ONE CLASS GENERA OF LATTICE CHAINS OVER NUMBER FIELDS

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Abstract. We classify all one-class genera of admissible lattice chains of length at least 2 in hermitian spaces over number fields. If L is a lattice in the chain and $\mathfrak{p}$ the prime ideal dividing the index of the lattices in the chain, then the $\{\mathfrak{p}\}$-arithmetic group $\mathrm{Aut}(L_{\{\mathfrak{p}\}})$ acts chamber transitively on the corresponding Bruhat-Tits building. So our classification provides a step forward to a complete classification of these chamber transitive groups which has been announced 1987 (without a detailed proof) by Kantor, Liebler and Tits. In fact we find all their groups over number fields and one additional building with a discrete chamber transitive group (see Table 1).

1. Introduction

Kantor, Liebler and Tits [11] classified discrete groups Γ with a type preserving chamber transitive action on the affine building $\mathcal{B}^+$ of a simple adjoint algebraic group of relative rank $r \geq 2$. Such groups are very rare and hence this situation is an interesting phenomenon. Except for two cases in characteristic 2 ([11, case (v)]) and the exceptional group $\mathrm{G}_2(\mathbb{Q}_2)$ ([11, case (iii)], Section 5.4) the groups arise from classical groups U_p over $\mathbb{Q}_p$ for $p = 2, 3$. Moreover Γ is a subgroup of the S-arithmetic group $\Gamma_{max} := \mathrm{Aut}(L \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}])$ (so $S = \{p\}$) for a suitable lattice L in some hermitian space (V, Φ) and $\mathrm{U}_p = \mathrm{U}(V_p, \Phi)$ is the completion of the unitary group $\mathrm{U}(V, \Phi)$ (see Remark 2.3). This paper uses the classification of one- and two-class genera of hermitian lattices in [16] to obtain these S-arithmetic groups Γ_{max}.

Instead of the thick building $\mathcal{B}^+$ we start with the affine building $\mathcal{B}$ of admissible lattice chains as defined in [1]. The points in the building $\mathcal{B}$ correspond to homothety classes of certain $\mathbb{Z}_p$-lattices in V_p. The lattices form a simplex in $\mathcal{B}$, if and only if representatives in these classes can be chosen to form an admissible chain of lattices in V_p. In particular the maximal simplices of $\mathcal{B}$ (the so called chambers) correspond to the fine admissible lattice chains in V_p (for the thick building $\mathcal{B}^+$ one might have to apply the oriflamme construction as explained in Remark 4.6).

Any fine admissible lattice chain $\mathcal{L}_p$ in V_p arises as the completion of a lattice chain $\mathcal{L}'$ in (V, Φ). After rescaling and applying the reduction operators from Section 2.3 we obtain a fine p-admissible lattice chain $\mathcal{L} = (L_0, \ldots, L_r)$ in (V, Φ) (see Definition 3.4) such that $\mathrm{Aut}(\mathcal{L}) \supseteq \mathrm{Aut}(\mathcal{L}')$ and such that the completion of $\mathcal{L}$ at p is $\mathcal{L}_p$. The S-arithmetic group $\mathrm{Aut}(L_0 \otimes \mathbb{Z}[\frac{1}{p}]) = \mathrm{Aut}(L_i \otimes \mathbb{Z}[\frac{1}{p}]) =: \mathrm{Aut}(\mathcal{L} \otimes \mathbb{Z}[\frac{1}{p}])$ contains $\mathrm{Aut}(\mathcal{L}' \otimes \mathbb{Z}[\frac{1}{p}])$. Therefore we call this group closed.

The closed $\{p\}$-arithmetic group $\mathrm{Aut}(L_0 \otimes \mathbb{Z}[\frac{1}{p}])$ acts chamber transitively on $\mathcal{B}$, if the lattice L_0 represents a genus of class number one and $\mathrm{Aut}(L_0)$ acts transitively on the fine flags of (isotropic) subspaces in the hermitian space $\overline{L_0}$ (see Theorem 4.4). If we only impose chamber transitivity on the thick building $\mathcal{B}^+$, then we also have to take two-class genera of lattices L_0 into account. To obtain a complete classification of all chamber transitive actions of closed S-arithmetic groups on the thick building $\mathcal{B}^+$ using this strategy there are two ingredients missing:

(a) By Theorem 4.9 we need the still unknown classification of proper special genera of lattices L_0 with class number one (see also Proposition 4.5 and [27, Proposition 1]).
(b) We should also include the skew hermitian forms over quaternion algebras for which a classification of one-class genera is still unknown.

Already taking only the one-class genera of lattices L_0 we find all the groups from [11] and one additional case (described in Proposition 5.3 (1)). Hence our computations correct an omission in the classification of [11]. A list of the corresponding buildings and groups U_p is given in Section 6.

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2. Lattices in hermitian spaces

Let K be a number field. Further, let E/K be a field extension of degree at most 2 or let E be a quaternion skewfield over K. The canonical involution of E/K will be denoted by $\sigma\colon E \to E$. In particular, K is the fixed field of σ and hence the involution σ is the identity if and only if $K = E$. A hermitian space over E is a finitely generated (left) vector space V over E equipped with a non-degenerate sesquilinear form $\Phi\colon V \times V \to E$ such that

- $\Phi(x + x', y) = \Phi(x, y) + \Phi(x', y)$ for all $x, x', y \in V$.
- $\Phi(\alpha x, \beta y) = \alpha\Phi(x, y)\sigma(\beta)$ for all $x, y \in V$ and $\alpha, \beta \in E$.
- $\Phi(y, x) = \sigma(\Phi(x, y))$ for all $x, y \in V$.

The unitary group $\mathrm{U}(V, \Phi)$ of Φ is the group of all E-linear endomorphisms of V that preserve the hermitian form Φ. The *special unitary group* is defined as

$$\mathrm{SU}(V, \Phi) := \{g \in \mathrm{U}(V, \Phi) \mid \det(g) = 1\}$$

if E is commutative and $\mathrm{SU}(V, \Phi) := \mathrm{U}(V, \Phi)$ if E is a quaternion algebra.

We denote by $\mathbb{Z}_K$ the ring of integers of the field K and we fix some maximal order $\mathcal{M}$ in E. Further, let d be the dimension of V over E.

Definition 2.1. *An* $\mathcal{M}$-lattice *in V is a finitely generated $\mathcal{M}$-submodule of V that contains an E-basis of V. If L is an $\mathcal{M}$-lattice in V then its* automorphism group *is*

$$\mathrm{Aut}(L) := \{g \in \mathrm{U}(V, \Phi) \mid Lg = L\}.$$

2.1. **Completion of lattices and groups.** Let $\mathfrak{P}$ be a maximal two sided ideal of $\mathcal{M}$ and let $\mathfrak{p} = \mathfrak{P} \cap K$. The completion $\mathrm{U}_\mathfrak{p} := \mathrm{U}(V \otimes_K K_\mathfrak{p}, \Phi)$ is an algebraic group over the $\mathfrak{p}$-adic completion $K_\mathfrak{p}$ of K.

Let $L \le V$ be some $\mathcal{M}$-lattice in V. We define the $\mathfrak{p}$-adic completion of L as $L_\mathfrak{p} := L \otimes_{\mathbb{Z}_K} \mathbb{Z}_{K_\mathfrak{p}}$ and we let

$$L(\mathfrak{p}) := \{X \le V \mid X_\mathfrak{q} = L_\mathfrak{q} \text{ for all prime ideals } \mathfrak{q} \ne \mathfrak{p}\},$$

be the set of all $\mathcal{M}$-lattices in V whose $\mathfrak{q}$-adic completion coincides with the one of L for all prime ideals $\mathfrak{q} \ne \mathfrak{p}$.

Remark 2.2. By the local global principle, given a lattice X in $V_\mathfrak{p}$, there is a unique lattice $M \in L(\mathfrak{p})$ with $M_\mathfrak{p} = X$.

