\}, j \in \{0, \ldots, m\}]$, we consider the homomorphism of $K$-algebras $\varphi\colon R \to K[X_0, \ldots, X_n, Y_0, \ldots, Y_m] \colon Z_{ij} \mapsto X_i Y_j$.

Show that $\ker(\varphi) = \langle Z_{ij} Z_{kl} - Z_{il} Z_{kj}; i, k \in \{0, \ldots, n\}, j, l \in \{0, \ldots, m\} \rangle \trianglelefteq R$ is a homogeneous prime ideal, such that $\sigma(\mathbf{P}^n \times \mathbf{P}^m) = \mathbf{V}_L^\sharp(\ker(\varphi)) \subseteq \mathbf{P}^s$.

**(12.34) Exercise: Quadric surface.**
Let $K \subseteq L$ be a field extension, where $L$ is infinite, and let $A^\sharp = K[W, X, Y, Z]$. We consider the projective **quadric surface** $\mathbf{V} := \mathbf{V}_L^\sharp(XY - ZW) \subseteq \mathbf{P}^3$.

**a)** Determine $\mathbf{I}_K^\sharp(\mathbf{V}) \trianglelefteq A^\sharp$, and show that $\mathbf{V}$ coincides with the image of the Segre embedding $\varphi\colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$, for a suitable choice of coordinates.

**b)** Show that $\mathbf{V}$ contains two sets of lines $\{\mathbf{L}_i; i \in \mathcal{I}\}$ and $\{\mathbf{M}_j; j \in \mathcal{J}\}$, each parameterized by $\mathbf{P}^1$, where $\mathcal{I}$ and $\mathcal{J}$ are index sets, such that $\mathbf{L}_i \cap \mathbf{L}_k = \emptyset$ whenever $i \neq k$, and $\mathbf{M}_j \cap \mathbf{L}_l = \emptyset$ whenever $j \neq l$, while $\mathbf{L}_i \cap \mathbf{M}_j$ is a singleton set, for all $i$ and $j$. Here, a **line** in $\mathbf{P}^3$ is an intersection of two distinct hyperplanes.

**c)** Show that $\mathbf{V}$ contains further space curves, distinct from the above lines. Deduce that the Zariski topology on $\mathbf{V}$ is not homeomorphic via $\varphi$ to the product topology of the Zariski topologies on $\mathbf{P}^1 \times \mathbf{P}^1$.

## 13 Exercises for Part II

**(13.1) Exercise: Functor categories.**
**a)** Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be categories, and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{E}$ be (covariant) functors. Show that assigning $A \mapsto \mathcal{G}(\mathcal{F}(A))$, for $A \in \mathcal{C}$, and $\alpha \mapsto$

$\big(\mathcal{G}(\mathcal{F}(\alpha))\colon \mathcal{G}(\mathcal{F}(A)) \to \mathcal{G}(\mathcal{F}(B))\big)$, for $\alpha\colon A \to B$ in $\mathcal{C}$, defines a (covariant) functor $\mathcal{G}\circ\mathcal{F}\colon \mathcal{C} \to \mathcal{E}$, the **concatenation** of $\mathcal{G}$ and $\mathcal{F}$. How is the concatenation of contravariant functors, or of covariant and contravariant functors defined?

**b)** Show that the collection of covariant functors, together with all natural transformations as morphisms, formally fulfills the requirements of a category. (Ignoring set-theoretic issues, this might be called the **category of functors**.)

**(13.2) Exercise: Duality.**
Let $K$ be a field, and let $K$**-vec** be the category of finite-dimensional $K$-vector spaces, having object set $\mathrm{Ob}(K\textbf{-vec}) := \{K^n; n \in \mathbb{N}_0\}$, and having the set $\mathrm{Hom}_K(?, ??)$ of all $K$-linear maps as morphisms.

**a)** Show that $K$**-vec** is a category indeed. Moreover, show that it is **skeletal**, that is its objects are pairwise non-isomorphic.

**b)** Show that assigning $V \mapsto V^* := \mathrm{Hom}_K(V, K)$ and

$$\mathrm{Hom}_K(V, W) \to \mathrm{Hom}_K(W^*, V^*)\colon \varphi \mapsto (\varphi^*\colon \lambda \mapsto \lambda \circ \varphi)$$

defines a contravariant **duality** (**endo-**)functor $^*\colon K\textbf{-vec} \to K\textbf{-vec}$.

