ONE-CLASS GENERA OF EXCEPTIONAL GROUPS OVER NUMBER FIELDS

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Abstract. We show that exceptional algebraic groups over number fields do not admit one-class genera of parahoric groups, except in the case G_2. For the group G_2, we enumerate all such one-class genera for the usual seven-dimensional representation.

1. Introduction

The enumeration of all one-class genera of definite quadratic forms has a long history. Over the rationals, Watson classified these genera in a long series of papers with some exceptions in dimensions 4 and 5 using some transformations which do not increase class numbers, see [Wat62]. His classification has recently been completed by D. Lorch and the author [KL13] using Watson's transformations and the explicit mass formula of Minkowski, Siegel and Smith. Recently, the author worked out the one-class genera of definite quadratic and hermitian forms over number fields, see [Kir16].

The purpose of this note is to enumerate the one-class genera of parahoric subgroups of the exceptional algebraic groups. This yields a new proof of the result of Kantor, Liebler and Tits [KLT87] for exceptional groups in characteristic 0. Instead of requiring chamber-transitivity on the associated affine building, our one-class hypothesis allows for significantly less transitivity. For groups of type G_2, we find several examples in addition to the one in [KLT87]. For the remaining exceptional groups, as in [KLT87], we prove that there are no examples even with our weaker hypothesis.

The paper is organized as follows. In Section 2, we recall some basic facts on parahoric subgroups of algebraic groups. In Section 3, we state Prasad's mass formula. In the last Section, we use his mass formula and obtain a list of all one-class genera of parahoric families in exceptional groups over number fields.

2. Preliminaries

Let k be a number field of degree n and let $\mathfrak{o}_k$ be its ring of integers. The set of all finite (infinite) places of k will be denoted by V_f (V_∞). For any $v \in V := V_f \cup V_\infty$, let k_v be the completion of k at v and let $\mathfrak{o}_{k_v}$ its ring of integers. Further, we will write $\mathfrak{f}_v$ for the residue class field of k_v and we set $q_v = \#\mathfrak{f}_v$.

Let G be an absolutely quasi-simple, simply connected algebraic group defined over k. We always assume that $\prod_{v\in V_\infty} G(k_v)$ is compact. Then k is totally real.

We are mostly interested in the exceptional groups, i.e. G will be a k-form of G_2, F_4, E_6, E_7, E_8 or a triality form of D_4, (cf. [Spr98, Chapter 17]).

We will also assume that G is a subgroup of GL_m for some m. Let L be an $\mathfrak{o}_k$-lattice in k^m , i.e. a finitely generated $\mathfrak{o}_k$-submodule of k^m of full rank. The

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schematic closure of G in the group scheme $\underline{\mathrm{GL}}(L)$ yields an integral group scheme $\underline{G}$, cf. [Bor63, CNP98].

Let $A = \{(\alpha_v)_{v\in V} \mid \alpha_v \notin \mathfrak{o}_{k_v}$ for only finitely many $v \in V_f\}$ be the adele ring of k. Suppose $\alpha \in G(A)$. Then $L \cdot \alpha$ denotes the $\mathfrak{o}_k$-lattice L' with $L'_v = L_v\alpha_v$ for all $v \in V_f$. Similarly, we define $\underline{G} \cdot \alpha$ to be the stabilizer of $L \cdot \alpha$ in $G(k)$. Then $(\underline{G} \cdot \alpha)(\mathfrak{o}_{k_v}) = \alpha_v^{-1}\underline{G}(\mathfrak{o}_{k_v})\alpha_v$ for all $v \in V_f$.

Definition 2.1. *Two integral forms $\underline{G}$ and $\underline{G}'$ of G are isomorphic if $\underline{G} \cdot \alpha = \underline{G}'$ for some $\alpha \in G(k)$. Similarly, they are said to be in the same genus if $\underline{G} \cdot \alpha = \underline{G}'$ for some $\alpha \in G(A)$.*

Let $C = \prod_{v\in V_\infty} G(k_v) \times \prod_{v\in V_f} \underline{G}(\mathfrak{o}_{k_v})$. Then $\alpha^{-1}\underline{G}\alpha$ is the stabilizer of $\underline{G} \cdot \alpha$ in $G(A)$. Thus

$$C\alpha G(k) \mapsto \underline{G} \cdot \alpha$$

induces a bijection between the double cosets $C\backslash G(A)/G(k)$ and the isomorphism classes in the genus of $\underline{G}$.