To describe the groups $\mathrm{U}_\mathfrak{p}$ in the respective cases, we need some notation: Let R be one of $E, K, \mathbb{Z}_K, \mathcal{M}$ or a suitable completion. A hermitian module $\mathbb{H}(R)$ with R-basis (e, f) satisfying $\Phi(e, f) = 1, \Phi(e, e) = \Phi(f, f) = 0$ is called a *hyperbolic plane*. By [18, Theorem (2.22)] any hermitian space over E is either anisotropic (i.e. $\Phi(x, x) \neq 0$ for all $x \neq 0$) or it has a hyperbolic plane as an orthogonal direct summand.

Remark 2.3. In our situation the following cases are possible:

- $E = K$: Then $(V \otimes_K K_\mathfrak{p}, \Phi)$ is a quadratic space and hence isometric to $\mathbb{H}(K_\mathfrak{p})^r \perp (V_0, \Phi_0)$ with (V_0, Φ_0) anisotropic. The rank of $\mathrm{U}_\mathfrak{p}$ is r. The group that acts type preservingly on the thick Bruhat-Tits building $\mathcal{B}^+$ defined in Section 4.3 is
$$\mathrm{U}_\mathfrak{p}^+ := \{g \in \mathrm{U}_\mathfrak{p} \mid \det(g) = 1, \theta(g) \in K^2\}$$
the subgroup of the special orthogonal group with trivial spinor norm θ.
- $\mathfrak{P} \neq \sigma(\mathfrak{P})$. Then $E \otimes_K K_\mathfrak{p} \cong K_\mathfrak{p} \oplus K_\mathfrak{p}$ where the involution interchanges the two components and $\mathrm{U}_\mathfrak{p} \cong \mathrm{GL}_d(K_\mathfrak{p})$ has rank $r = d - 1$. As $\mathfrak{P}$ is assumed to be a maximal 2-sided ideal of $\mathcal{M}$, the case that E is a quaternion algebra is not possible here. Here we let
$$\mathrm{U}_\mathfrak{p}^+ = \{g \in \mathrm{U}_\mathfrak{p} \mid \det(g) = 1\} = \mathrm{SL}_d(K_\mathfrak{p}).$$
- $[E : K] = 4$ and $\mathfrak{P} = \mathfrak{p}\mathcal{M}$. Then $E_\mathfrak{p} \cong K_\mathfrak{p}^{2\times 2}$ and for $x \in E_\mathfrak{p}$, $\sigma(x)$ is simply the adjugate of x as $\sigma(x)x \in K$. Let $e^2 = e \in E_\mathfrak{p}$ such that $\sigma(e) = 1-e$. Then $V_\mathfrak{p} = eV_\mathfrak{p} \bigoplus (1 - e)V_\mathfrak{p}$. The hermitian form Φ gives rise to a skew-symmetric form
$$\begin{aligned}\Psi\colon eV_\mathfrak{p} \times eV_\mathfrak{p} &\to eE_\mathfrak{p}(1 - e) \cong K_\mathfrak{p},\\ (ex, ey) &\mapsto \Phi(ex, ey) = e\Phi(x, y)(1 - e)\,.\end{aligned}$$
From $E_\mathfrak{p} = E_\mathfrak{p} e E_\mathfrak{p}$ we conclude that $V_\mathfrak{p} = E_\mathfrak{p} e E_\mathfrak{p} V$. Hence we can recover the form Φ from Ψ and thus $\mathrm{U}_\mathfrak{p} \cong \mathrm{U}(eV, \Psi) \cong \mathrm{Sp}_{2d}(K_\mathfrak{p})$ has rank $r = d$. Here the full group $\mathrm{U}_\mathfrak{p}$ acts type preservingly on $\mathcal{B}^+$ and we put $\mathrm{U}_\mathfrak{p}^+ := \mathrm{U}_\mathfrak{p}$.
- In the remaining cases $E \otimes K_\mathfrak{p} = E_\mathfrak{P}$ is a skewfield, which is ramified over $K_\mathfrak{p}$ if and only if $\mathfrak{P}^2 = \mathfrak{p}\mathcal{M}$. In all cases $\mathrm{U}_\mathfrak{p}$ is isomorphic to a unitary group over $E_\mathfrak{P}$. Hence it admits a decomposition $\mathbb{H}(E_\mathfrak{P})^r \perp (V_0, \Phi_0)$ with (V_0, Φ_0) anisotropic where r is the rank of $\mathrm{U}_\mathfrak{p}$. If $E_\mathfrak{p}$ is commutative, we define
$$\mathrm{U}_\mathfrak{p}^+ := \{g \in \mathrm{U}_\mathfrak{p} \mid \det(g) = 1\} = \mathrm{SU}_\mathfrak{p}$$
and put $\mathrm{U}_\mathfrak{p}^+ = \mathrm{SU}_\mathfrak{p} := \mathrm{U}_\mathfrak{p}$ in the non-commutative case.

2.2. **The genus of a lattice.** To shorten notation, we introduce the adelic ring $A = A(K) = \prod_v K_v$ where v runs over the set of all places of K. We denote the adelic unitary group of the $A \otimes_K E$-module $V_A = A \otimes_K V$ by $\mathrm{U}(V_A, \Phi)$. The normal subgroup
$$\mathrm{U}^+(V_A, \Phi) := \{(g_\mathfrak{p})_\mathfrak{p} \in \mathrm{U}(V_A, \Phi) \mid g_\mathfrak{p} \in \mathrm{U}_\mathfrak{p}^+\} \leq \mathrm{U}(V_A, \Phi)$$
is called the special adelic unitary group.

The adelic unitary group acts on the set of all $\mathcal{M}$-lattices in V by letting $Lg = L'$ where L' is the unique lattice in V such that its $\mathfrak{p}$-adic completion $(L')_\mathfrak{p} = L_\mathfrak{p} g_\mathfrak{p}$ for all maximal ideals $\mathfrak{p}$ of $\mathbb{Z}_K$.

Definition 2.4. *Let L be an $\mathcal{M}$-lattice in V. Then*
$$\mathrm{genus}(L) := \{Lg \mid g \in \mathrm{U}(V_A, \Phi)\}$$

is called the genus *of* L.
Two lattices L *and* M *are said to be* isometric *(respectively* properly isometric*), if* $L = Mg$ *for some* $g \in \mathrm{U}(V,\Phi)$ *(resp.* $g \in \mathrm{SU}(V,\Phi)$*).*
Two lattices L *and* M *are said to be in the same* proper special genus*, if there exist* $g \in \mathrm{SU}(V,\Phi)$ *and* $h \in \mathrm{U}^+(V_A,\Phi)$ *such that* $Lgh = M$*. The proper special genus of* L *will be denoted by* $\mathrm{genus}^+(L)$*.*

Let L be an $\mathcal{M}$-lattice in V. It is well known that $\mathrm{genus}(L)$ is a finite union of isometry classes, c.f. [4, Theorem 5.1]. The number of isometry classes in $\mathrm{genus}(L)$ is called the class number $h(L)$ of (the genus of) L. Similarly the proper special genus is a finite union of proper isometry classes, the proper class number will be denoted by $h^+(L)$.

2.3. **Normalised genera.**

Definition 2.5. *Let* L *be an* $\mathcal{M}$*-lattice in* V*. Then* $L^\# = \{x \in V \mid \Phi(x,L) \subseteq \mathcal{M}\}$ *is called the* dual *lattice of* L*. If* $\mathfrak{p}$ *is a maximal ideal of* $\mathbb{Z}_K$*, then the unique* $\mathcal{M}$*-lattice* $X \in L(\mathfrak{p})$ *such that* $X_\mathfrak{p} = L_\mathfrak{p}^\#$ *is called the* partial dual of L at $\mathfrak{p}$*. It will be denoted by* $L^{\#,\mathfrak{p}}$*.*

Definition 2.6. *Let* L *be an* $\mathcal{M}$*-lattice in* V*. Further, let* $\mathfrak{P}$ *be a maximal two sided ideal of* $\mathcal{M}$ *and set* $\mathfrak{p} = \mathfrak{P} \cap K$*. If* $E_\mathfrak{p} \cong K_\mathfrak{p} \oplus K_\mathfrak{p}$ *then* $L_\mathfrak{p}$ *is called* square-free *if* $L_\mathfrak{p} = L_\mathfrak{p}^\#$*. In all other cases,* $L_\mathfrak{p}$ *is called square-free if* $\mathfrak{P}L_\mathfrak{p}^\# \subseteq L_\mathfrak{p} \subseteq L_\mathfrak{p}^\#$*. The lattice* L *is called square-free if* $L_\mathfrak{p}$ *is square-free for all maximal ideals* $\mathfrak{p}$ *of* $\mathbb{Z}_K$*.*

