On the multiplicity-free actions of the sporadic simple groups

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Abstract

We introduce a database containing the character tables of the endomorphism rings of the multiplicity-free permutation modules of the sporadic simple groups, their automorphism groups, their Schur covers, and their bicyclic extensions. We describe the techniques used to compile the data, and present a couple of applications to orbital graphs.
MSC: 20C34, 20C40, 20B25, 20B40, 20D08, 05C25.

1 Introduction

The purpose of the present paper is to introduce the database [5] containing the character tables of the endomorphism rings of the multiplicity-free permutation modules of the sporadic simple groups, their automorphism groups, their Schur covers, and their bicyclic extensions, thereby covering all almost quasi-simple groups related to the sporadic simple groups. The database is planned to become a contributed package of the computer algebra system GAP [11].

The subject has a certain history: The multiplicity-free actions of the sporadic simple groups and their automorphism groups have been classified in [4], their Schur covers have been considered in [21], and independently including their bicyclic extensions in [2]. The work of systematically computing collapsed adjacency matrices for the sporadic simple groups and their automorphism groups has been begun in [32], by a consideration of the necessarily multiplicity-free actions up to rank 5. In [14] these and other earlier results scattered in the literature have been collected, and therefrom the character tables associated to the multiplicity-free actions of degree $\leq 10^7$ have been determined. The remaining cases for the sporadic simple groups and their automorphism groups, having degree up to $\sim 10^{15}$, have been dealt with in [25], requiring considerable computational efforts. Finally, we have now been able to deal with the Schur covers and the bicyclic extension of the sporadic simple groups as well, completing the programme laid out above, up to a single open case, see (3.3).

Besides their own importance in the representation theory of finite groups, another reason to look at endomorphism rings of permutation modules is their connection to orbital graphs, relating group theory to notions of algebraic graph theory, such as distance-transitivity, see e. g. [16]. Actually, this has been the original motivation to compile the database, to have easy and complete access to the relevant data for the sporadic simple groups and their extensions.

The present paper is organised as follows: In Section 2 we recall the necessary facts about permutation modules, in Section 3 we explain how the database is organised, and how to actually access the data through GAP, in Section 4 we describe the techniques used to compile the data, and in Section 5 we indicate a couple of applications to orbital graphs.

Acknowledgements. I am grateful to I. Höhler and S. Linton for providing lots of explicit results, and to T. Breuer and M. Neunhöffer for valuable computational advice and help.

2 Theoretical background

We recall the necessary facts about permutation modules and their endomorphism rings, thereby fixing the notation used throughout; as general references see e. g. [25, 37, 1].

(2.1) Let G be a finite group, let $H \leq G$ and let $n := [G\colon H]$. Let $\Omega \neq \emptyset$ be a transitive G-set such that $\mathrm{Stab}_G(\omega_1) = H$, for some $\omega_1 \in \Omega$. Moreover, let $\Omega = \coprod_{i=1}^{r} \Omega_i$, be the decomposition of Ω into H-orbits, called the associated **suborbits**. The numbers $n \in \mathbb{N}$ and $r \in \mathbb{N}$ are called the **degree** and the **rank** of Ω, respectively. For $i \in \{1, \ldots, r\}$ we choose $\omega_i \in \Omega_i$ and $g_i \in G$ such that $\omega_1 g_i = \omega_i$, where we assume $g_1 = 1$ and $\Omega_1 = \{\omega_1\}$, and we let $H_i := \mathrm{Stab}_H(\omega_i) \leq H$ be the **stabiliser** of ω_i in H, and $k_i := |\Omega_i| = |H|/|H_i|$ be the associated **subdegrees**.

For $i \in \{1, \ldots, r\}$ the orbits $\mathcal{O}_i := [\omega_1, \omega_i] \cdot G \subseteq \Omega \times \Omega$ of the diagonal action of G on $\Omega \times \Omega$ are called the associated **orbitals**, hence $|\mathcal{O}_i| = |G|/|H_i| = nk_i$. For $i \in \{1, \ldots, r\}$ let $i^* \in \{1, \ldots, r\}$ be defined by $\mathcal{O}_{i^*} = [\omega_i, \omega_1] \cdot G \subseteq \Omega \times \Omega$. Then Ω_{i^*} is called the suborbit **paired** to Ω_i; note that $k_{i^*} = k_i$. For $i \in \{1, \ldots, r\}$ let $A_i = [a_{\omega,\omega'}] \in \{0, 1\}^{n \times n}$, with row index $\omega \in \Omega$ and column index $\omega' \in \Omega$, be defined by $a_{\omega,\omega'} := 1$ if $[\omega, \omega'] \in \mathcal{O}_i$, and $a_{\omega,\omega'} := 0$ otherwise. Hence in particular A_1 is the identity matrix, and we have $A_i^{\mathrm{tr}} = A_{i^*}$.

For a subset $\mathcal{I} \subseteq \{2, \ldots, r\}$ being closed under taking paired suborbits, i. e. for all $i \in \mathcal{I}$ we also have $i^* \in \mathcal{I}$, let the **generalised orbital graph** $\mathfrak{G}_{\mathcal{I}}$ be the simple undirected graph having vertex set Ω and adjacency matrix $A_{\mathcal{I}} := \sum_{i \in \mathcal{I}} A_i$; if $\mathcal{I} = \{i\}$ or $\mathcal{I} = \{i, i^*\}$ then $\mathfrak{G}_{\mathcal{I}}$ is called an **orbital graph**. Hence $\mathfrak{G}_{\mathcal{I}}$ is a regular graph with valency $\sum_{i \in \mathcal{I}} k_i$.

(2.2) Let $\mathbb{Z}\Omega$ be the permutation $\mathbb{Z}G$-module associated to Ω, and let $E := \mathrm{End}_{\mathbb{Z}G}(\mathbb{Z}\Omega)$ be its endomorphism ring, i. e. the set of all $\mathbb{Z}$-linear maps $\mathbb{Z}\Omega \to \mathbb{Z}\Omega$ commuting with the action of G. Via the $\mathbb{Z}$-basis Ω of $\mathbb{Z}\Omega$, the matrices $A_i \in \mathbb{Z}^{n \times n}$ can be interpreted as $\mathbb{Z}$-linear maps, and by Schur's Theorem [34], see also [18, Ch.II.12], the set $\mathcal{A} := \{A_i; i \in \{1, \ldots, r\}\}$ is indeed a $\mathbb{Z}$-basis of E, called its **Schur basis**. Moreover, $\mathcal{A}$ also is a $\mathbb{C}$-basis of $E_{\mathbb{C}} := E \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathrm{End}_{\mathbb{C}G}(\mathbb{C}\Omega)$, which is a split semisimple $\mathbb{C}$-algebra.

For $i \in \{1, \ldots, r\}$ let $P^{(i)} = [p_{h,j}^{(i)}] \in \mathbb{Z}^{r \times r}$, with row index $h \in \{1, \ldots, r\}$

and column index $j \in \{1, \ldots, r\}$, be the representing matrix of A_i for its right regular action on E, with respect to the Schur basis $\mathcal{A}$, i. e. we have $A_h A_i = \sum_{j=1}^{r} p_{h,j}^{(i)} A_j$. Hence the map $E \to \mathbb{Z}^{r \times r} : A_i \mapsto P^{(i)}$, for $i \in \{1, \ldots, r\}$, is a faithful $\mathbb{Z}$-representation of E. The matrices $P^{(i)}$ are called **collapsed adjacency matrices** or **intersection matrices**.

The matrix entries $p_{h,j}^{(i)} \in \mathbb{Z}$ are called **intersection numbers** and are given as $p_{h,j}^{(i)} = |\Omega_h \cap \Omega_{i^*} g_j| \in \mathbb{N}_0$. From $k_j \cdot |\Omega_h \cap \Omega_{i^*} g_j| = k_h \cdot |\Omega_j \cap \Omega_i g_h|$, for $h, i, j \in \{1, \ldots, r\}$, we get the identity $p_{h,j}^{(i)} k_j = |\Omega_j \cap \Omega_i g_h| \cdot k_h = p_{j,h}^{(i^*)} k_h$. Moreover, the first row and the first column of $P^{(i)}$ are given as $p_{1,j}^{(i)} = \delta_{i,j}$ and $p_{h,1}^{(i)} = k_h \cdot \delta_{h,i^*}$, respectively, where $\delta_{\cdot,\cdot} \in \{0, 1\}$ denotes the Kronecker function. The column sums of $P^{(i)}$ are identically given as $\sum_{h=1}^{r} p_{h,j}^{(i)} = \sum_{h=1}^{r} |\Omega_h \cap \Omega_{i^*} g_j| = k_i$, and the row sums of $P^{(i)}$, weighted with the subdegrees, are depending on h given as $\sum_{j=1}^{r} k_j p_{h,j}^{(i)} = k_h k_i$.