Lemma 2.2 ([CNP98, Proposition 3.3])**.** *Let $\underline{G}$ be an integral group scheme as above. Then $\underline{G}(\mathfrak{o}_{k_v})$ is a subgroup of finite index in a maximal compact subgroup of $G(k_v)$ and $\underline{G}(\mathfrak{o}_{k_v})$ is a hyperspecial maximal compact subgroup at all but finitely many places $v \in V_f$.*

The most important integral group schemes $\underline{G}$ are those for which $\underline{G}(\mathfrak{o}_{k_v})$ is a parahoric subgroup P_v of $G(k_v)$ at each finite place v. The genus of such a scheme is uniquely determined by the family $P = (P_v)_{v\in V_f}$. By the previous remark, P_v is hyperspecial almost everywhere. Such a family P is called *coherent* in [Pra89].

It is well known ([Bor63, Theorem 5.1]) that the genus of integral forms corresponding to P decomposes into finitely many isomorphism classes represented by $\underline{G}_1, \dots, \underline{G}_{\mathfrak{c}(P)}$ say.

Then the rational number $\mathfrak{M}(P) = \sum_{i=1}^{\mathfrak{c}(P)} (\#\underline{G}_i(\mathfrak{o}_k))^{-1}$ is called the *mass* of P. We clearly have $\mathfrak{c}(P) \geq \mathfrak{M}(P)$ and $\mathfrak{c}(P) = 1$ implies $\mathfrak{M}(P)^{-1} \in \mathbb{Z}$.

3. The mass formula

Let P be a coherent family of parahoric subgroups of G and let $\mathcal{G}$ be the unique quasi-split inner k-form of G. If $\mathcal{G}$ is of type ${}^6D_4$ (cf. [Spr98, Chapter 17.9]), let ℓ/k be a cubic extension contained in a Galois extension of k of degree 6 over which $\mathcal{G}$ splits. In all other cases let ℓ be the minimal extension of k over which $\mathcal{G}$ splits. The absolute values of the absolute discriminants of k and ℓ will be denoted by D_k and D_ℓ respectively. If $\mathcal{G}$ splits over k, let $s(\mathcal{G}) = 0$. Otherwise let $s(\mathcal{G})$ be the sum of the number of short roots and the number of short simple roots of the relative roots system of $\mathcal{G}$ over k. In particular, if $\mathcal{G}$ is a triality form of D_4, then $s(\mathcal{G}) = 7$ and if $\mathcal{G}$ is an outer form of E_6 then $s(\mathcal{G}) = 6$. For more details, see Section 0.4 of [Pra89].

We fix a family $\mathcal{P} = (\mathcal{P}_v)_{v\in V_f}$ of maximal parahoric subgroups of $\mathcal{G}$ such that $\mathcal{P}_v$ is hyperspecial (special) if $\mathcal{G}$ splits (does not split) over the maximal unramified extension of k_v and $\prod_{v\in V_\infty} \mathcal{G}(k_v) \times \prod_{v\in V_f} \mathcal{P}_v$ is an open subgroup of $G(A)$. See [Pra89, Section 1.2] for more details.

Let $\overline{\mathcal{G}}_v$ and $\overline{G}_v$ be the groups $\mathcal{G} \otimes_{\mathfrak{o}_k} \mathfrak{f}_v$ and $G_v \otimes_{\mathfrak{o}_k} \mathfrak{f}_v$. By [Tit79, Section 3.5], both these groups admit a Levi decomposition over $\mathfrak{f}_v$. Hence we may fix some maximal connected reductive $\mathfrak{f}_v$-subgroups $\overline{\mathcal{M}}_v$ and $\overline{\mathrm{M}}_v$ such that $\overline{\mathcal{G}}_v = \overline{\mathcal{M}}_v.R_u(\overline{\mathcal{G}}_v)$ and $\overline{G}_v = \overline{\mathrm{M}}_v.R_u(\overline{G}_v)$. Here R_u denotes the unipotent radical.

In his seminal paper [Pra89], Prasad gave the following explicit formula for $\mathfrak{M}(P)$.