Given a maximal two sided ideal $\mathfrak{P}$ of $\mathcal{M}$, we define an operator $\rho_\mathfrak{P}$ on the set of all $\mathcal{M}$-lattices as follows:

$$\rho_\mathfrak{P}(L) = \begin{cases} L + (\mathfrak{P}^{-1}L \cap L^\#) & \text{if } \mathfrak{P} \neq \sigma(\mathfrak{P}), \\ L + (\mathfrak{P}^{-1}L \cap \mathfrak{P}L^\#) & \text{otherwise.} \end{cases}$$

The operators generalise the maps defined by L. Gerstein in [7] for quadratic spaces. They are similar in nature to the *p-mappings* introduced by G. Watson in [32]. The maps satisfy the following properties:

Remark 2.7. Let L be an $\mathcal{M}$-lattice in V. Let $\mathfrak{P}$ be a maximal two sided ideal of $\mathcal{M}$ and set $\mathfrak{p} = \mathfrak{P} \cap \mathbb{Z}_K$.

1. $\rho_\mathfrak{P}(L) \in L(\mathfrak{p})$.
2. If $L_\mathfrak{p}$ is integral, then $(\rho_\mathfrak{P}(L))_\mathfrak{p} = L_\mathfrak{p} \iff L_\mathfrak{p}$ is square-free.
3. If $\mathfrak{Q}$ is a maximal two sided ideal of $\mathcal{M}$, then $\rho_\mathfrak{P} \circ \rho_\mathfrak{Q} = \rho_\mathfrak{Q} \circ \rho_\mathfrak{P}$.
4. If L is integral, there exist a sequence of not necessarily distinct maximal two sided ideals $\mathfrak{P}_1, \dots, \mathfrak{P}_s$ of $\mathcal{M}$ such that
$$L' := (\rho_{\mathfrak{P}_1} \circ \ldots \circ \rho_{\mathfrak{P}_s})(L)$$
is square-free. Moreover, the genus of L' is uniquely determined by the genus of L.

Proposition 2.8. *Let* L *be an* $\mathcal{M}$*-lattice in* V *and let* $\mathfrak{P}$ *be a maximal two sided ideal of* $\mathcal{M}$*. Then the class number of* $\rho_\mathfrak{P}(L)$ *is at most the class number of* L*.*

Proof. The definition of $\rho_\mathfrak{P}(L)$ only involves taking sums and intersections of multiples of L and its dual. Hence $\rho_\mathfrak{P}(L)g = \rho_\mathfrak{P}(Lg)$ for all $g \in \mathrm{U}(V,\Phi)$ and similar for $g \in \mathrm{U}(V_A,\Phi)$. In particular, $\rho_\mathfrak{P}$ maps lattices in the same genus (isometry class) to ones in the same genus (isometry class). The result follows. □

Definition 2.9. *Let $\mathfrak{A}$ be a two sided $\mathcal{M}$-ideal. An $\mathcal{M}$-lattice L is called $\mathfrak{A}$-maximal, if $\Phi(x,x) \in \mathfrak{A}$ for all $x \in L$ and no proper overlattice of L has that property. Similarly, one defines maximal lattices in $V_\mathfrak{p}$ for a maximal ideal $\mathfrak{p}$ of $\mathbb{Z}_K$.*

Definition 2.10. *Let $\mathfrak{P}$ be a maximal two sided ideal of $\mathcal{M}$ and set $\mathfrak{p} = \mathfrak{P} \cap K$. We say that an $\mathcal{M}$-lattice L is* $\mathfrak{p}$-normalised *if L satisfies the following conditions:*

- *L is square-free.*
- *If $E = K$ then $L_\mathfrak{p} \cong \mathbb{H}(\mathbb{Z}_{K_\mathfrak{p}})^r \perp M_0$ where $M_0 = \rho_\mathfrak{p}^\infty(M)$ and M denotes a $2\mathbb{Z}_{K_\mathfrak{p}}$-maximal lattice in an anisotropic quadratic space over $K_\mathfrak{p}$.*
- *If $E_\mathfrak{p}/K_\mathfrak{p}$ is a quadratic field extension with different $\mathcal{D}(E_\mathfrak{p}/K_\mathfrak{p})$, then $L_\mathfrak{p} \cong \mathbb{H}(\mathcal{M}_\mathfrak{p})^r \perp M_0$ where $M_0 = \rho_\mathfrak{p}^\infty(M)$ and M denotes a $\mathcal{D}(E_\mathfrak{p}/K_\mathfrak{p})$-maximal lattice in an anisotropic hermitian space over $E_\mathfrak{p}$.*
- *If $[E : K] = 4$, then $L_\mathfrak{p} = L_\mathfrak{p}^\#$.*

Here $\rho_\mathfrak{P}^\infty(M)$ denotes the image of M under repeated application of $\rho_\mathfrak{P}$ until this process becomes stable.

Remark 2.11. Let $\mathfrak{P}, \mathfrak{p}$ and L be as in Definition 2.10. Then the isometry class of $L_\mathfrak{p}$ is uniquely determined by $(V_\mathfrak{p}, \Phi)$.

Proof. There is nothing to show if $[E : K] = 4$. Suppose now $E = K$. The space KM_0 is a maximal anisotropic subspace of $(V_\mathfrak{p}, \Phi)$. By Witt's theorem [25, Theorem 42:17] its isometry type is uniquely determined by $(V_\mathfrak{p}, \Phi)$. Further, M_0 is the unique $2\mathbb{Z}_{K_\mathfrak{p}}$-maximal $\mathbb{Z}_{K_\mathfrak{p}}$-lattice in KM_0, see [25, Theorem 91:1]. Hence the isometry type of $\rho_\mathfrak{p}^\infty(M_0)$ depends only on $(V_\mathfrak{p}, \Phi)$. The case $[E : K] = 2$ is proved similarly. □

3. Genera of lattice chains

Definition 3.1. *Let $\mathcal{L} := (L_1, \dots, L_m)$ and $\mathcal{L}' := (L'_1, \dots, L'_m)$ be two m-tuples of $\mathcal{M}$-lattices in V. Then $\mathcal{L}$ and $\mathcal{L}'$ are* isometric, *if there is some $g \in \mathrm{U}(V, \Phi)$ such that $L_i g = L'_i$ for all $i = 1, \dots, m$. They are in the same* genus *if there is such an element $g \in \mathrm{U}(V_A, \Phi)$. Let*

$$[\mathcal{L}] := \{\mathcal{L}' \mid \mathcal{L}' \text{ is isometric to } \mathcal{L}\}$$

and

$$\mathrm{genus}(\mathcal{L}) := \{\mathcal{L}' \mid \mathcal{L}' \text{ and } \mathcal{L} \text{ are in the same genus}\}$$

denote the isometry class and the genus of $\mathcal{L}$, respectively. The automorphism group *of $\mathcal{L}$ is the stabiliser of $\mathcal{L}$ in $\mathrm{U}(V, \Phi)$, i.e.*

$$\mathrm{Aut}(\mathcal{L}) = \bigcap_{i=1}^m \mathrm{Aut}(L_i).$$

It is well known [4, Theorem 5.1] that any genus of a single lattice contains only finitely many isometry classes. This is also true for finite tuples of lattices in V:

Lemma 3.2. *Let $\mathcal{L} = (L_1, \dots, L_m)$ be an m-tuple of $\mathcal{M}$-lattices in V. Then* $\mathrm{genus}(\mathcal{L})$ *is the disjoint union of finitely many isometry classes. The number of isometry classes in* $\mathrm{genus}(\mathcal{L})$ *is called the class number of $\mathcal{L}$.*