Moreover, if E is commutative, then we have $p_{i,j}^{(h)} k_j = p_{h,j}^{(i)} k_j = |\Omega_j \cap \Omega_i g_h| \cdot k_h$. Thus in this case, if the subdegrees are known, the determination of $P^{(h)}$ essentially boils down to computing the **orbit counting matrices** $C(g_h) := [c_{i,j}(g_h)] \in \mathbb{Z}^{r \times r}$, with row index $i \in \{1, \ldots, r\}$ and column index $j \in \{1, \ldots, r\}$, where the matrix entries $c_{i,j}(g_h) := |\Omega_j \cap \Omega_i g_h| \in \mathbb{N}_0$ are called **orbit counting numbers**. Note that the column sums and row sums of $C(g_h)$ are identically given as $\sum_{i=1}^{r} c_{i,j}(g_h) = k_j$ and $\sum_{j=1}^{r} c_{i,j}(g_h) = k_i$, respectively.

(2.3) Let $\mathrm{Irr}_{\mathbb{C}}(E)$ be the set of irreducible complex characters of $E_{\mathbb{C}}$. Then the **character table** of E is defined as the matrix $\Phi_E := [\varphi(A_i)] \in \mathbb{C}^{|\mathrm{Irr}_{\mathbb{C}}(E)| \times r}$, with row index $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E)$ and column index $i \in \{1, \ldots, r\}$.

By [9, Ch.1.11.D] there is a natural bijection, called the **Fitting correspondence**, between $\mathrm{Irr}_{\mathbb{C}}(E)$ and the irreducible constituents of the permutation character $1_H^G \in \mathbb{Z}\mathrm{Irr}_{\mathbb{C}}(G)$ associated to Ω, where $\mathrm{Irr}_{\mathbb{C}}(G)$ denotes the set of irreducible complex characters of G. The Fitting correspondent of $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E)$ is denoted by $\chi_\varphi \in \mathrm{Irr}_{\mathbb{C}}(G)$, and for its degree $\chi_\varphi(1) \in \mathbb{N}$ we have $\frac{n}{\chi_\varphi(1)} = \sum_{i=1}^{r} \frac{\|\varphi(A_i)\|^2}{k_i}$, where $\|\cdot\| \colon \mathbb{C} \to \mathbb{R}$ denotes the absolute value function. Thus the degrees of the Fitting correspondents can be easily computed from Φ_E.

The $\mathbb{C}$-algebra $E_{\mathbb{C}}$, and hence equivalently E, is commutative if and only if the permutation character $1_H^G \in \mathbb{Z}\mathrm{Irr}_{\mathbb{C}}(G)$ is **multiplicity-free**, i. e. if all irreducible constituents of 1_H^G occur with multiplicity 1. In this case, we have $|\mathrm{Irr}_{\mathbb{C}}(E)| = r$ and $\varphi(A_1) = 1$, for all $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E)$. Moreover, the rows of $\Phi_E \in \mathbb{C}^{r \times r}$ fulfil **orthogonality relations**, which can be written as the matrix identity $\overline{\Phi_E} \cdot \mathrm{diag}[k_i; i \in \{1, \ldots, r\}]^{-1} \cdot \Phi_E^{\mathrm{tr}} = n \cdot \mathrm{diag}[\chi_\varphi(1); \varphi \in \mathrm{Irr}_{\mathbb{C}}(E)]^{-1}$, where $\overline{} \colon \mathbb{C} \to \mathbb{C}$ denotes complex conjugation, and $\mathrm{diag}[\cdot] \in \mathbb{C}^{r \times r}$ denotes the diagonal matrix having the indicated entries; in particular Φ_E is invertible.

(2.4) Let E be commutative. Then the character table Φ_E and the collapsed adjacency matrices $P^{(i)}$ are related as follows: If Φ_E is given, the matrices $P^{(i)}$ are easily computed using the formula $P^{(i)} = \Phi_E^{\mathrm{tr}} \cdot \mathrm{diag}[\varphi(A_i); \varphi \in \mathrm{Irr}_{\mathbb{C}}(E)] \cdot \Phi_E^{-\mathrm{tr}}$.

Conversely, if the $P^{(i)}$ are given, the set $\{[\varphi(A_i); i \in \{1, \ldots, r\}]; \varphi \in \mathrm{Irr}_{\mathbb{C}}(E)\} \subseteq \mathbb{C}^r$, i. e. the set of the rows of Φ_E still to be computed, is characterised as the unique $\mathbb{C}$-basis of the row space $\mathbb{C}^r$ consisting of common eigenvectors of all the matrices $P^{(i)\mathrm{tr}} \in \mathbb{C}^{r \times r}$, for $i \in \{1, \ldots, r\}$, and having 1 as their first entry. Thus we have to find the complex eigenvalues of the $P^{(i)\mathrm{tr}}$, which is done as follows:

Note first that by Schur's Lemma the Schur basis elements $A_i \in \mathbb{Z}^{n \times n}$, for $i \in \{1, \ldots, r\}$, are diagonalisable over any common splitting field $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$ of the irreducible constituents of 1_H^G. Thus K may be chosen as a suitable abelian algebraic number field being contained in $\mathbb{Q}(\mathrm{Irr}_{\mathbb{C}}(G))$, where for the cases considered here K typically has very low degree. Anyway, this implies that the matrices $P^{(i)} \in \mathbb{Z}^{r \times r}$ are also diagonalisable over K. Hence the minimum polynomial $\mu_{P^{(i)}} \in \mathbb{Q}[X]$ of $P^{(i)}$ is found as the square-free part of the characteristic polynomial of $P^{(i)}$, and splits into linear factors in $K[X]$. Since we actually have $\mu_{P^{(i)}} \in \mathbb{Z}[X]$, we may first factorise $\mu_{P^{(i)}}$ in $\mathbb{Z}[X]$, and subsequently factorise further in $K[X]$; algorithms to find square-free parts, as well as to factorise polynomials in $\mathbb{Z}[X]$ and in $K[X]$ are well-known, see [7].

This shows that we have determined a certain row of Φ_E whenever we have found a subset $\mathcal{I} \subseteq \{1, \ldots, r\}$ such that the $P^{(i)\mathrm{tr}}$, for $i \in \mathcal{I}$, already have a 1-dimensional common eigenspace. Actually, it often turns out that a single collapsed adjacency matrix already determines most of or even all of Φ_E.

(2.5) Let still E be commutative. Then the character table Φ_E is related to the character table of G as follows: Let $\mathrm{Cl}(G)$ denote the set of conjugacy classes of G, and let $\mathcal{X}_{G,H} := [\chi(C)] \in \mathbb{C}^{r \times |\mathrm{Cl}(G)|}$, with row index χ and column index $C \in \mathrm{Cl}(G)$, be the rows of the character table of G consisting of the irreducible constituents of 1_H^G. Let $\Gamma := [\gamma_i(C)] \in \mathbb{Z}^{r \times |\mathrm{Cl}(G)|}$, with row index $i \in \{1, \ldots, r\}$ and column index $C \in \mathrm{Cl}(G)$, where $\gamma_i(C) := |C \cap Hg_i| \in \mathbb{N}_0$. Then Φ_E is determined by $\mathcal{X}_{G,H}$ and Γ: We have $\Phi_E = \frac{1}{|H|} \cdot \mathcal{X}_{G,H} \cdot \Gamma^{\mathrm{tr}} \cdot \mathrm{diag}[k_i; i \in \{1, \ldots, r\}]$, where the Fitting correspondent of the i-th row of Φ_E is the i-th row of $\mathcal{X}_{G,H}$.

Letting $\varphi_1 \in \mathrm{Irr}_{\mathbb{C}}(E)$ be the Fitting correspondent of the trivial constituent of 1_H^G, this shows that $\varphi_1(A_i) = k_i \in \mathbb{N}_0$, for $i \in \{1, \ldots, r\}$. By the orthogonality relations φ_1 is the unique character all of whose values are non-negative rational integers, hence the subdegrees are also easily computed from Φ_E.

Moreover, the matrix Γ is determined by $\mathcal{X}_{G,H}$ and Φ_E as follows: By Ree's formula, see [9, Thm.1.11.28], and the orthogonality relations we have $\Gamma = \frac{1}{n} \cdot \mathrm{diag}[k_i; i \in \{1, \ldots, r\}]^{-1} \cdot \Phi_E^{\mathrm{tr}} \cdot \overline{\mathcal{X}_{G,H}} \cdot \mathrm{diag}[|C|; C \in \mathrm{Cl}(G)]$.

3 The database

(3.1) Contents and accessing the data. The database can be accessed through GAP [11], where the data and the necessary helper code is stored in the GAP-readable file mferctbl.gap, which can be downloaded from [5].