Theorem 3.1 ([Pra89])**.**

$$\mathfrak{M}(P) = D_k^{\frac{1}{2}\dim G} \left(D_\ell / D_k^{[\ell:k]}\right)^{s(\mathcal{G})/2} \left(\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}}\right)^n \zeta(P)$$

where $m_1, \dots, m_r$ are the exponents of the simple, simply-connected compact real-analytic Lie group of the same type as $\mathcal{G}$ and $\zeta(P) = \prod_{v \in V_f} \zeta(P_v)$ with $\zeta(P_v) := \frac{q_v^{(\dim \overline{\mathrm{M}}_v + \dim \overline{\mathcal{M}}_v)/2}}{\# \overline{\mathrm{M}}_v(\mathfrak{f}_v)} > 1$.

For computational purposes, it is usually more convenient to express $\mathfrak{M}(P)$ in terms of $\mathfrak{M}(\mathcal{P})$ which is a product of special values of certain L-series of k. For $v \in V_f$ let

$$z(P_v) := \zeta(P_v)/\zeta(\mathcal{P}_v) = q_v^{(\dim \overline{\mathrm{M}}_v - \dim \overline{\mathcal{M}}_v)/2} \frac{\# \overline{\mathcal{M}}_v(\mathfrak{f}_v)}{\# \overline{\mathrm{M}}_v(\mathfrak{f}_v)} \,.$$

Then $\mathfrak{M}(P) = \mathfrak{M}(\mathcal{P}) \cdot \prod_{v \in V_f} z(P_v)$. Moreover, we have the following empirical fact.

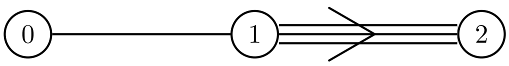
Lemma 3.2 ([PY12, 2.5])**.** *The correction factors $z(P_v)$ are integral.*

Proof. This follows from explicit computations using Bruhat-Tits theory. The groups of type A_n are discussed in [MG12, Lemma 2]. Without loss of generality, P_v is maximal parahoric. Since we are only interested in exceptional groups, we discuss the case ${}^3D_4$. The other cases are handled similarly. The comment after [PY12, 2.5] shows that the result holds whenever $G(k_v)$ contains a hyperspecial parahoric subgroup. So only the case that v ramifies in ℓ remains. Then G is of type G_2^1 (using the notation of [Tit79, Tables 4.2 and 4.3]). From [Ono66, Table 1] and [Pra89, (1.5)] we see that $q_v^{-\dim \overline{\mathcal{M}}_v/2} \overline{\mathcal{M}}_v(\mathfrak{f}_v) = q_v^{-1}(q_v^2 - 1)(q_v^6 - 1)$. The theory of Bruhat-Tits shows that every maximal parahoric subgroup of $G(k_v)$ is of type G_2, A_2 or $A_1 \times A_1$. Hence $q_v^{-\dim \overline{\mathrm{M}}_v/2} \overline{\mathrm{M}}_v(\mathfrak{f}_v)$ is either $q_v^{-1}(q_v^2-1)(q_v^6-1)$, $q_v^{-1}(q_v^2-1)^2$ or $q_v^{-1}(q_v^2-1)(q_v^3-1)$. In particular, $z(P_v)$ is integral. $\square$

4. The exceptional groups

4.1. **The case G_2.** Let $\mathbb{O}$ be the octonion algebra over k with totally definite norm form and denote by $\mathbb{O}^0$ its trace zero subspace. The automorphism group $\mathrm{Aut}(\mathbb{O})$ of $\mathbb{O}$, i.e. the stabilizer of the octonion multiplication in the special orthogonal group of $\mathbb{O}$ yields an algebraic group of type G_2 and $\mathbb{O}^0$ is an invariant subspace (cf. [SV00, Chapter 2]). Thus we obtain an algebraic group $G < \mathrm{GL}_7$ of type G_2. Further, the construction shows that $G(k_v)$ is of type G_2 for all finite places v.

The extended Dynkin diagram of G_2 is as follows.