Proof. The case $m = 1$ is the classical case. So assume that $m \geq 2$ and let $\mathrm{genus}(L_1) := [M_1] \uplus \dots \uplus [M_h]$, with $M_i = L_1 g_i$ for suitable $g_i \in \mathrm{U}(V_A, \Phi)$. We decompose $\mathrm{genus}(\mathcal{L}) = \mathcal{G}_1 \uplus \dots \uplus \mathcal{G}_h$ where $\mathcal{G}_i := \{(L'_1, \dots, L'_m) \in \mathrm{genus}(\mathcal{L}) \mid L'_1 \cong M_i\}$. It is clearly enough to show that each $\mathcal{G}_i$ is the union of finitely many isometry classes. By construction, any isometry class in $\mathcal{G}_i$ contains a representative of the form $(M_i, L'_2, \dots, L'_m)$ for some lattices L'_j in the genus of L_j. As all the L_j are lattices in the same vector space V, there are $a, b \in \mathbb{Z}_K$ such that

$$bL_1 \subseteq L_j \subseteq \frac{1}{a} L_1 \text{ for all } 1 \le j \le m.$$

As $(M_i, L'_2, \dots, L'_m) = \mathcal{L}g$ for some $g \in \mathrm{U}(V_A, \Phi)$ we also have

$$bM_i \subseteq L'_j \subseteq \frac{1}{a} M_i \text{ for all } 2 \le j \le m.$$

So there are only finitely many possibilities for such lattices L'_j. Hence the set of all m-tuples $(M_i, L'_2, \dots, L'_m) \in \mathrm{genus}(\mathcal{L})$ is finite and so is the class number. □

Remark 3.3. If $\mathcal{L}' \subseteq \mathcal{L}$ then the class number of $\mathcal{L}'$ is at most the class number of $\mathcal{L}$.

3.1. **Admissible lattice chains.**

Definition 3.4. *Let $\mathfrak{P}$ be a maximal 2-sided ideal of $\mathcal{M}$ and $\mathfrak{p} := K \cap \mathfrak{P}$. A lattice chain*

$$\mathcal{L} := \{L_0 \supset L_1 \supset \ldots \supset L_{m-1} \supset L_m\}$$

is called admissible *for $\mathfrak{P}$, if*

(1) $L_0 \subseteq L_0^{\#,\mathfrak{p}}$,
(2) $\mathfrak{P} L_0 \subset L_m$,
(3) $\mathfrak{P} L_m^{\#,\mathfrak{p}} \subseteq L_m$ *if* $\mathfrak{P} = \sigma(\mathfrak{P})$.

We call a $\mathfrak{P}$-admissible chain fine, *if L_0 is normalised for $\mathfrak{p}$ in the sense of Definition 2.10, L_i is a maximal sublattice of L_{i-1} for all $i = 1, \dots, m$ and either*

(a) $\mathfrak{P} = \sigma(\mathfrak{P})$ *and* $L_m / \mathfrak{P} L_m^{\#,\mathfrak{p}}$ *is an anisotropic space over* $\mathcal{M}/\mathfrak{P}$
(b) $\mathfrak{P} \neq \sigma(\mathfrak{P})$ *and* $\mathfrak{P} L_0$ *is a maximal sublattice of* L_m.

Remark 3.5. In the case that $\mathfrak{P} \neq \sigma(\mathfrak{P})$ the *length* m of a fine admissible lattice chain is just $m = r = \dim_E(V) - 1$. Also if $\mathfrak{P} = \sigma(\mathfrak{P})$, then $m = r$, where r is the rank of the $\mathfrak{p}$-adic group defined in Remark 2.3.

Note that any admissible chain $\mathcal{L}$ contains a unique maximal integral lattice which we will always denote by L_0.

Remark 3.6. Let $\mathcal{L} = (L_0, \dots, L_r)$ be a fine admissible lattice chain for $\mathfrak{P}$.

(a) If $\mathfrak{P} = \sigma(\mathfrak{P})$ then $\overline{L_0} := L_0 / \mathfrak{P} L_0^{\#,\mathfrak{p}}$ is a hermitian space over $\mathcal{M}/\mathfrak{P}$ and the spaces $V_j := \mathfrak{P} L_j^{\#,\mathfrak{p}} / \mathfrak{P} L_0^{\#,\mathfrak{p}}$ $(j = 1, \dots, r)$ define a maximal chain of isotropic subspaces of this hermitian space. We call the chain $(V_1, \dots, V_{r-1})$ *truncated.*
(b) If $\mathfrak{P} \neq \sigma(\mathfrak{P})$ then $\overline{L_0} := L_0 / \mathfrak{P} L_0$ is a vector space over $\mathcal{M}/\mathfrak{P}$ and the spaces $V_j := L_j / \mathfrak{P} L_0$ $(j = r, \dots, 1)$ form a maximal chain of subspaces. Here we call the chain $(V_{r-1}, \dots, V_1)$ *truncated.*

For the different hermitian spaces $\overline{L_0}$, the number of such chains of isotropic subspaces can be found by recursively applying the formulas in [29, Exercises 8.1, 10.4, 11.3].

Lemma 3.7. *The fine admissible lattice chain $\mathcal{L}$ represents a one-class genus of lattice chains if and only if L_0 represents a one-class genus of lattices and* $\mathrm{Aut}(L_0)$ *is transitive on the maximal chains of (isotropic) subspaces of $\overline{L_0}$.*

Proof. If $\mathcal{L}$ has class number one, so has any lattice in the chain $\mathcal{L}$. Suppose now L_0 has class number one. Let $\mathcal{L}'$ be any other lattice chain in the genus of $\mathcal{L}$. We have to show that $\mathcal{L}$ and $\mathcal{L}'$ are isometric. To that end, let L'_0 be the unique maximal integral lattice in $\mathcal{L}'$. Then L_0 and L_0' are isometric, as they are in the same genus. So without loss of generality, $L_0 = L_0'$. Then $\mathcal{L}$ and $\mathcal{L}'$ correspond to unique maximal chains of (isotropic) subspaces of $\overline{L_0}$. Since $\mathrm{Aut}(L_0)$ acts transitively on these chains of subspaces, it yields an isometry from $\mathcal{L}$ to $\mathcal{L}'$. □

4. Chamber transitive actions on affine buildings.

Kantor, Liebler and Tits [11] classified discrete groups acting chamber transitively and type preservingly on the affine building of a simple adjoint algebraic group of relative rank ≥ 2 over a locally compact local field. Such groups are very rare and hence this situation is an interesting phenomenon, further studied in [9], [10], [19], [22], [12], and [21] (and many more papers by these authors) where explicit constructions of the groups are given. One major disadvantage of the existing literature is that the proof in [11] is very sketchy, essentially the authors limit the possibilities that need to be checked to a finite number.

From the classification of the one-class genera of admissible fine lattice chains in Section 5, we obtain a number theoretic construction of the groups in [11] over fields of characteristic 0. It turns out that we find essentially all these groups and that our construction allows to find one more case: The building of $\mathrm{U}_5(\mathbb{Q}_3(\sqrt{-3}))$ of type $C - BC_2$, see Proposition 5.3 (1), which, to our best knowledge, has not appeared in the literature before.

4.1. **S-arithmetic groups.** We assume that (V, Φ) is a totally positive definite hermitian space, i.e. K is totally real and $\Phi(x, x) \in K$ is totally positive for all non-zero $x \in V$.

Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ be a finite set of prime ideals of $\mathbb{Z}_K$. For a prime ideal $\mathfrak{p}$ we denote by $\nu_\mathfrak{p}$ the $\mathfrak{p}$-adic valuation of K. Then the ring of S-integers in K is

$$\mathbb{Z}_S := \{a \in K \mid \nu_\mathfrak{q}(a) \geq 0 \text{ for all prime ideals } \mathfrak{q} \notin S\}.$$

Let L be some $\mathcal{M}$-lattice in (V, Φ) and put $L_S := L \otimes_{\mathbb{Z}_K} \mathbb{Z}_S$. Then the group

$$\mathrm{Aut}(L_S) := \{g \in \mathrm{U}(V, \Phi) \mid L_S g = L_S\}$$

is an S-arithmetic subgroup of $\mathrm{U}(V, \Phi)$.