The data comprises into the record MULTFREEINFO: For each group G which is a sporadic simple group or one of its cyclic or bicyclic extensions, MULTFREEINFO contains a description of all conjugacy classes of subgroups $H < G$ such that the action of G on the right cosets of H is faithful and multiplicity-free, and the character tables Φ_E of the endomorphism rings of the associated permutation modules. Let us just look at the following example:

(3.2) Example. By [4] there are precisely seven equivalence classes of faithful multiplicity-free actions of the sporadic simple Mathieu group $G := M_{11}$:

```
gap> info := MultFreeEndoRingCharacterTables("M11");;
gap> Length(info);
7
gap> info[1];
rec( name := "M11",
     group := "$M_{11}$",
     subgroup := "$A_6.2_3$",
     character := Character( CharacterTable( "M11" ),
                          [ 11, 3, 2, 3, 1, 0, 1, 1, 0, 0 ] ),
     rank := 2,
     charnmbs := [ 1, 2 ],
     ATLAS := "1a+10a",
     ctbl := [ [ 1, 10 ], [ 1, -1 ] ],
     mats := [ [ [ 1, 0 ], [ 0, 1 ] ],
               [ [ 0, 1 ], [ 10, 9 ] ] ] )
```

Here, the record components are as follows: name is the identifier of the GAP character table of the group G, which is available in the character table library CTblLib [3] of GAP and essentially coincides with the one given in [8]; group is a LaTeX format of name; subgroup is a LaTeX format of the name of the stabiliser H; character is the list of values of the permutation character $1_H^G \in \mathbb{Z}\mathrm{Irr}_{\mathbb{C}}(G)$; rank is its rank r; charnmbs lists the positions of the irreducible constituents of 1_H^G in the character table of G; ATLAS is a LaTeX format string describing these constituents; ctbl is the character table Φ_E of the endomorphism ring, whose rows by Fitting correspondence are ordered according to charnmbs; and finally mats is the list of the corresponding collapsed adjacency matrices $P^{(i)}$.

Note that the record component mats containing the collapsed adjacency matrices $P^{(i)}$ is not stored in the database, but the $P^{(i)}$ are computed from the character table Φ_E at runtime, invoking the formula given in (2.4). This can also be done on user demand as follows:

```
gap> CollapsedAdjacencyMatricesFromCharacterTable(info[1].ctbl);
[ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 0, 1 ], [ 10, 9 ] ] ]
```

The character table Φ_E can be displayed as follows, where the header gives the names of the group G and of the stabiliser H, the rows are by Fitting correspondence labelled by the irreducible constituents of the permutation character 1_H^G, and the columns are labelled by the suborbits Ω_i. Recall that the entries in the first row, which always corresponds to the trivial representation 1a, are exactly the subdegrees k_i. Irrational character values are displayed in a format as in [8], where a legend shown after the table explains the values; e. g. ER is the GAP function returning a square root of a rational integer:

```
gap> DisplayMultFreeEndoRingCharacterTable(info[4]);
G = M_{11}, H = 11:5 < L_2(11)

    | O_1      O_2      O_3 O_4 O_5 O_6
-------------------------------------
1a  |   1       11       11  11  55  55
11a |   1       -1       -1  11  -5  -5
16a |   1 -4-3b11 -1+3b11  -1  -5  10
16b |   1 -1+3b11 -4-3b11  -1  -5  10
45a |   1        3        3  -1  -5  -1
55a |   1       -1       -1  -1   7  -5

-1+3b11 = (-5+3*ER(-11))/2
-4-3b11 = (-5-3*ER(-11))/2
```

Moreover, if G is a simple group or an associated automorphism group, and H is not a maximal subgroup of G, then also the name of a maximal subgroup containing H is given, while for the Schur covers and the bicyclic extensions, the action of the associated simple group or its automorphism group being covered by the action under consideration is given:

```
gap> info2 := MultFreeEndoRingCharacterTables("2.M12");;
gap> DisplayMultFreeEndoRingCharacterTable(info2[1]);
G = 2.M_{12}, H = M_{11} ---> (M_{12},1)
...
```

(3.3) By [4, 21, 2] there are 147 equivalence classes of faithful multiplicity-free actions of the sporadic simple groups, 120 of their automorphism groups, 62 of their Schur covers and 68 of their bicyclic extensions, accounting for a total of 397 equivalence classes. The complete list is accessed as follows:

```
gap> infoall := MultFreeEndoRingCharacterTables("all");;
gap> Length(infoall);
397
```

There is a single case remaining for which the character table Φ_E is not known; it is characterized by the value fail for the record component ctbl:

```
gap> unknown := Filtered(infoall, x -> x.ctbl = fail);;
gap> Length(unknown);
1
gap> DisplayMultFreeEndoRingCharacterTable(unknown[1]);
G = 2.B, H = Fi_{23} ---> (B,4)
(character table not yet known)
```

(3.4) Explicit permutation groups. Finally, there is a couple of programs dealing with explicit permutation groups:

For a group G whose table of marks is contained in the table of marks library TomLib [24] of GAP, a list containing its multiplicity-free permutation characters and the associated permutation representations can be computed using the function MultFreeFromTOM, which works as follows: Together with the table of marks, GAP provides the smallest faithful permutation representation of G, given as standard generators in the sense of [35] whenever the latter are defined, and a representative H for each conjugacy class of subgroups. Using the facilities dealing with permutation groups available in GAP, it is straightforward to compute the action of G on the right cosets of H. Note that the ordering of the actions does not necessarily coincide with the one given by MultFreeEndoRingCharacterTables, and that the trivial representation is returned as well.

```
gap> multfree := MultFreeFromTOM("M11");;
gap> Length(multfree);
8
gap> multfree[8];
[ Character( CharacterTable( "M11" ),
  [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ] ), Group(()) ]
gap> multfree[7];
[ Character( CharacterTable( "M11" ),
  [ 11, 3, 2, 3, 1, 0, 1, 1, 0, 0 ] ),
  Group([ (1,7)(2,9)(3,11)(5,6), (1,8,5,3)(2,10,6,4) ]) ]
```

Moreover, for a transitive permutation group G with stabiliser H, the associated collapsed adjacency matrices $P^{(i)}$ can be computed, invoking the formula in (2.2), as follows:

```
gap> CollapsedAdjacencyMatricesInfo(multfree[7][2]);
rec( mats := [ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 0, 1 ], [ 10, 9 ] ] ],
     points := [ 1, 2 ],
     representatives := [ (), (1,2)(5,7)(6,9)(8,11) ] )
```

Here, the record component mats is the list of the collapsed adjacency matrices $P^{(i)}$, points are representatives of the suborbits, and representatives is a list of elements of G, the i-th element mapping points[1] to points[i]. Note that the ordering of the suborbits does not necessarily coincide with the one given by MultFreeEndoRingCharacterTables.

4 Compiling the data

We briefly describe the techniques, which are of computational as well as of theoretical nature, used to collect the data. For a complete account of all the details, which of course would be very technical and too lengthy for the present paper, the reader is referred to [25].

(4.1) For the 'small' cases of degree $n \leq 10^7$, which for the sporadic simple groups and their automorphism groups have already been dealt with in [14], typically the associated permutation representations have been used explicitly:

For the sporadic simple groups up to the Held group He, and their automorphism groups except $HS.2$ and $He.2$, the tables of marks are available in GAP, hence explicit permutations are accessible through the function MultFreeFromTOM described in (3.4). The cases up to rank 5 for all sporadic simple groups and their automorphism groups have been dealt with in [32]. For the Schur covers of the sporadic simple groups explicit permutations have been constructed in [19]. For many of the remaining cases for the sporadic simple groups and their extensions, explicit permutations are accessible in the database [36], given as standard generators in the sense of [35].

For quite a few cases we have constructed permutation representations using the 'vector permutation' technique implemented in the MeatAxe [33]. Given G and $H < G$, we look for a matrix representation of G over a small finite field F, such that there is $0 \neq v \in M$, where M is the underlying FG-module, which is fixed under the action of H, but not fixed under any proper overgroup of H. Hence the action of G on $v \cdot G \subseteq M$ is equivalent to its action on the right cosets of H. Here again the database [36] has been a rich source of explicitly given matrices to start with, and suitable vectors are searched for using the algorithms to compute submodule lattices in [23], which are also available in the MeatAxe.

Having explicit permutation representations in hand, the technique described in (3.4) yields all the collapsed adjacency matrices $P^{(i)}$, from which the character tables Φ_E are obtained using the method described in (2.4). Admittedly, for a few exceptional bicyclic extension cases, instead of using explicit constructions it turned out to be easier to apply the ideas described in (4.3).

(4.2) The 'large' cases of degree $n > 10^7$ are collected in Table 1. As is also indicated in Table 1, a few collapsed adjacency matrices have already been available in the literature. In many of these cases the available matrices have been sufficient to obtain a complete splitting of $\mathbb{C}^r$ into 1-dimensional common eigenspaces, and thus to determine the character table Φ_E. But in a few cases

Table 1: 'Large' multiplicity-free permutation representations.