**c)** Show that the **biduality functor** $^{**}$ is isomorphic to the identity functor.

**(13.3) Exercise: Limits.**
**a)** Let $\mathcal{I}$ be a partially ordered set, let $\mathcal{C}$ be a category, let $\{A_i\}$ be an inverse system in $\mathcal{C}$, with respect to $\psi_{ji}\colon A_j \to A_i$ for $j \geq i$, and let $P$ together with morphisms $\{\pi_i\}$ be a limit. Show that $P$ is unique up to isomorphism, and that $\{\pi_i\}$ induces an equivalence of contravariant functors $\mathrm{Hom}_{\mathcal{C}}(?, P) \cong \mathrm{Hom}_{\mathcal{C}}(?, \{A_i\})$.

**b)** Now let $\mathcal{C}$ be the category of sets, and let $\mathcal{P} := \prod_{i\in\mathcal{I}} A_i$ be the Cartesian product, with projection maps $\pi_i$. Show that $P := \{x \in \mathcal{P}; \pi_i(x) = \psi_{ji}(\pi_j(x)) \text{ for } j \geq i\}$ is a limit in $\mathcal{C}$. What happens if $\mathcal{I}$ carries the trivial partial order, or if $\mathcal{I} = \emptyset$?

**c)** Let additionally $\mathcal{I} := \{0, 1, 2\}$ with non-trivial relations $1 \geq 0$ and $2 \geq 0$. Show that the pullback $A_1 \times_{A_0} A_2 := \{[x_1, x_2] \in A_1 \times A_2; \psi_1(x_1) = \psi_2(x_2)\}$ is a limit, with respect to the maps $\psi_i\colon A_i \to A_0$. Moreover, show that if both $\psi_i$ are surjective, then both $\pi_i\colon A_1 \times_{A_0} A_2 \to A_i$ are surjective as well.

**d)** Let $\mathcal{I} := \mathbb{N}$, partially ordered by divisibility, let $\psi_{mn}\colon \mathbb{Z}/\langle m\rangle \to \mathbb{Z}/\langle n\rangle$ be the natural maps, for $n \mid m$, and let $\widehat{\mathbb{Z}} := \varprojlim\{\mathbb{Z}/\langle n\rangle\}$ in the category of sets. Show that the profinite completion $\widehat{\mathbb{Z}}$ is a limit in the category of rings.

Let $p \in \mathbb{Z}$ be a prime, let $\mathcal{I}_p := \{p^n \in \mathbb{N}; n \in \mathbb{N}_0\}$, and let $\widehat{\mathbb{Z}}_p := \varprojlim\{\mathbb{Z}/\langle p^n\rangle\}$ in the category of sets. Show that the $p$-adic completion $\widehat{\mathbb{Z}}_p$ is a limit in the category of rings. Moreover, show that $\widehat{\mathbb{Z}}_p$ is a domain, having $\langle p\rangle$ as its unique maximal ideal. Finally, show that $\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \widehat{\mathbb{Z}}_p$ as rings.

**(13.4) Exercise: Colimits.**
**a)** Let $\mathcal{I}$ be a partially ordered set, let $\mathcal{C}$ be a category, let $\{A_i\}$ be a direct system in $\mathcal{C}$, with respect to $\psi_{ij}\colon A_i \to A_j$ for $i \leq j$, and let $I$ together with morphisms $\{\iota_i\}$ be a direct limit. Show that $I$ is unique up to isomorphism, and that $\{\iota_i\}$ induces an equivalence of (covariant) functors $\mathrm{Hom}_{\mathcal{C}}(I, ?) \cong \mathrm{Hom}_{\mathcal{C}}(\{A_i\}, ?)$.

**b)** Now let $\mathcal{C}$ be the category of sets, and let $\mathcal{Q} := \coprod_{i\in\mathcal{I}} A_i$ be the disjoint union, with inclusion maps $\iota_i$. Show that there is an equivalence relation $\sim$ on $\mathcal{Q}$, such that $I := \mathcal{Q}/_\sim$, together with the induced maps $\{\iota_i\}$, is a colimit in $\mathcal{C}$. What happens if $\mathcal{I}$ carries the trivial partial order, or if $\mathcal{I} = \emptyset$?