Remark 4.1. For any prime ideal $\mathfrak{p}$, the group $\mathrm{U}(V, \Phi)$ (being a subgroup of $\mathrm{U}_\mathfrak{p}$) acts on the Bruhat-Tits building $\mathcal{B}$ of the group $\mathrm{U}_\mathfrak{p}$ defined in Remark 2.3. Assume that the rank of $\mathrm{U}_\mathfrak{p}$ is at least 1. The action of the subgroup $\mathrm{Aut}(L_S)$ is discrete and cocompact on $\mathcal{B}$, if and only if $\mathfrak{p} \in S$ and $(V_\mathfrak{q}, \Phi)$ is anisotropic for all $\mathfrak{p} \neq \mathfrak{q} \in S$.

4.2. **The action on the building of $\mathrm{U}_\mathfrak{p}$.** In the following we fix a prime ideal $\mathfrak{p}$ and assume that $S = \{\mathfrak{p}\}$.

A lattice class model for the affine building $\mathcal{B}$ has been described in [1]. Note that [1] imposes the assumption that the residue characteristic of $K_\mathfrak{p}$ is $p \neq 2$. This is only necessary to obtain a proof of the building axioms that is independent from Bruhat-Tits theory. For $p = 2$, the dissertation [6] contains the analogous description of the Bruhat-Tits building for orthogonal groups. For all residue characteristics, the chambers in $\mathcal{B}$ correspond to certain fine lattice chains in the natural $\mathrm{U}_\mathfrak{p}$-module $W_\mathfrak{p}$.

Let L be a fixed $\mathfrak{p}$-normalised lattice in V and put $V_\mathfrak{p} := V \otimes_K K_\mathfrak{p}$.

In the case that $E \otimes_K K_\mathfrak{p}$ is a skewfield, we decompose the completion

$$L_\mathfrak{p} = \mathbb{H}(\mathcal{M}_\mathfrak{p})^r \perp M_0 = \mathop{\perp}_{i=1}^{r} \langle e_i, f_i \rangle_{\mathcal{M}_\mathfrak{p}} \perp M_0$$

as in Definition 2.10. Then $V_\mathfrak{p} = V_0 \perp \langle e_1, \ldots, e_r, f_1, \ldots, f_r \rangle_{K_\mathfrak{p}}$ where $V_0 = K_\mathfrak{p} M_0$ is anisotropic. Then the standard chamber corresponding to L and the choice of this hyperbolic basis is represented by the admissible fine lattice chain

$$\mathcal{L} = (L = L_0, L_1, \ldots, L_r)$$

where $L_j \in L(\mathfrak{p})$ is the unique lattice in V such that

$$(L_j)_\mathfrak{p} = \mathop{\perp}_{i=1}^{j} \langle \pi e_i, f_i \rangle_{\mathcal{M}_\mathfrak{p}} \perp \mathop{\perp}_{i=j+1}^{r} \langle e_i, f_i \rangle_{\mathcal{M}_\mathfrak{p}} \perp M_0.$$

Now assume that $E \otimes_K K_\mathfrak{p} \cong K_\mathfrak{p}^{2\times 2}$ and $W_\mathfrak{p} = eV_\mathfrak{p}$ for some primitive idempotent e such that $\sigma(e) = 1 - e$ as in Remark 2.3. Then $W_\mathfrak{p}$ carries a symplectic form Ψ and the lattice $L_\mathfrak{p} e$ has a symplectic basis $(e_1, f_1, \ldots, e_r, f_r)$, i.e.

$$L_\mathfrak{p} e = \mathop{\perp}_{i=1}^{r} \langle e_i, f_i \rangle_{\mathbb{Z}_{K_\mathfrak{p}}}$$

with $\Psi(e_i, f_i) = 1$. The standard chamber corresponding to L and the choice of this symplectic basis is represented by the admissible fine lattice chain

$$\mathcal{L} = (L = L_0, L_1, \ldots, L_r)$$

where $L_j \in L(\mathfrak{p})$ is the unique lattice in V such that

$$(L_j)_\mathfrak{p} = \mathop{\perp}_{i=1}^{j} \langle \pi e_i, f_i \rangle_{\mathcal{M}_\mathfrak{p}} \perp \mathop{\perp}_{i=j+1}^{r} \langle e_i, f_i \rangle_{\mathcal{M}_\mathfrak{p}}.$$

In the last and most tricky case $E \otimes_K K_\mathfrak{p} \cong K_\mathfrak{p} \oplus K_\mathfrak{p}$. Then $W_\mathfrak{p} = V_\mathfrak{p} e_\mathfrak{P}$ for any of the two maximal ideals $\mathfrak{P}$ of $\mathcal{M}$ that contain $\mathfrak{p}$, $\mathrm{U}_\mathfrak{p} \supseteq \mathrm{SL}(W_\mathfrak{p})$ and $M_\mathfrak{p} := L_\mathfrak{p} e_\mathfrak{P}$ is a lattice in $W_\mathfrak{p}$. To define the standard chamber fix some $\mathbb{Z}_{K_\mathfrak{p}}$-basis $(e_1, \ldots, e_r)$ of $M_\mathfrak{p}$. Then the fine admissible lattice chain

$$\mathcal{L} = (L = L_0, L_1, \ldots, L_r)$$

where L_j is the unique lattice in V such that

- $(L_j)_\mathfrak{Q} = L_\mathfrak{Q}$ for all prime ideals $\mathfrak{Q} \neq \mathfrak{P}$ of $\mathcal{M}$
- $(L_j)_\mathfrak{P} = \bigoplus_{i=1}^{j} \langle \pi e_i \rangle_{\mathcal{M}_\mathfrak{P}} \oplus \bigoplus_{i=j+1}^{r} \langle e_i \rangle_{\mathcal{M}_\mathfrak{P}}$.

Lemma 4.2. *Assume that $\mathfrak{P} \neq \sigma(\mathfrak{P})$, so $E \otimes_K K_\mathfrak{p} \cong K_\mathfrak{p} \oplus K_\mathfrak{p}$ and keep the notation from above. Let M be some $\mathcal{M}$-lattice in V. Then $\{X \in M(\mathfrak{p}) \mid e_\mathfrak{P} X_\mathfrak{p} = e_\mathfrak{P} M_\mathfrak{p}\}$ contains a unique lattice Y with $Y = Y^{\#,\mathfrak{p}}$.*

Proof. As $Y \in M(\mathfrak{p})$ it is enough to define $Y_\mathfrak{p} = e_\mathfrak{P} M_\mathfrak{p} \oplus (1 - e_\mathfrak{P}) X_\mathfrak{p}$. This $\mathcal{M}_\mathfrak{p}$-lattice is unimodular if and only if

$$(1 - e_\mathfrak{P}) X_\mathfrak{p} = \{x \in (1 - e_\mathfrak{P}) V \mid \Phi(e_\mathfrak{P} M_\mathfrak{p}, x) \subseteq \mathcal{M}_\mathfrak{p}\}.$$

□

Thus for $\mathfrak{P} \neq \sigma(\mathfrak{P})$ the stabiliser in the S-arithmetic group $\mathrm{Aut}(L_S)$ of a vertex in the building $\mathcal{B}$ is the automorphism group of a $\mathfrak{p}$-unimodular lattice. Also if $\mathfrak{P} = \sigma(\mathfrak{P})$, any vertex in the building $\mathcal{B}$ corresponds to a unique homothety class of lattices $[M_\mathfrak{p}] = \{aM_\mathfrak{p} \mid a \in K_\mathfrak{p}^*\}$. So by Remark 2.2 there is a unique lattice $X \in L(\mathfrak{p})$ with $X_\mathfrak{p} = M_\mathfrak{p}$. Hence the stabilisers of the vertices in $\mathcal{B}$ are exactly the automorphism groups of the respective lattices in V. In particular these are finite groups.

Remark 4.3. As $\mathrm{U}_{\mathfrak{p}}$ acts transitively on the chambers of $\mathcal{B}$, any other chamber (i.e. r-dimensional simplex) in $\mathcal{B}$ corresponds to some lattice chain in the genus of $\mathcal{L} = (L_0, \dots, L_r)$. The $(r-1)$-dimensional simplices are the $\mathrm{U}_{\mathfrak{p}}$-orbits of the subchains $\mathcal{L}_j := (L_i \mid i \neq j)$ of $\mathcal{L}$ for $j = 0, \dots, r$. We call these simplices *panels* and j the cotype of the panel $\mathcal{L}_j$.