By [Tit79, 3.5.2], the parahoric subgroups P_v of $G(k_v)$ are in one-to-one correspondence with the non-empty subsets of $\{0, 1, 2\}$. For any non-empty subset T of $\{0, 1, 2\}$ let P_v^T be the parahoric subgroup of $G(k_v)$ whose Dynkin diagram is obtained from the extended Dynkin diagram of G_2 by omitting the vertices in T. For example, $P_v^{\{0\}}$ is hyperspecial and $P_v^{\{2\}}$ is of type A_2.

Theorem 4.1. *Suppose P is a coherent family of parahoric subgroups of G such that $\mathfrak{c}(P) = 1$. Then $k = \mathbb{Q}$ and P_p is hyperspecial for all primes $p \notin \{2, 3, 5\}$. The possible combinations (T_2, T_3, T_5) such that $P_p = P_p^{T_p}$ for $p \in \{2, 3, 5\}$ are given in Table 1.*

T_2	T_3	T_5	$\mathfrak{M}(P)^{-1}$	$\underline{G}(\mathbb{Z})$	*sgdb*
$\{0\}$	$\{0\}$	$\{0\}$	$2^6\cdot 3^3\cdot 7$	$G_2(2)$	–
$\{0\}$	$\{0\}$	$\{2\}$	$2^5\cdot 3$	$(C_4\times C_4).S_3$	64
$\{0\}$	$\{2\}$	$\{0\}$	$2^4\cdot 3^3$	$3_+^{1+2}.\mathrm{QD}_{16}$	520
$\{2\}$	$\{0\}$	$\{0\}$	$2^6\cdot 3\cdot 7$	$2^3.\mathrm{GL}_3(2)$	814
$\{2\}$	$\{2\}$	$\{0\}$	$2^4\cdot 3$	$\mathrm{GL}_2(3)$	29
$\{1\}$	$\{0\}$	$\{0\}$	$2^6\cdot 3^2$	$2_+^{1+4}.((C_3\times C_3).2)$	8282[1]
$\{1,2\}$	$\{0\}$	$\{0\}$	$2^6\cdot 3$	$2_+^{1+4}.S_3$	1494
$\{0,2\}$	$\{0\}$	$\{0\}$	$2^6\cdot 3$	$((C_4\times C_4).2).S_3$	956
$\{0,1\}$	$\{0\}$	$\{0\}$	$2^6\cdot 3$	$2_+^{1+4}.S_3$	988
$\{0,1,2\}$	$\{0\}$	$\{0\}$	2^6	$\mathrm{Syl}_2(G_2(2))$	134

TABLE 1. The one-class genera of G_2.

The last column of Table 1 gives the label of the group $\underline{G}(\mathbb{Z})$ in the list of all groups of order $\mathfrak{M}(P)^{-1} = \#\underline{G}(\mathbb{Z})$ as defined by the small group database [BEO01].

Proof. Using the notation of Section 3, we have $\ell = k$, $r = 2$, $(m_1, m_2) = (1, 5)$ and $\dim G = r + 2(m_1 + m_2) = 14$. Thus Theorem 3.1 shows

$$\mathfrak{c}(P) \geq D_k^7 \left(\frac{15}{32\pi^8}\right)^n .$$

Hence $\mathfrak{c}(P) = 1$ implies

$$D_k^{1/n} \leq \left(\frac{32\pi^8}{15}\right)^{1/7} < 4.123\,.$$

Voight's tables [Voi08] now show that k is one of $\mathbb{Q}$, $\mathbb{Q}(\sqrt{d})$ with $d \in \{2, 3, 5, 13\}$ or the maximal totally real subfield $\mathbb{Q}(\theta_7)$ of the seventh cyclotomic field $\mathbb{Q}(\zeta_7)$.

The assumption $\mathfrak{c}(P) = 1$ forces $\mathfrak{M}(P)^{-1} \in \mathbb{Z}$. Hence $\mathfrak{M}(\mathcal{P})^{-1} \in \mathbb{Z}$ by Lemma 3.2. The exact values of $\mathfrak{M}(\mathcal{P}) = 2^{-2n}\zeta_k(-1)\zeta_k(-5)$ for the various possible base fields k is given in the following table.

k	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{13})$	$\mathbb{Q}(\theta_7)$
$\mathfrak{M}(\mathcal{P})$	$\frac{1}{2^6\cdot 3^3\cdot 7} = \frac{1}{\#G_2(2)}$	$\frac{361}{48384}$	$\frac{1681}{12096}$	$\frac{67}{302400}$	$\frac{33463}{157248}$	$\frac{7393}{84672}$