G	H	n	r	
$3.Fi_{22}$	${}^2F_4(2)'$	10 777 536	25	[19]
$3.Fi_{22}.2$	${}^2F_4(2)'.2$	10 777 536	17	(4.3)
$6.Fi_{22}.2$	${}^2F_4(2)'.2$	21 555 072	22	(4.3)
$6.Fi_{22}.2$	${}^2F_4(2)'.2$	21 555 072	22	(4.3)
$HN.2$	S_{11}	13 680 000	17	(4.5)
HN	A_{11}	13 680 000	19	(4.3)
$HN.2$	$U_3(8).6$	16 500 000	15	(4.5)
HN	$U_3(8).3_1$	16 500 000	19	(4.3)
Ly	$3.McL$	19 212 250	8	(4.5), (4.6)
Th	${}^3D_4(2).3$	143 127 000	11	(4.5)
Th	$2^5.L_5(2)$	283 599 225	11	(4.5)
Fi_{23}	$S_8(2)$	86 316 516	13	[20]
Fi_{23}	$2^{11}.M_{23}$	195 747 435	16	[20]
Co_1	$2_+^{1+8}.O_8^+(2)$	46 621 575	11	[16]
$2.Co_1$	Co_3	16 773 120	12	[19]
J_4	$2^{11}: M_{24}$	173 067 389	7	[17]
J_4	$2^{11}: M_{23}$	4 153 617 336	11	(4.5)
$Fi_{24}'.2$	$O_{10}^-(2).2$	50 177 360 142	17	[20]
Fi_{24}'	$O_{10}^-(2)$	50 177 360 142	17	(4.3)
$Fi_{24}'.2$	$O_{10}^-(2)$	100 354 720 284	34	(4.3)
$Fi_{24}'.2$	$3^7.O_7(3).2$	125 168 046 080	17	[20], (4.2)
Fi_{24}'	$3^7.O_7(3)$	125 168 046 080	18	(4.3)
$3.Fi_{24}'$	$O_{10}^-(2)$	150 532 080 426	43	[19], (4.2)
$3.Fi_{24}'.2$	$O_{10}^-(2).2$	150 532 080 426	30	(4.3)
B	$2.{}^2E_6(2).2$	13 571 955 000	5	[12]
B	$2.{}^2E_6(2)$	27 143 910 000	8	(4.3), (4.4)
B	$2^{1+22}.Co_2$	11 707 448 673 375	10	(4.7)
B	Fi_{23}	1 015 970 529 280 000	23	(4.7)
$2.B$	Fi_{23}	2 031 941 058 560 000	34	unsolved
M	$2.B$	97 239 461 142 009 186 000	9	[29]

there are some higher-dimensional common eigenspaces left, which have to be split further into 1-dimensional ones; an example computation how to proceed in these situations is included in (4.6) below.

In the sequel we comment on the cases in Table 1 not covered by earlier results. The major techniques used, as also indicated in Table 1, are as follows:

(4.3) For quite a few cases there are different, but closely related groups, having different, but again closely related actions, and it turns out that the associated character tables are closely related as well. This allows us to determine certain character tables without explicitly constructing any permutations. Here, we only present the simplest of these situations, and just note that the idea to relate various character tables indeed can be applied in several, more complicated situations. For the quite technical details we refer the reader to [25, Ch.5].

Let still $H \leq G$ such that 1_H^G is multiplicity-free. Moreover, assume there is an intermediate group $H \leq H' \leq G$, hence $1_{H'}^G$ is multiplicity-free as well. Let $\Omega' = \coprod_{j=1}^{s} \Omega'_j$ be the G-set of right cosets of H', together with its decomposition into H'-orbits and subdegrees k'_j, and let E' be the associated endomorphism ring with Schur basis $\mathcal{A}' = \{A'_1, \ldots, A'_s\} \subseteq E'$. Since any irreducible constituent of $1_{H'}^G$ also occurs in 1_H^G we may consider E' as a non-unitary subring of E, and $\mathrm{Irr}_{\mathbb{C}}(E')$ as a subset of $\mathrm{Irr}_{\mathbb{C}}(E)$. Moreover, the natural homomorphism of G-sets $\overline{}\colon \Omega \to \Omega'\colon Hg \mapsto H'g$ induces a map $\alpha\colon \{1, \ldots, r\} \to \{1, \ldots, s\}\colon i \mapsto j$, being defined by $\overline{\Omega_i} \subseteq \Omega'_j$.

Then by [25, Cor.5.13] the character tables Φ_E and $\Phi_{E'}$ are related as follows: Let $j \in \{1, \ldots, s\}$. Firstly, for $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E') \subseteq \mathrm{Irr}_{\mathbb{C}}(E)$ and $i \in \alpha^{-1}(j)$ we have $\varphi(A_i) = \frac{k_i}{k'_j} \cdot \varphi(A'_j)$; thus implying $\sum_{i \in \alpha^{-1}(j)} \varphi(A_i) = [H'\colon H] \cdot \varphi(A'_j)$. Secondly, for $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E) \setminus \mathrm{Irr}_{\mathbb{C}}(E')$ we have $\sum_{i \in \alpha^{-1}(j)} \varphi(A_i) = 0$; thus in particular if $\alpha^{-1}(j) = \{i\}$ is a singleton set then we have $\varphi(A_i) = 0$.

A particularly nice situation occurs if additionally $[H'\colon H] = 2$, implying that $H \lhd H'$. Hence we have $1_H^G = 1_{H'}^G + (1^-)_{H'}^G \in \mathbb{Z}\mathrm{Irr}_{\mathbb{C}}(G)$, where $1^- \in \mathrm{Irr}_{\mathbb{C}}(H')$ is the linear character obtained by inflation from the non-trivial linear character of H'/H. The $\mathbb{Z}G$-module affording $(1^-)_{H'}^G$ can be identified with $\mathbb{Z}\Omega'$, but to indicate the different, monomial G-action it is denoted by $\mathbb{Z}\Omega''$ instead. Let $E'' := \mathrm{End}_{\mathbb{Z}G}(\mathbb{Z}\Omega'')$ be its endomorphism ring. Generalising from permutation representations to monomial representations, inspired by ideas in [13], by [25, Ch.3] the ring E'' also has a natural Schur basis, being parametrised by a subset of $\{1, \ldots, s\}$. Moreover, there are a notion of a character table $\Phi_{E''}$ and a generalised version of [25, Cor.5.13], relating the character tables Φ_E and $\Phi_{E''}$.

(4.4) Example. Let $G := B$ be the sporadic simple Baby Monster group, let $H := 2.^2E_6(2)$ and let $H' := 2.^2E_6(2).2$. The character table $\Phi_{E''}$ associated to $(1^-)_{H'}^G$ has been determined in [12], where also the character table $\Phi_{E'}$ associated to $1_{H'}^G$ is given. Using the above remarks, from these character tables the character table Φ_E associated to 1_H^G can be determined. The character tables $\Phi_{E'}$, $\Phi_{E''}$ and Φ_E are shown in Table 2: The Schur basis of E'' is parametrised by $\{1,2,4\} \subset \{1,\ldots,5\}$, the preimages $\alpha^{-1}(j)$, for $j \in \{1,\ldots,5\}$, are indicated in the header of Φ_E as well, and the rows of Φ_E are arranged according to the partition $\mathrm{Irr}_{\mathbb{C}}(E) = \mathrm{Irr}_{\mathbb{C}}(E') \,\dot\cup\, \mathrm{Irr}_{\mathbb{C}}(E'')$.

(4.5) Still, for the remaining cases we need explicit constructions. But instead of computing permutations first, and to use them subsequently to find the collapsed adjacency matrices $P^{(i)}$, we now make use of the 'direct condense' technique invented in [30], which has been elaborated in [22] into an efficient distributed computing technique to directly determine the orbit counting matrices $C(g) \in \mathbb{N}_0$, for arbitrary elements $g \in G$, without computing any permutations. Moreover, since in these cases Ω typically is too large to be stored completely, only a part of Ω is actually stored, in a way still allowing us to get an overview over all of Ω. This is controlled by a suitably chosen 'small' helper subgroup $U < G$, where as a rule of thumb only a fraction of roughly $\frac{1}{|U|}$ of Ω is stored. Again, Ω is realised as a set of vectors in an FG-module, or as a set of 1-dimensional F-subspaces of an FG-module. Moreover, we make use of the modification of this technique described in [27], where Ω is realised as a set of higher dimensional F-subspaces of an FG-module.