Theorem 4.4. *Let $\mathcal{L} = (L_0, \dots, L_r)$ be a fine admissible lattice chain for $\mathfrak{P}$ of class number one. Put $L := L_0$ and $S := \{\mathfrak{p}\}$. Then $\mathrm{Aut}(L_S)$ acts chamber transitively on the (weak) Bruhat-Tits building $\mathcal{B}$ of the completion $\mathrm{U}_{\mathfrak{p}}$.*

Proof. We use the characterisation of Lemma 3.7. Let $\mathcal{C}$ be the chamber of $\mathcal{B}$ that corresponds to $\mathcal{L}$ by the construction above and let $\mathcal{D}$ be some other chamber in $\mathcal{B}$. Then there is some element $g \in \mathrm{U}_{\mathfrak{p}}$ with $\mathcal{C}g = \mathcal{D}$. As the genus of L consists only of one class, there is some $h \in \mathrm{Aut}(L_S)$ such that $gh \in \mathrm{U}_{\mathfrak{p}}$ stabilises the vertex v that corresponds to L. So $gh \in \mathrm{Stab}_{\mathrm{U}_{\mathfrak{p}}}(L_{\mathfrak{p}})$ and $\mathcal{D}h$ is some chamber in $\mathcal{B}$ containing the vertex v. Now $\mathrm{Aut}(L)$ acts transitively on the set of all fine admissible lattice chains for $\mathfrak{P}$ starting in L, so there is some $h' \in \mathrm{Aut}(L)$ such that $\mathcal{D}hh' = \mathcal{C}$. Thus the element $hh' \in \mathrm{Aut}(L_S)$ maps $\mathcal{D}$ to $\mathcal{C}$. □

As in [27, Proposition 1] we obtain the following if and only if statement:

Proposition 4.5. *The group $\mathrm{Aut}^+(L_S)$ acts chamber transitively on the (weak) Bruhat-Tits building $\mathcal{B}$ if and only if the special class number $h^+(\mathcal{L}) = 1$ or equivalently if $h^+(L_0) = 1$ and $\mathrm{Aut}^+(L_0)$ is transitive on the maximal chains of (isotropic) subspaces of $\overline{L_0}$.*

For the maximal S-arithmetic group $\mathrm{Aut}(L_S)$ an if and only if statement is technically more involved due to the fact that $U(V, \Phi)$ is not necessarily connected and so we do not have strong approximation for this group. Here we obtain that $\mathrm{Aut}(L_S)$ acts chamber transitively on $\mathcal{B}$ if and only if $\mathrm{Aut}(L_0)$ acts transitively on the maximal chains of (isotropic) subspaces of $\overline{L_0}$ (see Lemma 3.7) and all $\mathfrak{p}$-neighbours of L_0 (i.e. all lattices L in the genus of L_0 with $L/(L \cap L_0) \cong \mathcal{M}/\mathfrak{P}$ for some maximal twosided ideal $\mathfrak{P}$ of $\mathcal{M}$ over $\mathfrak{p}$) are isometric to L_0.

For the orthogonal groups we can further characterise the transitivity of $\mathrm{Aut}(L_S)$ on $\mathcal{B}$: Let $g \in U(V, \Phi)$ be some isometry of determinant -1. Then the union of the proper special genera of L and $g(L)$ consists of exactly $h^+(L)$ isometry classes. Let

$$\mathcal{N}_{\mathfrak{p}}^+(L) := \{Mh \mid M \text{ is an iterated } \mathfrak{p}\text{-neighbour}, h \in U^+(V, \Phi)\}.$$

Then by [2] the set $\mathcal{N}_{\mathfrak{p}}^+(L)$ consists of $a \leq 2$ proper special genera. The exact value of a is given by some local condition, see [2, Equation (1.1)]. In particular, the union of all isometry classes of iterated $\mathfrak{p}$-neighbours is the following unig of proper special genera

$$\mathrm{genus}^+(L) \cup \mathrm{genus}^+(Lg) \cup \mathrm{genus}^+(L') \cup \mathrm{genus}^+(L'g).$$

where L' denotes any $\mathfrak{p}$-neighbour of L. The above union consists of a single isometry class, if and only if $h^+(L) = 1$ and $a = 1$.

4.3. **The oriflamme construction.** The buildings $\mathcal{B}$ described above are in general not thick buildings, i.e. there are panels that are only contained in exactly two chambers. Such panels are called thin. To obtain a thick building $\mathcal{B}^+$ (with a type preserving action by the group $\mathrm{U}_{\mathfrak{p}}^+$ defined in Remark 2.3) we need to apply a generalisation of the oriflamme construction as described in [1, Section 8]. In particular [1, Section 8.1] gives the precise situations which panels are thin for the case that $p \neq 2$. Also for $p = 2$ only the panels of cotype 0 and r can be thin. We refrain from describing the situations for $p = 2$ in general, but refer to the individual examples below.

Remark 4.6. Assume that $\mathfrak{P} = \sigma(\mathfrak{P})$.

(a) Assume that there are only two lattices L_0 and L_0' in the genus of L_0 such that
$$L_1 \subseteq L_0, L_0' \subseteq L_1^{\#,\mathfrak{p}}.$$
Then $\mathcal{L}$ and $\mathcal{L}' := (L_0', L_1, \ldots, L_r)$ are the only chambers in $\mathcal{B}$ that contain the panel $\mathcal{L}_0 = (L_1, \ldots, L_r)$ and hence this panel is thin. Then we replace the vertex represented by L_1 by the one represented by L_0'.

(b) Assume that there are only two lattices L_r and L_r' in the genus of L_r such that
$$\mathfrak{P}L_{r-1}^{\#,\mathfrak{p}} \subseteq L_r, L_r' \subseteq L_{r-1}.$$
Then $\mathcal{L}$ and $\mathcal{L}' := (L_0, L_1, \ldots, L_r')$ are the only chambers in $\mathcal{B}$ that contain the panel $\mathcal{L}_r = (L_0, \ldots, L_{r-1})$ and hence this panel is thin. Then we replace the vertex represented by L_{r-1} by the one represented by L_r'.

(c) After this construction the standard chamber $\mathcal{L}^+$ in the thick building $\mathcal{B}^+$ is either represented by $\mathcal{L}$, $(L_0, L_0', L_2, \ldots, L_r)$, $(L_0, L_1, \ldots, L_{r-2}, L_r, L_r')$, or $(L_0, L_0', L_2, \ldots, L_{r-2}, L_r, L_r')$. Note that by construction the chain $\mathcal{L}$ can be recovered from $\mathcal{L}^+$, so the stabiliser of $\mathcal{L}$ is equal to the stabiliser of all lattices in $\mathcal{L}^+$. Moreover every element in $\mathrm{U}_\mathfrak{p}$ mapping the chain $\mathcal{L}$ to some other chain $\mathcal{L}'$ maps the chamber $\mathcal{L}^+$ to the chamber $(\mathcal{L}')^+$.

For more details we refer to [1, Section 8.3].

In particular by part (c) of the previous remark we find the important corollary.

Corollary 4.7. *In the situation of Theorem 4.4 the group* $\mathrm{Aut}(L_S)$ *also acts chamber transitively (not necessarily type preservingly) on the thick building* $\mathcal{B}^+$.

Remark 4.8. Also in the situation where $\mathfrak{P} \neq \sigma(\mathfrak{P})$, i.e. $E_\mathfrak{p} = K_\mathfrak{p} \oplus K_\mathfrak{p}$, the stabilisers of the points in the building are not the stabilisers of the lattices in the lattice chain. By Lemma 4.2 the lattices L_i $(i = 1, \ldots, r)$ need to be replaced by the uniquely defined lattices $Y_i \in L_i(\mathfrak{p})$, such that $(Y_i)_\mathfrak{p}$ is unimodular (as in Lemma 4.2) and $Y_i \cap L_0 = L_i$. We refer to this construction as a variant of the oriflamme construction in the examples below.