This shows that $k = \mathbb{Q}$ as claimed. For any given prime p, the local correction factor $z(P_p)$ is given by the following table.

root system of P_p	$\emptyset$	A_1	A_2	$A_1\times A_1$	G_2
$z(P_p)$	$p^8 - p^6 - p^2 + 1$	$p^6 - 1$	$p^3 + 1$	$p^4 + p^2 + 1$	1

If $p \geq 23$ then $\#G_2(2)\cdot(p^3+1) > 1$ and therefore P_p is hyperspecial. For $p < 23$ we can simply check all possible combinations of P_p which yield $\mathfrak{M}(P)^{-1} \in \mathbb{Z}$. This yields precisely the claimed combinations.

Let B be an Iwahori subgroup of G. The set of all $\mathbb{Z}_pB$-invariant lattices in $\mathbb{O}_p^0$ have been worked out in [CNP98]. For each candidate P, Theorem 4.1 of [CNP98] yields a lattice L in $\mathbb{O}^0$ such that the stabilizer $\underline{G}(\mathbb{Z})$ of L in $G_2 < \mathrm{GL}(\mathbb{O}^0)$ is of type P. One checks that $\mathfrak{M}(P)^{-1} = \#\underline{G}(\mathbb{Z})$ in all cases. □

[1]The group is isomorphic to a index 2 subgroup of the automorphism group of the root lattice $\mathbb{F}_4$.

Let P be the parahoric family corresponding to the last entry of Table 1. Then P_2 is an Iwahori subgroup of G, i.e. the stabilizer of a chamber in the affine building B of $G_2(\mathbb{Q}_2)$. This family P yields the chamber-transitive action of $G_2(\mathbb{Z}[1/2])$ on B from [Kan85] and [KLT87, case (iii)].

4.2. **The case F_4.**

Proposition 4.2. *Suppose G is of type F_4. Then there exists no coherent family P of parahoric subgroups of G with class number one.*

Proof. We have $r = 4$, $(m_1, \dots, m_4) = (1, 5, 7, 11)$ and $\dim G = r + 2\sum_i m_i = 52$. Thus Theorem 3.1 shows

$$\mathfrak{c}(P) \geq D_k^{26} \left(\frac{736745625}{8192\pi^{28}} \right)^n .$$

Hence $\mathfrak{c}(P) = 1$ implies

$$D_k^{1/n} \leq \left(\frac{8192\pi^{28}}{736745625} \right)^{1/26} < 2.213 < \sqrt{5} .$$

Hence $k = \mathbb{Q}$. But for $k = \mathbb{Q}$ we have

$$\mathfrak{M}(\mathcal{P}) = \frac{736745625}{8192\pi^{28}} \cdot \prod_{i=1}^{4} \zeta_{\mathbb{Q}}(m_i + 1) = \frac{1}{4} \prod_{i=1}^{4} \zeta_{\mathbb{Q}}(-m_i) = \frac{691}{2^{15} 3^6 5^2 7^2 13} .$$

In particular, $\mathfrak{M}(P)^{-1} \notin \mathbb{Z}$. □

If $k = \mathbb{Q}$, then $\mathcal{P}$ is the model in the sense of Gross and it actually has class number 2 (see [Gro96, Proposition 5.3]).

4.3. **Triality forms of D_4.** Let G be of type ${}^3D_4$ or ${}^6D_4$. The field ℓ is a totally real cubic extension of k. The extension is normal (and thus cyclic) if and only if G is of type ${}^3D_4$.