To actually run the 'direct condense' technique efficiently using distributed computing, Ω has to be partitioned into 'many' pieces, while in the present cases $r \in \mathbb{N}$ typically is 'small'. Hence we choose a 'medium sized' subgroup $V < H$, and consider the decomposition $\Omega = \coprod_{j=1}^{s} \Omega'_j$ of Ω into V-orbits instead, where now $s \in \mathbb{N}$ is 'large enough'. Then orbit counting matrices $C(g) \in \mathbb{Z}^{s\times s}$ with respect to this decomposition are computed for a few elements $g \in G$. Finally, the fusion of the Ω'_j, for $j \in \{1,\ldots,s\}$, into the Ω_i, for $i \in \{1,\ldots,r\}$, has to be determined, which is just done by computing orbit counting matrices $C(h) \in \mathbb{Z}^{s\times s}$ for a few elements $h \in H$.

(4.6) Example. Let $G := Ly$ be the sporadic simple Lyons group, and let $H := 3.McL < H' := N_G(H) = 3.McL.2 < G$, where $H' < G$ is a maximal subgroup. Hence we have $n = |\Omega| = 19\,212\,250$ and $r = 8$.

The G-set Ω is realised as follows: Let M be the absolutely irreducible $\mathbb{F}_5G$-module of $\mathbb{F}_5$-dimension 517. Representing matrices of standard generators of G in the sense of [35], as well as generators of H as words in the standard generators of G, are available in the database [36]. Using the MeatAxe we find that the subspace $\mathrm{Fix}_H(M) \leq M$ of vectors fixed by H is 1-dimensional, and that for any $0 \neq v \in \mathrm{Fix}_H(M)$ we in turn have $\mathrm{Stab}_G(v) = H$, hence we let $\Omega := v \cdot G \subseteq M$. Thus, using 'compressed vectors' as are available in GAP and the MeatAxe, a single vector in M needs 173 Bytes of memory space, hence to store all of Ω would need $3\,323\,719\,250 \sim 3 \cdot 10^9$ Bytes, which is slightly too much to be comfortable.

The setting for the 'direct condense' programs is as follows: Letting $V := 3 \times$

Table 2: The character table for $G := B$ and $H := 2.{}^2E_6(2)$.

φ	χ_φ	1	2	3	4	5
1	$1a$	1	3968055	23113728	2370830336	11174042880
3	$96255a$	1	228735	−709632	14483456	−14002560
4	$9458750a$	1	50895	133056	124928	−308880
6	$4275362520a$	1	1935	−4032	−31744	33840
7	$9287037474a$	1	−945	1728	14336	−15120

φ	χ_φ	1	2	4
2	$4371a$	1	566865	84672512
5	$63532485a$	1	28665	−114688
8	$13508418144a$	1	−135	512

φ	χ_φ	$1'$	$1''$	$2'$	$2''$	3	$4'$	$4''$	5
1	$1a$	1	1	3968055	3968055	46227456	2370830336	2370830336	22348085760
3	$96255a$	1	1	228735	228735	−1419264	14483456	14483456	−28005120
4	$9458750a$	1	1	50895	50895	266112	124928	124928	−617760
6	$4275362520a$	1	1	1935	1935	−8064	−31744	−31744	67680
7	$9287037474a$	1	1	−945	−945	3456	14336	14336	−30240
2	$4371a$	1	−1	566865	−566865	0	84672512	−84672512	0
5	$63532485a$	1	−1	28665	−28665	0	−114688	114688	0
8	$13508418144a$	1	−1	−135	135	0	512	−512	0

$M_{11} < H$, using GAP we find $s = \langle 1_H^G, 1_V^G \rangle_G = 837$, where $\langle \cdot, \cdot \rangle_G$ denotes the usual hermitian product on the complex class function on G. This number of V-orbits in Ω is 'large enough' to make full use of distributed computing. Furthermore, we choose $C_{11} \cong U < V < H$, a cyclic group of order 11. Thus by the above rule of thumb this reduces the amount of memory space needed to comfortable $\sim 3 \cdot 10^8$ Bytes.

By choosing random elements of G we compute a few of the collapsed adjacency matrices, and as described in (2.4) we quickly obtain a splitting of $\mathbb{C}^8$ into 6 common 1-dimensional eigenspaces, and a 2-dimensional common eigenspace. Of course we could choose more random elements of G, until we have found sufficiently many distinct collapsed adjacency matrices also yielding a further splitting of the 2-dimensional common eigenspaces. But it turns out to be easier to proceed as follows:

Let $\langle \psi_1, \psi_2 \rangle_{\mathbb{C}} \leq \mathbb{C}^8$ be the 2-dimensional common eigenspace, where we may assume that the first entries of ψ_1 and ψ_2 are 1 and 0, respectively. As the irreducible constituents of 1_H^G have pairwise distinct degrees, using the degree formula in (2.3) we conclude that we have found $\varphi_i \in \mathrm{Irr}_{\mathbb{C}}(E)$, for $i \in \{1, 2, 4, 5, 7, 8\}$, while we have $\langle \psi_1, \psi_2 \rangle_{\mathbb{C}} = \langle \varphi_3, \varphi_6 \rangle_{\mathbb{C}}$. Hence there are $a, b \in \mathbb{C}$ such that $\varphi_3 = \psi_1 + a\psi_2$ and $\varphi_6 = \psi_1 + b\psi_2$. Actually, since $\mathbb{Q}$ is a common splitting field of the irreducible constituents of 1_H^G, we necessarily have $a, b \in \mathbb{Q}$. Thus from the orthogonality relations for φ_3 we get $\sum_{i=1}^{r} \frac{(\psi_1 + a\psi_2)(A_i)^2}{k_i} = \frac{n}{\chi_{\varphi_3}(1)}$, which is a quadratic equation for the unknown $a \in \mathbb{Q}$, having the solutions $a \in \{\pm 1800\}$. Similarly, for φ_6 we obtain $b \in \{\pm 675\}$. Finally, the orthogonality relations imply $a \cdot b < 0$. This completes the character table Φ_E; see Table 3, where rows and column again have been reordered to exhibit the phenomena described in (4.3).

(4.7) Finally, the 'direct condense' technique as described above was not efficient enough to tackle the orbits of the the sporadic simple Baby Monster group $G := B$. Actually, these problems have been the original motivation to generalise the technique in [30, 22] to a fully grown divide-and-conquer technique, see [25, 28], which instead of a 'small' helper subgroup $U < G$ utilises a whole chain $U_1 < U_2 < \cdots < U_k < G$ of helper subgroups.

Details for the cases $H := 2^{1+22}.Co_2$ and $H := Fi_{23}$ are to be found in [26] and [28], respectively. Finally, in [25, Ch.17] also partial results on the presently unsolved case $G := 2.B$ and $H := Fi_{23}$ have been collected.

(4.8) Given the character table Φ_E, we proceed as follows to determine the Fitting correspondence, which we consider as an injective map $\mathcal{F} \colon \mathrm{Irr}_{\mathbb{C}}(E) \to \mathrm{Irr}_{\mathbb{C}}(G) \colon \varphi \mapsto \chi_\varphi$, whose image is the set of irreducible constituents of 1_H^G. By (2.3) the degree $\chi_\varphi(1)$ of the Fitting correspondent of $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E)$ can be easily determined from Φ_E. In particular, if the degrees of the irreducible constituents of 1_H^G are pairwise distinct, then $\mathcal{F}$ already is uniquely determined. Still, for a candidate map $\mathcal{F} \colon \mathrm{Irr}_{\mathbb{C}}(E) \to \mathrm{Irr}_{\mathbb{C}}(G)$, fulfilling $\varphi^{\mathcal{F}}(1) = \chi_\varphi(1)$ for $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E)$,

Table 3: The character table for $G := Ly$ and $H := 3.McL$.

φ	χ_φ	$1'$	$1''$	2	$3'$	$3''$	4	$5'$	$5''$
1	$1a$	1	1	30800	534600	534600	3742200	7185024	7185024
2	$45694a$	1	1	-650	10125	10125	-4050	-7776	-7776
4	$1534500a$	1	1	-20	-60	-60	5610	-2736	-2736
5	$3028266a$	1	1	350	125	125	-1050	224	224
7	$4997664a$	1	1	-200	-150	-150	-1050	774	774
3	$1152735a$	1	-1	0	1800	-1800	0	3024	-3024
6	$3073960a$	1	-1	0	-675	675	0	3024	-3024
8	$5379430a$	1	-1	0	0	0	0	-2376	2376
ψ_1		1	-1	0	0	0	0	3024	-3024
ψ_2		0	0	0	1	-1	0	0	0

let $\mathcal{X}_{\mathcal{F}} := [\varphi^{\mathcal{F}}(C)] \in \mathbb{C}^{r \times |\mathrm{Cl}(G)|}$, with row index $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E)$ and column index $C \in \mathrm{Cl}(G)$, be the rows of the character table of G as prescribed by $\mathcal{F}$, and let $\Gamma_{\mathcal{F}} := \frac{1}{n} \cdot \mathrm{diag}[\frac{1}{k_i}; i \in \{1, \ldots, r\}] \cdot \Phi_E^{\mathrm{tr}} \cdot \overline{\mathcal{X}_{\mathcal{F}}} \cdot \mathrm{diag}[|C|; C \in \mathrm{Cl}(G)] \in \mathbb{C}^{r \times |\mathrm{Cl}(G)|}$; recall that the subdegrees k_i are also easily determined from Φ_E. Thus by (2.5) we may discard $\mathcal{F}$ whenever $\Gamma_{\mathcal{F}}$ has an entry which is not a non-negative rational integer, otherwise $\mathcal{F}$ is called **admissible**.