Theorem 4.9. *Let* $\mathcal{L} = (L_0, \ldots, L_r)$ *be a fine* $\mathfrak{P}$*-admissible lattice chain for some maximal two sided ideal* $\mathfrak{P}$ *of* $\mathcal{M}$ *such that* $\sigma(\mathfrak{P}) = \mathfrak{P}$. *Suppose that the oriflamme construction replaces* $\mathcal{L}$ *by some sequence of lattices* $\mathcal{L}^+$ *which is one of*
$$\mathcal{L}, (L_0, L_0', L_2, \ldots, L_r), (L_0, L_1, \ldots, L_{r-2}, L_r, L_r') \text{ or } (L_0, L_0', L_2, \ldots, L_{r-2}, L_r, L_r').$$
Then L_0 *and* L_0' *as well as* L_r *and* L_r' *are in the same genus but not in the same proper special genus. Put* $L := L_0$ *and* $S := \{\mathfrak{p}\}$. *Then* $\mathrm{Aut}^+(L_S) := \mathrm{Aut}(L_S) \cap \mathrm{SU}(V, \Phi)$ *acts type preservingly on the thick building* $\mathcal{B}^+$. *This action is chamber transitive if and only if* $h^+(L) = 1$ *and* $\mathrm{Aut}^+(L)$ *is transitive on the maximal chains (in the first two cases) respectively truncated maximal chains (in the last two cases) of isotropic subspaces of* $\overline{L}$ *defined in Remark 3.6.*

Proof. The proof that the action is chamber transitive in all cases is completely analogous to the proof of Theorem 4.4. We only need to show that $h^+(L) = 1$. So let M be some lattice in the same proper special genus as L. By strong approximation for $\mathrm{U}^+(V_A, \Phi)$ (see [17]), there is some element $g \in \mathrm{U}_\mathfrak{p}^+$ and $h \in \mathrm{SU}(V, \Phi)$ such that $Mh = Lg$. As $\mathrm{Aut}^+(L_S)$ is chamber transitive and type preserving, there is some $f \in \mathrm{Aut}^+(L_S)$ such that $Lf = Mh$ so $M = Lfh^{-1}$ is properly isometric to L. $\square$

To obtain a classification of all chamber transitive discrete actions on $\mathcal{B}^+$ we hence need a classification of all proper spinor genera with proper class number one. The thesis [16] only lists the genera of class number one and two. In some cases, $h(L) = h^+(L)$ for every square-free lattice L, for example if:

(a) $E = K$, $\dim(V) \geq 5$ and K has narrow class number one ([25, Theorem 102.9]),
(b) $[E : K] = 2$ and $\dim_E(V)$ is odd ([28]),
(c) or $[E : K] = 4$.

5. The one-class genera of fine admissible lattice chains

We split this section into three subsections dealing with the different types of hermitian spaces ($[E : K] = 1, 2, 4$). The fourth subsection comments on the exceptional groups.

Suppose $\mathcal{L} = (L_0, \dots, L_r)$ is a fine $\mathfrak{P}$-admissible lattice chain of class number one, where $\mathfrak{P}$ is a maximal two sided ideal of $\mathcal{M}$. Then $\mathfrak{p} := \mathfrak{P} \cap \mathbb{Z}_K$ together with L_0 determines the isometry class of $\mathcal{L} := \mathcal{L}(L_0, \mathfrak{p})$. Moreover L_0 is a $\mathfrak{p}$-normalised lattice in (V, Φ) of class number one and by Corollary 3.7 the finite group $\mathrm{Aut}(L_0)$ acts transitively on the fine chains of (isotropic) subspaces of $\overline{L_0}$ as in Remark 3.6. The one- and two-class genera of lattices in hermitian spaces (V, Φ) have been classified in [16]. For all such lattices L_0 and all prime ideals $\mathfrak{p}$, for which L_0 is $\mathfrak{p}$-normalised, we check by computer if $\mathrm{Aut}(L_0)$ acts transitively on the fine chains of (isotropic) subspaces of $\overline{L_0}$. Note that the number of such chains grows with the norm of $\mathfrak{p}$, so the order of $\mathrm{Aut}(L_0)$ gives us a bound on the possible prime ideals $\mathfrak{p}$. We also checked weaker conditions (similar to the ones in Theorem 4.9) that would imply a chamber transitive action on the thick building $\mathcal{B}^+$, i.e. $h(L_0) \leq 2$ and transitivity only on the truncated maximal chains. The cases $h(L_0) = 2$ never gave a transitive action on the chambers of $\mathcal{B}^+$.

For any non-empty subset T of $\{1, 2, \dots, r\}$ we list the automorphism group G_T of the subchain $(L_i)_{i \in T}$. With our applications on the action on buildings in mind, we also give the order of

$$G_T^+ := G_T \cap \mathrm{U}_{\mathfrak{p}}^+$$

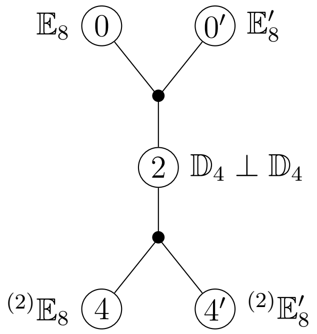
where $\mathrm{U}_{\mathfrak{p}}^+$ is given in Remark 2.3. Note that we will always assume that the rank of the group $\mathrm{U}_{\mathfrak{p}}$ is $r \geq 2$.

5.1. **Quadratic forms.** In this section suppose that $E = K$. We denote by $\mathbb{A}_n, \mathbb{B}_n, \mathbb{D}_n, \mathbb{E}_n$ the root lattices of the same type over $\mathbb{Z}_K$. If L is a lattice and $a \in K$ we denote by ${}^{(a)}L$ the lattice L with form rescaled by a. Sometimes we identify lattices over number fields using the trace lattice. For instance $(\mathbb{E}_8)_{\sqrt{-3}}$ denotes a hermitian lattice over $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ of dimension 4 whose trace lattice over $\mathbb{Z}$ is isometric to $\mathbb{E}_8$.

5.1.1. *Quadratic forms in more than four variables.* If $E = K$, $\dim_K(V) \geq 5$ and (V, Φ) contains a one-class genus of lattices, then by [16, Section 7.4] either $K = \mathbb{Q}$ or $K = \mathbb{Q}[\sqrt{5}]$ where one has essentially one one-class genus of lattices of dimension 5 and 6 each. The rational lattices have been classified in [20] and are available electronically from [13].

Proposition 5.1. *If $E = K$, $\dim_K(V) \geq 5$ and (V, Φ) contains a fine $\mathfrak{p}$-admissible lattice chain $\mathcal{L}(L_0, \mathfrak{p})$ of class number one for some prime ideal $\mathfrak{p}$, then $K = \mathbb{Q}$ and $\mathcal{L}(L_0, \mathfrak{p})$ is one of the following nine essentially different chains:*

(1) $\mathcal{L}(\mathbb{E}_8, 2) = (\mathbb{E}_8, \mathbb{D}_8, \mathbb{D}_4 \perp \mathbb{D}_4, {}^{(2)}\mathbb{D}_8^{\#}, {}^{(2)}\mathbb{E}_8)$. *After applying the oriflamme construction, the lattice chain becomes*

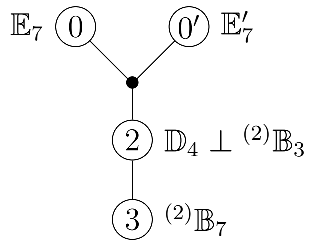


The automorphism groups are as follows

T	G_T	$\#G_T^+$
$\{i\}$	$2.\mathrm{O}_8^+(2).2$	$2^{13}\cdot 3^5 \cdot 5^2 \cdot 7$
$\{2\}$	$\mathrm{Aut}(\mathbb{D}_4) \wr C_2$	$2^{13}\cdot 3^4$
$\{i,j\}$	$2_+^{1+6}.S_8$	$2^{13}\cdot 3^2 \cdot 5 \cdot 7$
$\{2,i\}$	$N.(S_3 \times S_3 \wr C_2)$	$2^{13}\cdot 3^3$
$\{i,j,k\}$	$2_+^{1+6}.(C_3^2.\mathrm{PSL}_2(7))$	$2^{13}\cdot 3 \cdot 7$
$\{2,i,j\}$	$N.(C_2 \times S_3 \wr C_2)$	$2^{13}\cdot 3^2$
$\{0,0',4,4'\}$	$2_+^{1+6}.(C_2^3 : S_4)$	$2^{13}\cdot 3$
$\{2,i,j,k\}$	$N.(C_2^3 \times S_3)$	$2^{13}\cdot 3$
$\{0,0',2,4,4'\}$	$N.C_2^3$	2^{13}

where $N = O_2(G_{\{2\}}) \cong 2_+^{1+4} \times 2_+^{1+4}$ *and* $i,j,k \in \{0,0',2,4,4'\}$ *with* $\#\{i,j,k\} = 3$.