Lemma 4.3. *Suppose G is a k-form of D_4 and P a parahoric family of G with class number one. Then the base field k is either $\mathbb{Q}$, $\mathbb{Q}(\sqrt{d})$ with $d \in \{2, 3, 5, 13, 17\}$ or the maximal totally real subfield $\mathbb{Q}(\theta_e)$ of $\mathbb{Q}(\zeta_e)$ for $e \in \{7, 9\}$.*

Proof. If G is any form of D_4, then $r = 4$, $(m_1, \dots, m_4) = (1, 3, 3, 5)$ and $\dim G = r + 2\sum_i m_i = 28$. Thus Theorem 3.1 shows that

$$\mathfrak{c}(P) \geq \mathfrak{M}(P) \geq D_k^{14} \left(\frac{135}{2^{11}\pi^{16}} \right)^n .$$

Hence $\mathfrak{c}(P) = 1$ implies

$$D_k^{1/n} \leq \left(\frac{2^{11}\pi^{16}}{135} \right)^{1/14} < 4.493 .$$

The result follows from Voight's tables of totally real number fields [Voi08]. □

Given a finite place $v \in V_f$ let $\ell_v = \ell \otimes_k k_v$. By [Tit79, Section 4], the type of G at v is (using the notation of [Tit79, Tables 4.2 and 4.3])

$$\begin{cases} {}^1D_4 & \text{if } v \text{ is completely split in } \ell, \\ {}^3D_4 & \text{if } \ell_v/k_v \text{ is an unramified cubic field extension,} \\ G_2^1 & \text{if } \ell_v/k_v \text{ is a ramified cubic field extension,} \\ {}^2D_4 & \text{if } \ell_v \cong k_v \oplus m_v \text{ for some unramified quadratic extension } m_v/k_v, \\ B\!-\!C_3 & \text{if } \ell_v \cong k_v \oplus m_v \text{ for some ramified quadratic extension } m_v/k_v. \end{cases}$$

Therefore $\zeta(\mathcal{P}_v) = \left(1 - \frac{1}{q_v^2}\right)\left(1 - \frac{1}{q_v^6}\right) \cdot \lambda_v$ where λ_v is given by

$$\begin{cases} \left(1 - \frac{1}{q_v^4}\right)^2 & \text{if } v \text{ is completely split in } \ell, \\ 1 + \frac{1}{q_v^4} + \frac{1}{q_v^8} & \text{if } \ell_v/k_v \text{ is an unramified cubic field extension,} \\ 1 & \text{if } \ell_v/k_v \text{ is a ramified cubic field extension,} \\ \left(1 + \frac{1}{q_v^4}\right)\left(1 - \frac{1}{q_v^4}\right) & \text{if } \ell_v = k_v \oplus m_v \text{ for some unramified extension } m_v/k_v, \\ 1 - \frac{1}{q_v^4} & \text{if } \ell_v = k_v \oplus m_v \text{ for some ramified extension } m_v/k_v. \end{cases}$$

Using the functional equation for L-series, we obtain

$$\mathfrak{M}(\mathcal{P}) = 2^{-4n} \cdot |\zeta_k(-1)\zeta_{\ell/k}(-3)\zeta_k(-5)| \tag{1}$$

(see also [PY12, Section 2.8]).

Proposition 4.4. *If G is of type ${}^3D_4$ or ${}^6D_4$ then there exists no coherent parahoric family of class number one.*

Proof. If G admits a one-class parahoric family P, then

$$1 \geq \mathfrak{M}(P) \geq \mathfrak{M}(\mathcal{P}) > D_k^{7/2} D_\ell^{7/2} \left(\frac{135}{2^{11}\pi^{16}}\right)^n$$

or equivalently, $D_\ell \leq D_k^{-1} \cdot \left(\frac{2^{11}\pi^{16}}{135}\right)^{2n/7}$. By Lemma 4.3, there are only finitely many candidates for k. For each such field k, [Voi08] lists all possible cubic extensions ℓ that satisfy the previous inequality. We only find the possibility $k = \mathbb{Q}$ and $\ell = k[x]/(f(x))$ where $f(x)$ is one of the ten polynomials given below. In each case, we can now evaluate $\mathfrak{M}(\mathcal{P})$ explicitly using equation (1).

$f(x)$	$\mathfrak{M}(\mathcal{P})$
$x^3 - x^2 - 2x + 1$	79/84672
$x^3 - 3x - 1$	199/36288
$x^3 - x^2 - 3x + 1$	577/12096
$x^3 - x^2 - 4x - 1$	11227/157248
$x^3 - 4x - 1$	1333/6048
$x^3 - x^2 - 4x + 3$	1891/6048
$x^3 - x^2 - 4x + 2$	2185/3024
$x^3 - x^2 - 4x + 1$	925/1344
$x^3 - x^2 - 6x + 7$	4087/4032
$x^3 - x^2 - 5x - 1$	19613/12096

The result now follows from the fact that $\mathfrak{M}(P)$ is an integral multiple of $\mathfrak{M}(\mathcal{P})$ and therefore never the reciprocal of an integer. $\square$

4.4. **The case E_6.** Let G be a form of E_6. The assumption that $G(k_v)$ is anisotropic for all $v \in V_\infty$ forces G to be of type ${}^2E_6$, see for example [Gro96, Proposition 2.2] for details. Thus the splitting field ℓ of G is a totally complex quadratic extension of k.