Typically, this still does not yield uniqueness, due to the symmetries of the character tables involved; these are described as follows: The symmetric group $\mathcal{S}_{\mathrm{Cl}(G)}$ on $\mathrm{Cl}(G)$ acts on the set of complex class functions $\chi\colon \mathrm{Cl}(G) \to \mathbb{C}$ by letting $\chi^{\sigma} := \sigma^{-1}\chi$, for $\sigma \in \mathcal{S}_{\mathrm{Cl}(G)}$. Then $\sigma \in \mathcal{S}_{\mathrm{Cl}(G)}$ is called a **table automorphism** of $\mathrm{Irr}_{\mathbb{C}}(G)$, if $\mathrm{Irr}_{\mathbb{C}}(G)^{\sigma} = \mathrm{Irr}_{C}(G)$, and $\sigma\pi_j = \pi_j\sigma$ for all $j \in \mathbb{Z}$, where $\pi_j\colon \mathrm{Cl}(G) \to \mathrm{Cl}(G)\colon g^G \mapsto (g^j)^G$ is the j-th power map. Let $A_G \leq \mathcal{S}_{\mathrm{Cl}(G)}$ denote the group of table automorphisms of $\mathrm{Irr}_{\mathbb{C}}(G)$, and let $A_{G,H} \leq A_G$ be the subgroup leaving the set of irreducible constituents of 1_H^G invariant. Note that by the orthogonality relations for the irreducible characters of G we have $|C^{\sigma}| = |C|$, for $C \in \mathrm{Cl}(G)$ and $\sigma \in A_G$. Given the character table of G, there are programs available in GAP to determine A_G.

Similarly, the symmetric group $\mathcal{S}_r$ acts on the set of functions $\varphi\colon \mathcal{A} \to \mathbb{C}$ by letting $\varphi^{\tau} := \tau^{-1}\varphi$, and $\tau \in \mathcal{S}_r$ is called a **table automorphism** of $\mathrm{Irr}_{\mathbb{C}}(E)$, if $\mathrm{Irr}_{\mathbb{C}}(E)^{\tau} = \mathrm{Irr}_{\mathbb{C}}(E)$. Let $A_E \leq \mathcal{S}_r$ denote the group of table automorphisms of $\mathrm{Irr}_{\mathbb{C}}(E)$. Note that for the Fitting correspondent $\varphi_1 \in \mathrm{Irr}_{\mathbb{C}}(E)$ of the trivial constituent of 1_H^G we have $\varphi_1^{\tau} = \varphi_1$, for $\tau \in A_E$. Hence we have $k_{i\tau} = k_i$ for $i \in \{1, \ldots, r\}$, and thus $\chi_{\varphi^{\tau}}(1) = \chi_{\varphi}(1)$ for $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E)$. Actually, for the cases considered here $A_E \leq \mathcal{S}_r$ typically is a 'tiny' group, whose elements can be enumerated directly. In particular, if the subdegrees k_i are pairwise distinct, then A_E is the trivial group.

The group $A_E \times A_{G,H}$ acts on the set of maps $\mathcal{F}\colon \mathrm{Irr}_{\mathbb{C}}(E) \to \mathrm{Irr}_{\mathbb{C}}(G)$ by letting $\mathcal{F}^{\tau,\sigma} := \tau^{-1}\mathcal{F}\sigma$, for $[\tau, \sigma] \in A_E \times A_{G,H}$. Since the A_E-action respects degrees of Fitting correspondents and subdegrees, and the A_G-action respects sizes of conjugacy classes, the set of admissible maps is a union of $(A_E \times A_{G,H})$-orbits.

Moreover, given a group G and its irreducible characters $\mathrm{Irr}_{\mathbb{C}}(G)$, we typically have several inequivalent multiplicity-free actions, hence we have to keep the associated Fitting correspondences consistent between the various actions: Let $1_{H_1}^G, \ldots, 1_{H_t}^G$ be distinct multiplicity-free permutation characters for suitable $H_j \leq G$, with associated endomorphism rings E_j, and let $\mathcal{F}_j\colon \mathrm{Irr}_{\mathbb{C}}(E_j) \to \mathrm{Irr}_{\mathbb{C}}(G)$, for $j \in \{1, \ldots, t\}$, be the sets of admissible maps. Now $\prod_{j=1}^t \mathcal{F}_j$ is acted on by $A := \left(\prod_{j=1}^t A_{E_j}\right) \times \left(\bigcap_{j=1}^t A_{G,H_j}\right)$, where the left hand factor acts componentwise, while the right hand factor acts diagonally. Hence $\prod_{j=1}^t \mathcal{F}_j$ is a union of A-orbits, and the aim is to determine the particular A-orbit describing the correct tuple of Fitting correspondences. We are done if $\prod_{j=1}^t \mathcal{F}_j$ consists of a single A-orbit, which in many of the cases considered here indeed occurs.

We proceed similarly for closely related groups, i. e. a simple group, its automorphism group, its Schur cover, and its bicyclic extensions, where the associated character tables are closely related as well, and thus we also have to keep the associated Fitting correspondences consistent between the various groups.

To discard erroneous candidate maps we proceed as follows: For closely related groups and closely related actions, the relations between the associated character tables as described in (4.3) are used. For the 'small' cases, where tables of marks are known, see (4.1), we proceed as follows: We use the faithful permutation representation of G provided by GAP to determine the conjugacy classes $\mathrm{Cl}(G)$ of G. Picking a subgroup $H \leq G$, we compute the numbers $\gamma_i(C) = |C \cap Hg_i| \in \mathbb{N}_0$, for $i \in \{1, \ldots, r\}$ and $C \in \mathrm{Cl}(G)$, by explicit counting. Letting $\Gamma := [\gamma_i(C)] \in \mathbb{Z}^{r \times k}$, comparing the matrix $\frac{1}{|H|} \cdot \mathcal{X}_{G,H} \cdot \Gamma^{\mathrm{tr}} \cdot \mathrm{diag}[k_i; i \in \{1, \ldots, r\}] \in \mathbb{C}^{r \times r}$ with the character table Φ_E, by (2.5) yields the Fitting correspondence. Thus doing so for the various subgroups $H_j \leq G$ yields a consistent choice of Fitting correspondences. Finally, for the 'large' cases we have to use ad hoc techniques; an example involving Krein parameters is detailed in [25, Ch.11.5].

5 Applications to orbital graphs

We give a couple of applications to generalised orbital graphs associated to multiplicity-free actions. We only briefly recall the necessary facts from algebraic graph theory to fix notation; as general references see [6, 10]. All graphs considered are finite, undirected and simple.

(5.1) Ramanujan property. Let $\mathfrak{G}$ be a regular graph with valency $k \in \mathbb{N}$, and let $\rho_1 > \ldots > \rho_t$, for some $t \in \mathbb{N}$, denote the eigenvalues of its adjacency matrix $A_{\mathfrak{G}}$; note that $A_{\mathfrak{G}}$ is diagonalisable over $\mathbb{R}$. The eigenvalues together with their multiplicities are called the **spectrum** of $\mathfrak{G}$. We have $\rho_1 = k$, its multiplicity being the number of connected components of $\mathfrak{G}$, as well as $|\rho_t| \leq k$, and $\mathfrak{G}$ is bipartite if and only if $\rho_t = -k$ and its multiplicity equals the multiplicity of ρ_1. The number $\rho_{\mathfrak{G}} := \max\{|\rho_l| \in \mathbb{R}; l \in \{1, \ldots, t\}, |\rho_l| < k\} \geq 0$ is called the **spectral radius** of $\mathfrak{G}$. If $\mathfrak{G}$ is connected such that $\rho_{\mathfrak{G}} \leq 2 \cdot \sqrt{k-1}$ then it is called a **Ramanujan graph**; this property is related to the notion of **expander graphs**, see e. g. [10, Ch.1], thus the most interesting Ramanujan graphs are those with small valency compared to their numbers of vertices.