**c)** Let additionally $\mathcal{I} := \{0, 1, 2\}$ with non-trivial relations $0 \leq 1$ and $0 \leq 2$. Show that the pushout $A_1 \sqcup_{A_0} A_2 := (A_1 \sqcup A_2)/_\sim$, where $\sim$ is the relation generated by reflexive closure from $\iota_1(\psi_1(x)) \sim \iota_2(\psi_2(x))$, for $x \in A_0$, is a colimit, with respect to the maps $\psi_i\colon A_0 \to A_i$. Moreover, show that if both $\psi_i$ are injective, then both $\pi_i\colon A_1 \times_{A_0} A_2 \to A_i$ are injective as well.

**d)** We consider the category of $K$-vector spaces, where $K$ is a field, let $V_i$ be $K$-vector spaces, for $i \in \{0, 1, 2\}$, and let $\psi_i\colon V_0 \to V_i$ be $K$-linear, for $i \in \{1, 2\}$. If $V_0 = \{0\}$, show that the direct sum $V_1 \oplus V_2$ of $K$-vector spaces is a fibre sum. Show that a fibre sum $V_1 \oplus_{V_0} V_2$ exists in general, and give an explicit description.

We consider the category of $K$-algebras, let $A_i$ be $K$-algebras, and let $\psi_i\colon K \to A_i$ be the structure homomorphisms, for $i \in \{1, 2\}$. Show that the tensor product $A_1 \otimes_K A_2$ of $K$-algebra is a fibre sum. What is its structure homomorphism?

**(13.5) Exercise: Locally constant sheaves.**
Let $V$ be a topological space.

**a)** Given a presheaf $\mathcal{F}$ on $V$, show that the following assertions are equivalent:

**i)** For any open $\emptyset \neq U \subseteq V$ the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is bijective.

**ii)** $\mathcal{F}$ is a **constant sheaf**, that is there is a set $A$, equipped with the discrete topology, such that $\mathcal{F}(U)$ consists of the locally constant functions $U \to A$.

**iii)** $\mathcal{F}$ is a **locally constant sheaf**, that is any point $v \in V$ has an open neighborhood $U \subseteq V$ such that the restriction $\mathcal{F}|_U$ is a constant sheaf.

**b)** Let $V$ be connected, and let $\mathcal{F}$ be a locally constant sheaf on $V$ such that $|\mathcal{F}(V)| \geq 2$. Show that $V$ is irreducible.

**c)** Let $\mathcal{F}$ be a locally constant sheaf on $V$, and let $\mathcal{F}_0$ be the **constant presheaf** on $V$ with value $A := \mathcal{F}(V)$, that is $\mathcal{F}(U)$ consists of the constant functions $U \to A$, for any $U \subseteq V$ open. Show that $\mathcal{F}$ is the sheafification of $\mathcal{F}_0$.

Moreover, show that the natural morphism of presheaves $\psi\colon \mathcal{F}_0 \Rightarrow \mathcal{F}$ has the following universal property: For any morphism $\varphi_0\colon \mathcal{F}_0 \Rightarrow \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there is a unique morphism $\varphi\colon \mathcal{F} \Rightarrow \mathcal{G}$ such that $\varphi_0 = \psi \cdot \varphi$.

**(13.6) Exercise: Gluing sheaves.**
Let $V$ be a topological space, and let $\{V_i \subseteq V; i \in \mathcal{I}\}$ be an open covering, where $\mathcal{I}$ is an index set. For all $i \in \mathcal{I}$ let $\mathcal{F}_i$ be a sheaf on $V_i$, with values in a fixed category, such that there are isomorphisms $\varphi_{ij} \colon (\mathcal{F}_i)|_{V_i \cap V_j} \Rightarrow (\mathcal{F}_j)|_{V_i \cap V_j}$ fulfilling $\varphi_{ij}\varphi_{jk} = \varphi_{ik} \colon (\mathcal{F}_i)|_{V_i \cap V_j \cap V_k} \Rightarrow (\mathcal{F}_k)|_{V_i \cap V_j \cap V_k}$, for all $i, j, k \in \mathcal{I}$.