Proposition 4.5. *There exists no coherent family P of parahoric subgroups of G with class number one.*

Proof. We have $r = 6$, $(m_1, \ldots, m_6) = (1, 4, 5, 7, 8, 11)$, $s(\mathcal{G}) = 26$ and $\dim G = 78$. Suppose P is a parahoric family of class number one. Then Theorem 3.1 implies

$$1 = \mathfrak{c}(P) \geq \mathfrak{M}(P) > D_k^{39} \cdot (D_\ell/D_k^2)^{13} \cdot \gamma^n \geq D_k^{39} \cdot \gamma^n \text{ where } \gamma := \prod_{i=1}^{6} \frac{m_i!}{(2\pi)^{m_i+1}}$$

and therefore $D_k^{1/n} < \gamma^{-1/39} < 2.31$. Hence k is either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$.

Suppose $k = \mathbb{Q}(\sqrt{5})$. Since the narrow class group of k is trivial, the extension ℓ/k ramifies at a finite place. Thus $D_\ell/D_k^2 \geq 4$ and hence $\mathfrak{c}(P) > 5^{39} \cdot 4^{13} \cdot \gamma^2 > 1$.

So we may suppose that $k = \mathbb{Q}$. Then $1 = \mathfrak{c}(P) > D_\ell^{13} \cdot \gamma$ implies that $D_\ell \leq 12$. Thus ℓ is $\mathbb{Q}(\sqrt{-d})$ for some $d \in \{1, 2, 3, 7, 11\}$. For any $v \in V_f$, the group G is quasi-split over k_v. Moreover, the type of G over k_v is ${}^1E_6$, ${}^2E_6$ or F_4^I (using the notation of [Tit79, Section 4]) depending on whether v is split, inert or ramified in ℓ. Thus $\zeta(\mathcal{P}_v)^{-1}$ equals

$$(1-q_v^{-2})(1-q_v^{-6})(1-q_v^{-8})(1-q_v^{-12}) \cdot \begin{cases} (1-q_v^{-5})(1-q_v^{-9}) & \text{if } v \text{ is split in } \ell, \\ (1+q_v^{-5})(1+q_v^{-9}) & \text{if } v \text{ is inert in } \ell, \\ 1 & \text{if } v \text{ is ramified in } \ell. \end{cases}$$

Using the functional equation for zeta functions, we obtain

$$\mathfrak{M}(\mathcal{P}) = 2^{-6} \cdot |\zeta_{\mathbb{Q}}(-1)\zeta_{\ell/\mathbb{Q}}(-4)\zeta_{\mathbb{Q}}(-5)\zeta_{\mathbb{Q}}(-7)\zeta_{\ell/\mathbb{Q}}(-8)\zeta_{\mathbb{Q}}(-11)|\,.$$

The values for $\mathfrak{M}(\mathcal{P})$ for all possible fields $\ell = \mathbb{Q}(\sqrt{-d})$ are

d	1	2	3	7	11
$\mathfrak{M}(\mathcal{P})$	$\frac{191407}{243465191424}$	$\frac{1097308691}{169073049600}$	$\frac{559019}{30813563289600}$	$\frac{6102221}{5200977600}$	$\frac{7340406625}{18598035456}$

In particular, there exists no parahoric family P such that $\mathfrak{M}(P)^{-1} \in \mathbb{Z}$. $\square$

4.5. **The case E_7.**

Proposition 4.6. *If G is of type E_7 then there exists no coherent family P of parahoric subgroups of G with class number one.*