The spectrum of a generalised orbital graph $\mathfrak{G}_{\mathcal{I}}$, for a subset $\mathcal{I} \subseteq \{2, \ldots, r\}$ closed under taking paired suborbits, associated to a multiplicity-free action is easily determined from the character table of the associated endomorphism ring: The eigenvalues of the adjacency matrix $A_{\mathcal{I}}$ are given by $\varphi(A_{\mathcal{I}}) = \sum_{i \in \mathcal{I}} \varphi(A_i)$, for $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E)$, where the multiplicity of $\varphi(A_{\mathcal{I}})$ equals the degree $\chi_\varphi(1)$ of the Fitting correspondent of φ. Hence the Ramanujan generalised orbital graphs of the multiplicity-free actions of the sporadic simple groups and their extensions are easily determined from the database. Those having 'small' valency $k \leq \sqrt{n}$ are given in Table 4, where also the spectral radii and the diameters are indicated. A complete list for the actions of the sporadic simple groups and their automorphism groups up to degree $n \leq 10^7$ is given in [14].

(5.2) Distance-regularity and -transitivity. Let $\mathfrak{G}$ be a connected graph of diameter $d_{\mathfrak{G}} \in \mathbb{N}_0$, having vertex set Ω. For $\omega \in \Omega$ and $d \in \{0, \ldots, d_{\mathfrak{G}}\}$ let $\mathfrak{G}_d(\omega) \subseteq \Omega$ be the set of all vertices of $\mathfrak{G}$ having distance d from ω, and let the **distance graph** $\mathfrak{G}_d$ be defined by having vertex set Ω, and vertices $\omega, \omega' \in \Omega$ being adjacent if and only if $\omega' \in \mathfrak{G}_d(\omega)$. The graph $\mathfrak{G}$ is called **distance-transitive**, if there is a group G of graph automorphisms acting transitively on the sets $\mathfrak{G}_d(\omega)$, for all $\omega \in \Omega$ and $d \in \{0, \ldots, d_{\mathfrak{G}}\}$; in this case the G-action on Ω necessarily is multiplicity-free.

The graph $\mathfrak{G}$ is called **distance-regular**, if for all $\omega \in \Omega$ and $\omega' \in \mathfrak{G}_d(\omega)$, where $d \in \{0, \ldots, d_{\mathfrak{G}}\}$, the cardinalities $b_{\omega,\omega'} := |\{\omega'' \in \mathfrak{G}_{d+1}(\omega); \omega'' \in \mathfrak{G}_1(\omega')\}| \in \mathbb{N}_0$, and $c_{\omega,\omega'} := |\{\omega'' \in \mathfrak{G}_{d-1}(\omega); \omega'' \in \mathfrak{G}_1(\omega')\}| \in \mathbb{N}_0$, only depend on d, but not on the particular choice of $\omega \in \Omega$ and $\omega' \in \mathfrak{G}_d(\omega)$. In this case, letting $b_d := b_{\omega,\omega'}$ and $c_d := c_{\omega,\omega'}$, for some $\omega' \in \mathfrak{G}_d(\omega)$, the sequence $[b_0, \ldots, b_{d_{\mathfrak{G}}-1}; c_1, \ldots, c_{d_{\mathfrak{G}}}]$ is called the **intersection array** of $\mathfrak{G}$. Note that distance-transitivity implies distance-regularity, which in turn implies regularity. A distance-regular graph $\mathfrak{G}$ is called **primitive**, if all distance graphs $\mathfrak{G}_d$, for $d \in \{1, \ldots, d_{\mathfrak{G}}\}$, are connected, otherwise it is called **imprimitive**; and it is called **antipodal**, if $d_{\mathfrak{G}} \geq 2$ and $\{[\omega, \omega'] \in \Omega \times \Omega; \omega' \in \mathfrak{G}_0(\omega) \mathbin{\dot\cup} \mathfrak{G}_{d_{\mathfrak{G}}}(\omega)\}$ is an equivalence relation on Ω. Note that by [6, Thm.4.2.1], if $\mathfrak{G}$ is imprimitive of valency $k \geq 3$, then it is bipartite or antipodal or both.

For a generalised orbital graph $\mathfrak{G}_{\mathcal{I}}$, for a subset $\mathcal{I} \subseteq \{2, \ldots, r\}$ closed under taking paired suborbits, associated to a multiplicity-free action the above properties are easily determined from the character table of the associated endomorphism ring: Recall first that connectedness and bipartiteness by (5.1) are spectral properties anyway.

The sets $(\mathfrak{G}_{\mathcal{I}})_d(\omega_1) \subseteq \Omega$, for $d \in \{0, \ldots, d_{\mathfrak{G}_{\mathcal{I}}}\}$, are unions of suborbits, hence give rise to a partition $\{1, \ldots, r\} = \coprod_{d=0}^{d_{\mathfrak{G}_{\mathcal{I}}}} \mathcal{I}_d$, where $\mathcal{I}_0 = \{1\}$ and $\mathcal{I}_1 = \mathcal{I}$. Thus $\mathfrak{G}_{\mathcal{I}}$ is distance-transitive if and only if all the sets $\mathcal{I}_d$, for $d \in \{0, \ldots, d_{\mathfrak{G}_{\mathcal{I}}}\}$, are singleton sets. Moreover, the cardinalities $b_{\omega_1,\omega} \in \mathbb{N}_0$ and $c_{\omega_1,\omega} \in \mathbb{N}_0$ only depend on the suborbit the element $\omega \in \Omega$ belongs to, but not on the particular choice of ω within that suborbit. For $j \in \mathcal{I}_d$ we have $b_{\omega_1,\omega_j} = \sum_{h \in \mathcal{I}_{d+1}} p_{h,j}^{(i)}$ and $c_{\omega_1,\omega_j} = \sum_{h \in \mathcal{I}_{d-1}} p_{h,j}^{(i)}$, where the $p_{h,j}^{(i)} \in \mathbb{N}_0$ are the associated intersection numbers. Hence distance-regularity is easily decided, thereby also providing the diameter $d_{\mathfrak{G}_{\mathcal{I}}}$ of $\mathfrak{G}_{\mathcal{I}}$ and its intersection array; see also [32, Thm.3.3].

It remains to consider primitivity and antipodality: Given $d \in \{0, \ldots, d_{\mathfrak{G}_{\mathcal{I}}}\}$, the distance graph $(\mathfrak{G}_{\mathcal{I}})_d$ has valency $\sum_{i \in \mathcal{I}_d} k_i$ and adjacency matrix $A_{\mathcal{I}_d} = \sum_{i \in \mathcal{I}_d} A_i$. Hence its eigenvalues are given as $\varphi(A_{\mathcal{I}_d})$, for $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E)$, with associated multiplicities $\chi_\varphi(1)$. Thus $(\mathfrak{G}_{\mathcal{I}})_d$ is connected if and only if the eigenvalue $\varphi_1(A_{\mathcal{I}_d}) = \sum_{i \in \mathcal{I}_d} k_i$ occurs with multiplicity 1. Moreover, the generalised orbital graph $\mathfrak{G}_{\mathcal{I}}$ is antipodal if and only if $(A_{\mathcal{I}_{d_{\mathfrak{G}_{\mathcal{I}}}}})^2 \in E$ is a $\mathbb{Z}$-linear combination of $\{A_i; i \in \mathcal{I}_0 \mathbin{\dot\cup} \mathcal{I}_{d_{\mathfrak{G}_{\mathcal{I}}}}\}$, which in turn by the non-negativity of the

Table 4: Ramanujan generalised orbital graphs of valency $k \leq \sqrt{n}$.

G	H	n	r	k	ρ	d
M_{12}	$3^2.2.S_4$	220	5	12	5	3
$M_{12}.2$	$3^2.2.S_4$	440	9	4	3	6
$M_{12}.2$	$4^2:(6\times 2)$	990	13	28	10	3
M_{22}	$2^3:L_3(2)$	330	5	7	4	4
$M_{22}.2$	A_7	352	6	15	7	4
$M_{22}.2$	$2^3:L_3(2)\times 2$	330	5	7	4	4
$M_{22}.2$	$2^3:L_3(2)$	660	10	7	4	5
$M_{22}.2$	$2^3:L_3(2)$	660	10	8	5	5
$2.M_{22}.2$	A_7	704	10	15	7	6
J_2	$2_-^{1+4}:A_5$	315	6	10	5	4
J_2	$A_4\times A_5$	840	7	15	6	4
J_2	$A_4\times A_5$	840	7	39	11	3
$J_2.2$	$2_-^{1+4}:S_5$	315	5	10	5	4
$J_2.2$	$2^{2+4}.(3\times S_3)$	1050	12	44	9	3
$J_2.2$	$2^{2+4}.(3\times S_3)$	1050	12	45	10	3
$J_2.2$	$2^{2+4}.(3\times S_3)$	1050	12	88	18	2
$J_2.2$	$2^{2+4}.(S_3\times 3)$	1050	9	56	14	3
$J_2.2$	$2^{2+4}.(S_3\times 3)$	1050	9	88	18	2
$J_2.2$	$(A_4\times A_5).2$	840	7	15	6	4
$J_2.2$	$(A_4\times A_5).2$	840	7	39	11	3
$J_2.2$	$A_4\times A_5$	1680	14	15	6	5
$J_2.2$	$A_4\times A_5$	1680	14	16	7	5
$J_2.2$	$A_4\times A_5$	1680	14	39	11	3
$J_2.2$	$A_4\times A_5$	1680	14	39	11	4
$J_2.2$	$A_4\times A_5$	1680	14	40	12	3
$J_2.2$	$A_4\times A_5$	1680	14	40	12	4
$J_2.2$	$(A_5\times D_{10}).2$	1008	8	12	6	5
$J_2.2$	$(A_5\times D_{10}).2$	1008	8	37	11	3
M_{24}	$2^6:(L_3(2)\times S_3)$	3795	5	56	10	3
M_{24}	$2^6:(L_3(2)\times 3)$	7590	8	56	14	3
M_{24}	$2^6:(L_3(2)\times 3)$	7590	8	57	13	3
M_{24}	$2^6:(L_3(2)\times 3)$	7590	8	112	20	3
M_{24}	$2^6:(L_3(2)\times 3)$	7590	8	113	21	3
$HS.2$	$5_+^{1+2}:[2^5]$	22176	15	50	14	3