**a)** Show that we have $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i} \colon \mathcal{F}_i \Rightarrow \mathcal{F}_i$, for all $i \in \mathcal{I}$.

**b)** Show that there is a sheaf $\mathcal{F}$ on $V$, together with isomorphisms $\psi_i \colon \mathcal{F}_i \Rightarrow \mathcal{F}|_{V_i}$, such that $\psi_i = \varphi_{ij}\psi_j \colon (\mathcal{F}_i)|_{V_i \cap V_j} \Rightarrow \mathcal{F}|_{V_i \cap V_j}$, for all $i, j \in \mathcal{I}$. Moreover, show that $\mathcal{F}$ and the $\psi_i$ are uniquely defined up to unique isomorphism.

**(13.7) Exercise: Direct image sheaves.**
**a)** Let $V$ be a topological space, and let $\mathcal{A}$ be a category. Show that $\mathbf{PSh}(V, \mathcal{A})$, with objects consisting of the presheaves on $V$ with values in $\mathcal{A}$, and having all morphism of presheaves as morphisms, is a category. Similarly, show that $\mathbf{Sh}(V, \mathcal{A})$, with objects consisting of the sheaves on $V$ with values in $\mathcal{A}$, and having all morphism of presheaves as morphisms, is a category.

**b)** Let $W$ be a topological space, let $\varphi \colon V \to W$ be continuous, and let $\mathcal{F} \in \mathbf{PSh}(V, \mathcal{A})$. Show that for any $U' \subseteq U \subseteq W$ open letting

$$U \mapsto \mathcal{F}(\varphi^{-1}(U)) \quad \text{and} \quad \rho^U_{U' \subseteq W} := \rho^{\varphi^{-1}(U)}_{\varphi^{-1}(U') \subseteq V}$$

defines a presheaf $\varphi_*\mathcal{F} \in \mathbf{PSh}(W, \mathcal{A})$, being called the **direct image** of $\mathcal{F}$ under $\varphi$. Show that this gives rise to a functor $\varphi_* \colon \mathbf{PSh}(V, \mathcal{A}) \to \mathbf{PSh}(W, \mathcal{A})$.

Moreover, if $\mathcal{F}$ is a sheaf, show that $\varphi_*\mathcal{F}$ is a sheaf as well. Conclude that this gives rise to a functor $\varphi_* \colon \mathbf{Sh}(V, \mathcal{A}) \to \mathbf{Sh}(W, \mathcal{A})$.

**(13.8) Exercise: Localization.**
Let $R$ be a ring, let $\mathcal{S} \subseteq R$ be multiplicatively closed, and let $M$ be an $R$-module.

**a)** Show that a localization $M_\mathcal{S}$ of $M$ at $\mathcal{S}$ is unique up to isomorphism.

**b)** Let $\sim$ be the relation on $M \times \mathcal{S}$ given by letting $[m, f] \sim [m', f']$ if there is $g \in \mathcal{S}$ such that $(mf' - m'f)g = 0$. Show that $\sim$ is an equivalence relation.

**c)** Show that the set of equivalence classes $M/\mathcal{S} := (M \times \mathcal{S})/\sim$ becomes an $R$-module, such that there is a natural $R$-module homomorphism $\sigma \colon M \to M/\mathcal{S}$. Moreover, show that $M/\mathcal{S}$, together with $\sigma$, is a localization of $M$ at $\mathcal{S}$ indeed.

**d)** Show that the localization $R_\mathcal{S}$ of $R$ at $\mathcal{S}$ becomes a ring again, and derive the universal property of $R_\mathcal{S}$ in the category of rings from its universal property in the category of $R$-modules.

**e)** Show that $\frac{f}{1} \in R_\mathcal{S}$, where $f \in \mathcal{S}$, acts bijectively on any $R_\mathcal{S}$-module. Conversely, if any $f \in \mathcal{S}$ acts bijectively on $M$, show that $M$ becomes an $R_\mathcal{S}$-module.

**(13.9) Exercise: Localization functors.**
Let $R$ be a ring, and let $\mathcal{S} \subseteq R$ be multiplicatively closed.