Proof. If G is of type E_7 then $r = 7$, $(m_1, \ldots, m_7) = (1, 5, 7, 9, 11, 13, 17)$ and $\dim G = 133$. If $\mathfrak{c}(P) = 1$, then Theorem 3.1 implies that

$$D_k^{1/n} < \left(\prod_{i=1}^{7} \frac{(2\pi)^{m_i+1}}{m_i!} \right)^{2/133} < 1.547 < \sqrt{5}\,.$$

Thus $k = \mathbb{Q}$. But then

$$\mathfrak{M}(\mathcal{P}) = 2^{-7} \prod_{i=1}^{7} |\zeta_{\mathbb{Q}}(-m_i)| = \frac{691 \cdot 43867}{2^{24}3^{11}5^2 7^3 11^1 13^1 19^1}$$

shows that $\mathfrak{c}(P) > 1$ for all parahoric families P. $\square$

4.6. **The case E_8.**

Proposition 4.7. *If G is of type E_8 and P is a coherent family of parahoric subgroups of G then $\mathfrak{c}(P) \geq 8435$.*

Proof. If G is of type E_8 then $r = 8$ and $(m_1, \ldots, m_8) = (1, 7, 11, 13, 17, 19, 23, 29)$. Thus Theorem 3.1 implies that

$$\mathfrak{c}(P) \geq \mathfrak{M}(P) > \prod_{i=1}^{8} \frac{m_i!}{(2\pi)^{m_i+1}} > 8434\,.$$

$\square$

References

[BEO01] H. U. Besche, B. Eick, and E. A. O'Brien. The groups of order at most 2000. *Electron. Res. Announc. Amer. Math. Soc.*, 7:1–4, 2001.

[Bor63] A. Borel. Some finiteness properties of adele groups over number fields. *Publ. Math. I.H.E.S.*, 16:5–30, 1963.

[CNP98] A. Cohen, G. Nebe, and W. Plesken. Maximal integral forms of the algebraic group G_2 defined by finite subgroups. *J. Number Theory*, 72(2):282–308, 1998.

[Gro96] B. H. Gross. Groups over $\mathbf{Z}$. *Invent. Math.*, 124(1-3):263–279, 1996.

[Kan85] W. M. Kantor. Some exceptional 2-adic buildings. *J. Algebra*, 92(1):208–223, 1985.

[Kir16] M. Kirschmer. *Definite quadratic and hermitian form with small class number*. Habilitation, RWTH Aachen University, 2016.

[KL13] M. Kirschmer and D. Lorch. Single-class genera of positive integral lattices. *LMS J. Comput. Math.*, 16:172–186, 2013.

[KLT87] W. M. Kantor, R. A. Liebler, and J. Tits. On discrete chamber-transitive automorphism groups of affine buildings. *Bull. Amer. Math. Soc. (N.S.)*, 16(1):129–133, 1987.

[MG12] A. Mohammadi and A. S. Golsefidy. Discrete subgroups acting transitively on vertices of a Bruhat-Tits building. *Duke Mathematical Journal*, 161(3):483–544, 2012.

[Ono66] T. Ono. On algebraic groups and discontinuous groups. *Nagoya Math. J.*, 27:279–322, 1966.

[Pra89] G. Prasad. Volumes of S-arithmetic quotients of semi-simple groups. *Inst. Hautes Études Sci. Publ. Math.*, 69:91–117, 1989.

[PY12] G. Prasad and S.-K. Yeung. Nonexistence of arithmetic fake compact Hermitian symmetric spaces of type other than A_n ($n \leq 4$). *J. Math. Soc. Japan*, 64(3):683–731, 2012.

[Spr98] T. A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1998.

[SV00] T. A. Springer and F. D. Veldkamp. *Octonions, Jordan Algebras and Exceptional Groups*. Springer Monographs in Mathematics. Springer-Verlag, 2000.

[Tit79] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math.)*, volume 33, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.

[Voi08] J. Voight. Enumeration of totally real number fields of bounded root discriminant. In A. van der Poorten and A. Stein, editors, *Algorithmic number theory (ANTS VIII, Banff, 2008)*, volume 5011 of *Lecture Notes in Comp. Sci.*, pages 268–281. Springer, Berlin, 2008. See `http://www.math.dartmouth.edu/~jvoight/nf-tables/index.html`.

[Wat62] G. L. Watson. Transformations of a quadratic form which do not increase the class-number. *Proc. Lond. Math. Soc.*, 12:577–587, 1962.

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