associated intersection numbers is the case if and only if we have $p_{h,j}^{(i)} = 0$, for all $h, i \in \mathcal{I}_{d_{\mathfrak{G}_\mathcal{I}}}$ and $j \in \{2, \ldots, r\} \setminus \mathcal{I}_{d_{\mathfrak{G}_\mathcal{I}}}$.

Hence the distance-regular generalised orbital graphs of the multiplicity-free actions of the sporadic simple groups and their extensions, and their properties, are easily determined from the database. As the primitive distance-transitive orbital graphs of the sporadic simple groups and their automorphism groups are listed in [16], they are not reproduced here, and by [31] there are no faithful primitive distance-transitive orbital graphs of their Schur covers and their bicyclic extensions anyway. The imprimitive distance-transitive generalised orbital graphs of the sporadic simple groups and their extensions are given in Table 5. Moreover, the 'interesting' non-distance-transitive but distance-regular generalised orbital graphs of these groups are given in Table 6, where we have excluded the 'uninteresting' graphs of diameter $d \leq 2$, which are the complete graphs and the strongly regular graphs, see [6, Ch.1.3] and [15], and those of diameter $d = 3$ having intersection array $[k, k-1, 1; 1, k-1, k]$, see [6, Cor.1.5.4]. For the graphs listed we also include their intersection arrays, and we indicate whether they are primitive, bipartite, or antipodal. Using this information the graphs can be identified, and we locate them in [6]; note that the generalised orbital graphs of the sporadic simple groups and their automorphism groups associated to actions of rank at most 5 are also listed in [32, Ch.4].

The graphs marked by $(*)$ and $(**)$ deserve some comment: The imprimitive distance-transitive orbital graph $(*)$ of $3.Suz.2$ having 5346 vertices is an antipodal triple cover of the primitive distance-transitive Suzuki graph of $Suz.2$, having 1782 vertices and intersection array $[416, 315; 1, 96]$, see [15]. Note that $(*)$ is not mentioned in [6, Ch.6.12], where a collection of non-bipartite antipodal distance-regular graphs is listed together with the problem to find more such graphs, which we hence have done. The intersection arrays of the primitive generalised orbital graph $(**)$ of $M_{22}.2$ having 672 vertices, and of its antipodal double cover having 1344 vertices, are marked as 'unproven' in the lists of feasible intersection arrays given in [6, Ch.14, p.430] and [6, Ch.14, p.424], respectively; assuming that these lists still reflect the current state of the art, the existence of such graphs hence is proven now.

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Table 5: Imprimitive distance-transitive orbital graphs.

G	H	n	r	intersection array		
M_{11}	A_6	22	3	$[20, 1; 1, 20]$	ba	[6, Ch.1.3]
$M_{12}.2$	M_{11}	24	3	$[12, 11; 1, 12]$	ba	[6, Ch.1.3]
$2.M_{12}$	M_{11}	24	3	$[22, 1; 1, 22]$	ba	[6, Ch.1.3]
$2.M_{12}.2$	M_{11}	48	5	$[12, 11, 6, 1; 1, 6, 11, 12]$	ba	[6, Ch.1.8]
$M_{22}.2$	$L_3(4)$	44	4	$[21, 20, 1; 1, 20, 21]$	ba	[6, Cor.1.5.4]
$M_{22}.2$	$2^4 : A_6$	154	6	$[16, 15, 12, 4, 1; 1, 4, 12, 15, 16]$	ba	[6, Ch.6.11]
$3.M_{22}.2$	$2^3 : L_3(2) \times 2$	990	9	$[7, 6, 4, 4, 4, 1, 1, 1; 1, 1, 1, 2, 4, 4, 6, 7]$	a	[6, Ch.6.12, 11.4.D]
HS	$U_3(5)$	352	4	$[175, 72, 1; 1, 72, 175]$	a	[6, Ch.11.4.E]
HS	$U_3(5)$	352	4	$[175, 102, 1; 1, 102, 175]$	a	[6, Ch.11.4.E]
$HS.2$	$U_3(5).2$	352	4	$[50, 49, 36; 1, 14, 50]$	b	[6, Ch.11.4.E]
$HS.2$	$U_3(5).2$	352	4	$[126, 125, 36; 1, 90, 126]$	b	[6, Ch.11.4.E]
$HS.2$	M_{22}	200	6	$[22, 21, 16, 6, 1; 1, 6, 16, 21, 22]$	ba	[6, Ch.6.11, 13.1]
$3.Suz.2$	$G_2(4).2$	5346	5	$[416, 315, 64, 1; 1, 32, 315, 416]$	a	$(*)$
Co_3	McL	552	4	$[275, 112, 1; 1, 112, 275]$	a	[6, Ch.11.4.H]
Co_3	McL	552	4	$[275, 162, 1; 1, 162, 275]$	a	[6, Ch.11.4.H]
$3.Fi'_{24}.2$	$Fi_{23} \times 2$	920808	5	$[31671, 28160, 2160, 1; 1, 1080, 28160, 31671]$	a	[6, Ch.6.12, 13.2]

Table 6: 'Interesting' distance-regular generalised orbital graphs.

G	H	n	r	intersection array		
M_{12}	$3^2.2.S_4$	220	5	$[27, 16, 7; 1, 4, 9]$	p	[6, Ch.6.1, 9.1]
M_{22}	$L_2(11)$	672	6	$[110, 81, 12; 1, 18, 90]$	p	(∗∗)
$M_{22}.2$	$L_2(11).2$	672	6	$[110, 81, 12; 1, 18, 90]$	p	(∗∗)
$M_{22}.2$	$L_2(11)$	1344	12	$[176, 135, 24, 1; 1, 24, 135, 176]$	a	(∗∗)
$M_{22}.2$	A_7	352	6	$[50, 49, 36; 1, 14, 50]$	b	[6, Ch.1.6]
$M_{22}.2$	A_7	352	6	$[126, 125, 36; 1, 90, 126]$	b	[6, Ch.1.6]
$3.M_{22}$	$2^3 : L_3(2)$	990	13	$[7, 6, 4, 4, 4, 1, 1, 1; 1, 1, 1, 2, 4, 4, 6, 7]$	a	[6, Ch.6.12, 11.4.D]
J_2	$2_-^{1+4} : A_5$	315	6	$[10, 8, 8, 2; 1, 1, 4, 5]$	p	[6, Ch.6.6, 13.6]
$J_2.2$	$3.A_6.2_2$	560	8	$[279, 150, 1; 1, 150, 279]$	a	[6, Ch.1.5]
$J_2.2$	$3.A_6.2_2$	560	8	$[279, 128, 1; 1, 128, 279]$	a	[6, Ch.1.5]
M_{24}	$L_3(4).3.2_2$	2024	5	$[63, 40, 19; 1, 4, 9]$	p	[6, Ch.6.1, 9.1]
$3.Suz$	$G_2(4)$	5346	7	$[416, 315, 64, 1; 1, 32, 315, 416]$	a	(∗)
$Fi_{22}.2$	$O_7(3)$	28160	6	$[1444, 1443, 1296; 1, 148, 1444]$	b	[6, Ch.1.6]
$Fi_{22}.2$	$O_7(3)$	28160	6	$[12636, 12635, 1296; 1, 11340, 12636]$	b	[6, Ch.1.6]
$3.Fi'_{24}$	Fi_{23}	920808	7	$[31671, 28160, 2160, 1; 1, 1080, 28160, 31671]$	a	[6, Ch.6.12, 13.2]

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