**a)** Let $\alpha\colon M \to N$ be a homomorphism of $R$-modules. Show that there is a unique homomorphism $\alpha_{\mathcal{S}}\colon M_{\mathcal{S}} \to N_{\mathcal{S}}$ of $R_{\mathcal{S}}$-modules, called the **localization** of $\alpha$ at $\mathcal{S}$, such that $\alpha \cdot \sigma_N = \sigma_M \cdot \alpha_{\mathcal{S}}$, where $\sigma_?$ denotes the natural map.

**b)** Show that localization at $\mathcal{S}$ induces covariant functors $?_{\mathcal{S}}\colon \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}R_{\mathcal{S}}$ and $?_{\mathcal{S}}\colon \mathsf{mod}\text{-}R \to \mathsf{mod}\text{-}R_{\mathcal{S}}$, where $\mathsf{Mod}$-? and $\mathsf{mod}$-? denotes the category of all modules and of finitely generated modules, respectively. Is the map $?_{\mathcal{S}}\colon \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{R_{\mathcal{S}}}(M_{R_{\mathcal{S}}}, N_{R_{\mathcal{S}}})$ injective? Is it surjective?

**c)** Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} P$ be an **exact sequence** of $R$-modules, that is we have $\mathrm{im}(\alpha) = \ker(\beta)$. Show that $M_{\mathcal{S}} \xrightarrow{\alpha_{\mathcal{S}}} N_{\mathcal{S}} \xrightarrow{\beta_{\mathcal{S}}} P_{\mathcal{S}}$ is an exact sequence of $R_{\mathcal{S}}$-modules; in other words, $?_{\mathcal{S}}$ is an **exact functor**. In particular, conclude that $\alpha_{\mathcal{S}}$ is injective if $\alpha$ is so, that $\alpha_{\mathcal{S}}$ is surjective if $\alpha$ is so, and that $M_{\mathcal{S}}/N_{\mathcal{S}} \cong (M/N)_{\mathcal{S}}$ as $R_{\mathcal{S}}$-modules.

**d)** If $M' \leq M$ and $M'' \leq M$ are $R$-submodules, show that $(M' \cap M'')_{\mathcal{S}} = (M')_{\mathcal{S}} \cap (M'')_{\mathcal{S}} \subseteq M_{\mathcal{S}}$. Similarly, if $\{M_i \leq M; i \in \mathcal{I}\}$ are $R$-submodules, where $\mathcal{I}$ is an index set, show that $(\sum_{i\in\mathcal{I}} M_i)_{\mathcal{S}} = \sum_{i\in\mathcal{I}}(M_i)_{\mathcal{S}} \leq M_{\mathcal{S}}$. In which sense can the various localized modules be considered as submodules of $M_{\mathcal{S}}$?

**(13.10) Exercise: Varying the denominator set.**
Let $R$ be a ring, let $\mathcal{S}, \mathcal{T} \subseteq R$ be multiplicatively closed subsets, let $\mathcal{S}' := \sigma_{\mathcal{T}}(\mathcal{S}) \subseteq R_{\mathcal{T}}$ and $\mathcal{T}' := \sigma_{\mathcal{S}}(\mathcal{T}) \subseteq R_{\mathcal{S}}$, and let $M$ be an $R$-module.

**a)** Show that there are a natural ring homomorphism $\rho\colon R_{\mathcal{T}} \to (R_{\mathcal{S}})_{\mathcal{T}'}$, and a natural homomorphism of $R_{\mathcal{T}}$-modules $\tau\colon M_{\mathcal{T}} \to (M_{\mathcal{S}})_{\mathcal{T}'}$. In which sense does $(M_{\mathcal{S}})_{\mathcal{T}'}$ become an $R_{\mathcal{T}}$-module?

**b)** Show that if $\mathcal{S}'$ consists of units in $R_{\mathcal{T}}$, then $\rho$ and $\tau$ are an isomorphism of rings and $R_{\mathcal{T}}$-modules, respectively. What happens in the case $\mathcal{S} \subseteq \mathcal{T}$? What happens if $\mathcal{T}'$ consists of units in $R_{\mathcal{S}}$ as well?

**c)** Let $f, g \in R$, and let $R_f$ denote the localization of $R$ at $\{f^k \in R; k \in \mathbb{N}_0\}$. Show that there are a natural ring isomorphism $(R_f)_g \cong R_{fg}$, and a natural isomorphism of $R_{fg}$-modules $(M_f)_g \cong M_{fg}$.

**(13.11) Exercise: Quasi-affine varieties.**
Show that the affine line is not isomorphic to any proper open subset of itself.

**(13.12) Exercise: Projective affine varieties.**
Show that any irreducible affine variety which is projective is a singleton set.

**(13.13) Exercise: Rational functions.**
Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed, and let $V$ be an irreducible variety.

**a)** Show that any affine open subset $\emptyset \neq U \subseteq V$ is irreducible, and that $K(U)$ is independent of the choice of $U$. Hence $K(U)$ is (unambiguously) called the field of **rational functions** on $V$; we again denote it by $K(V)$. Describe $K(V)$ if $V$ is quasi-affine.

**b)** Show that for any open subset $U \subseteq V$ the $K$-algebra $\mathcal{O}_V(U)$ can be identified with a $K$-subalgebra of $K(V)$, such that $\mathrm{Q}(\mathcal{O}_V(U)) = K(V)$. If $U' \subseteq U$ is open, describe the associated restriction map in terms of $K$-subalgebras of $K(V)$. If $U \subseteq V$ and $U' \subseteq V$ are open, show that $\mathcal{O}_V(U \cup U') = \mathcal{O}_V(U) \cap \mathcal{O}_V(U') \subseteq K(V)$.

**c)** Let $\varphi\colon V \to W$ be a morphism of irreducible varieties such that $\varphi(V)$ contains an open subset of $W$. (In particular $\varphi$ is dominant.) Show that $\varphi^*$ induces a field extension $K(W) \subseteq K(V)$. Can the assumption on $\varphi(V)$ be dispensed of?

**(13.14) Exercise: Rational functions on projective varieties.**
Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed, let $\mathbf{V} \subseteq \mathbf{P}$ be an irreducible projective variety, having (homogeneous prime) vanishing ideal $I \trianglelefteq A^\sharp$, and let $R := (A^\sharp_{A^\sharp \setminus I})_0 = \{\frac{f}{g} \in Q^\sharp(A^\sharp); g \notin I\} \subseteq Q^\sharp(A^\sharp)$ be the associated graded localization of $A^\sharp$. Show that $IR \trianglelefteq R$ is the unique maximal homogeneous ideal of $R$, and that we have $K(\mathbf{V}) \cong R/IR$.

**(13.15) Exercise: Morphisms of quasi-projective varieties.**
Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed, and let $V \subseteq \mathbf{P}^n$ be a quasi-projective variety.

**a)** Let $f_0, \ldots, f_m \in A^\sharp$, where $m \in \mathbb{N}_0$, be homogeneous of the same degree, such that $\mathbf{V}_L(f_0, \ldots, f_m) \cap V = \emptyset$. Show that $\varphi\colon V \to \mathbf{P}^m\colon v \mapsto [f_0(v)\colon \ldots \colon f_m(v)]$ is a morphism of varieties. Moreover, if $g_0, \ldots, g_m \in A^\sharp$ also fulfill the above properties, show that $V \to \mathbf{P}^m\colon v \mapsto [g_0(v)\colon \ldots \colon g_m(v)]$ equals $\varphi$ if and only if $f_i g_j|_V = f_j g_i|_V$, for all $i, j \in \{0, \ldots, m\}$.

**b)** Let $\psi\colon V \to \mathbf{P}^m$ be a morphism of varieties. Show that for any $v \in V$ there is an open neighborhood $v \in U \subseteq V$ such that $\psi|_U$ is of the above form.

**(13.16) Exercise: Veronese embeddings again.**
Let $K \subseteq L$ be a field extension, where $L$ algebraically closed, let $\varphi\colon \mathbf{P}^n \to \mathbf{P}^N$ be the $d$-fold Veronese embedding, where $N := \binom{n+d}{n} - 1$; see Exercise (12.32).
**a)** Show that $\varphi$ induces an isomorphism of varieties $\mathbf{P}^n \cong \varphi(\mathbf{P}^n)$.
**b)** Let $\varphi(\mathbf{P}^1) \subseteq \mathbf{P}^2$ be the image of the 2-fold Veronese embedding of $\mathbf{P}^1$. Show that for the homogeneous coordinate algebras we have $K[\mathbf{P}^1] \not\cong K[\varphi(\mathbf{P}^1)]$.

**(13.17) Exercise: Morphisms of projective varieties.**
Let $L$ be an algebraically closed field, and let $p \in \mathbf{P}^2$.
**a)** Show that the set of lines in $\mathbf{P}^2$ through $p$ can be identified with $\mathbf{P}^1$.
**b)** We consider the **conic** $\mathbf{V} := \mathbf{V}^\sharp(XZ - Y^2) \subseteq \mathbf{P}^2$, where $L[X, Y, Z]$ is the homogeneous coordinate algebra of $\mathbf{P}^2$. Show that $\mathbf{V}$ is irreducible
**c)** Let $p := [0\colon 0\colon 1] \in \mathbf{V}$. Show that any line in $\mathbf{P}^2$ through $p$ intersects $\mathbf{V}$ in precisely two points, where for the 'tangent' at $\mathbf{V}$ the point $p$ is counted twice.
**d)** Show that this yields an isomorphism of projective varieties $\mathbf{P}^1 \to \mathbf{V}$.

**(13.18) Exercise: Morphisms of projective varieties.**
Let $L$ be an algebraically closed field such that $\text{char}(L) \neq 2$, and let $\mathbf{P}$ be the projective plane having homogeneous coordinate algebra $L[X, Y, Z]$. We consider the curve $\mathbf{V} := \mathbf{V}(f) \subseteq \mathbf{P}$ of degree 3 given by $f := ZY^2 - X(X^2 - Z^2)$.

**a)** Let $V_Z := \mathbf{V} \cap D_Z$, $V_Y := \mathbf{V} \cap D_Y$, $V_X := \mathbf{V} \cap D_X$, and let $p_0 := [0\colon 0\colon 1]$, $p_\infty := [0\colon 1\colon 0]$, $p_1 := [1\colon 0\colon 1]$, and $p_{-1} := [(-1)\colon 0\colon 1]$. Show that $V_Z = \mathbf{V} \setminus \{p_\infty\}$, $V_Y = \mathbf{V} \setminus \{p_0, p_1, p_{-1}\}$, and $V_X = \mathbf{V} \setminus \{p_0, p_\infty\}$ are affine open.

**b)** Given any point $p_0 \neq p := [x\colon y\colon 1] \in V_Z$, show that the line in $D_Z$ through $p$ and $p_0$ intersects $V_Z$ in $p' := [\frac{-1}{x}\colon \frac{-y}{x^2}\colon 1] = [x\colon y\colon (-x^2)]$.

We aim at showing that the map $\varphi$ given by $p \mapsto p'$, and interchanging $p_0$ and $p_\infty$, defines an involutory automorphism of $\mathbf{V}$. Convince yourself that to do so it suffices to show that $\varphi$ is a morphism. To this end, let $V' := \mathbf{V} \setminus \{p_\infty, p_1, p_{-1}\}$.

**c)** Show that $\mathbf{V} = V' \cup V_X \cup V_Y = V_Z \cup V_Y$, and that $\varphi(V_X \cup V_Y) \subseteq V_Z$ and $\varphi(V') \subseteq V_Y$. Moreover, show that $\varphi|_{V_X}\colon V_X \to V_Z$, $\varphi|_{V_Y}\colon V_Y \to V_Z$, and $\varphi|_{V'}\colon V' \to V_Y$ are morphisms. Conclude that $\varphi$ is a morphism.

**Hint for c).** Show that $\frac{1}{X^2-1}$ is regular on $V'$.

**(13.19) Exercise: Graphs of morphisms.**
Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed, let $U$ and $V$ be prevarieties, and let $W \subseteq U \times V$ be a subprevariety. Show that $W$ is the graph of a morphism $\varphi\colon U \to V$, if and only if the restriction $\pi_U|_W\colon W \to U$ is an isomorphism. In this case, show that $\varphi = (p_U|_W)^{-1} \cdot \pi_V$.

**(13.20) Exercise: Affine open coverings.**
Let $K \subseteq L$ be a field extension, where $L$ is algebraically closed, and let $V$ be a prevariety. Assume that $V$ has an affine open covering $\{V_i; i \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set, such that $V_i \cap V_j$ is affine open having coordinate algebra $K[V_i \cap V_j] = K[V_i] \cdot K[V_j]$, for all $i, j \in \mathcal{I}$. Show that $V$ is a variety.

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