(2) $\mathcal{L}(\mathbb{E}_7, 2) = (\mathbb{E}_7, \mathbb{D}_6 \perp \mathbb{A}_1, \mathbb{D}_4 \perp {}^{(2)}\mathbb{B}_3, {}^{(2)}\mathbb{B}_7)$. *After applying the oriflamme construction, the lattice chain becomes*



The automorphism groups are as follows

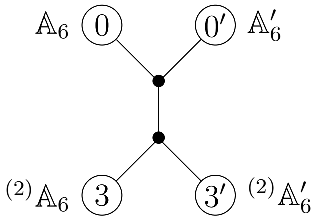
T	G_T	$\#G_T^+$
$\{i\}$	$C_2 \times \mathrm{PSp}_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$
$\{2\}$	$\mathrm{Aut}(\mathbb{D}_4) \times C_2 \wr S_3$	$2^9 \cdot 3^3$
$\{3\}$	$C_2 \wr S_7$	$2^9 \cdot 3^2 \cdot 5 \cdot 7$
$\{0,0'\}$	$C_2^6.S_6$	$2^9 \cdot 3^2 \cdot 5$
$\{i,2\}$	$N.S_3^2$	$2^9 \cdot 3^2$
$\{i,3\}$	$C_2^7.\mathrm{PSL}_2(7)$	$2^9 \cdot 3 \cdot 7$
$\{2,3\}$	$N.(C_2 \times S_3^2)$	$2^9 \cdot 3^2$
$\{0,0',2\}, \{i,2,3\}$	$N.D_{12}$	$2^9 \cdot 3$
$\{0,0',3\}$	$C_2^7.S_4$	$2^9 \cdot 3$
$\{0,0',2,3\}$	$N.C_2^2$	2^9

where $N := O_2(G_{\{2\}}) \cong 2_+^{1+4} \times Q_8$ *and* $i \in \{0,0'\}$. *The one-class chain*

$$\mathcal{L}(\mathbb{B}_7, 2) = \{\mathbb{B}_7, {}^{(2)}(\mathbb{D}_4^{\#} \perp B_3), {}^{(2)}\mathbb{D}_6^{\#} \perp \mathbb{B}_1, {}^{(2)}\mathbb{E}_7^{\#}\}$$

yields the same stabilisers.

(3) $\mathcal{L}(\mathbb{A}_6, 2) = \{\mathbb{A}_6, X, {}^{(2)}X^{\#,2}, {}^{(2)}\mathbb{A}_6\}$. *Here X is an indecomposable lattice with* $\mathrm{Aut}(X) = (C_2^4 \times C_3).D_{12}$. *After applying the oriflamme construction, the lattice chain becomes*



The automorphism groups are as follows

T	G_T	$\#G_T^+$	*sgdb*
$\#T = 1$	$C_2 \times S_7$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	–
$\{0, 0'\}, \{3, 3'\}$	$C_2 \times S_3 \times S_4$	$2^3 \cdot 3^2$	43
$\{0, 3\}, \{0, 3'\}, \{0', 3\}, \{0', 3'\}$	$C_2 \times \mathrm{PSL}_2(7)$	$2^3 \cdot 3 \cdot 7$	42
$\#T = 3$	$C_2 \times S_4$	$2^3 \cdot 3$	12
$\{0, 0', 3, 3'\}$	$C_2 \times D_8$	2^3	3

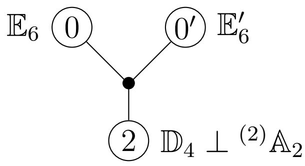
Here, and in the following tables, the column sgdb gives the label of G_T^+ as defined by the small group database ([3]*).*

The admissible one-class chain

$$\mathcal{L}({}^{(7)}\mathbb{A}_6^{\#}, 2) = \{{}^{(7)}\mathbb{A}_6^{\#}, {}^{(7)}X^{\#,7}, {}^{(14)}X^{\#}, {}^{(14)}\mathbb{A}_6^{\#}\}$$

yields the same groups.

(4) $\mathcal{L}(\mathbb{E}_6, 2) = \{\mathbb{E}_6, Y_0, \mathbb{D}_4 \perp {}^{(2)}\mathbb{A}_2\}$. *Here Y_0 is the even sublattice of* $\mathbb{B}_5 \perp {}^{(3)}\mathbb{B}_1$. *It is indecomposable and* $\mathrm{Aut}(Y_0) = \mathrm{Aut}(\mathbb{B}_5 \perp {}^{(3)}\mathbb{B}_1) \cong C_2 \times C_2 \wr S_5$. *After applying the oriflamme construction, the lattice chain becomes*



The automorphism groups are as follows

T	G_T	$\#G_T^+$	*sgdb*
$\{i\}$	$C_2 \times \mathrm{U}_4(2).2$	$2^6 \cdot 3^4 \cdot 5$	–
$\{2\}$	$\mathrm{Aut}(\mathbb{D}_4) \times D_{12}$	$2^6 \cdot 3^3$	–
$\{0, 0'\}$	$C_2 \wr S_5$	$2^6 \cdot 3 \cdot 5$	11358
$\{i, 2\}$	$N.S_3^2$	$2^6 \cdot 3^2$	8277
$\{0, 0', 2\}$	$N.D_{12}$	$2^6 \cdot 3$	201

where $N = O_2(G_3) \cong 2_+^{1+4} \times C_2$ *and* $i \in \{0, 0'\}$. *The admissible one-class chains*

$$\mathcal{L}(\mathbb{A}_2 \perp \mathbb{D}_4, 2) = \{\mathbb{A}_2 \perp \mathbb{D}_4, {}^{(2)}Y^{\#,2}, {}^{(2)}\mathbb{E}_6\}$$
$$\mathcal{L}({}^{(3)}(\mathbb{A}_2^{\#} \perp \mathbb{D}_4), 2) = \{{}^{(3)}(\mathbb{A}_2^{\#} \perp \mathbb{D}_4), {}^{(6)}Y^{\#}, {}^{(6)}\mathbb{E}_6\}$$
$$\mathcal{L}({}^{(3)}\mathbb{E}_6^{\#}, 2) = \{{}^{(3)}\mathbb{E}_6^{\#}, {}^{(3)}Y^{\#,3}, {}^{(3)}(\mathbb{A}_2 \perp \mathbb{D}_4)^{\#,3}\}$$

yield the same stabilisers.

(5) $\mathcal{L}(\mathbb{D}_6, 2) = \{\mathbb{D}_6, \mathbb{D}_4 \perp {}^{(2)}\mathbb{B}_2, {}^{(2)}\mathbb{B}_6\}$. *Here the application of the oriflamme construction is not necessary. The automorphism groups are as follows*

T	G_T	$\#G_T^+$	*sgdb*
$\{0\}, \{2\}$	$C_2 \wr S_6$	$2^8 \cdot 3^2 \cdot 5$	–
$\{1\}$	$\mathrm{Aut}(\mathbb{D}_4) \perp C_2 \wr S_2$	$2^8 \cdot 3^2$	–
$\{0, 1\}, \{1, 2\}$	$C_2^6.(C_2 \times S_4)$	$2^8 \cdot 3$	1086007
$\{0, 2\}$	$C_2^6.(C_2 \times S_4)$	$2^8 \cdot 3$	1088660
$\{0, 1, 2\}$	$C_2^6.(C_2 \times D_8)$	2^8	6331

(6) $\mathcal{L}(\mathbb{E}_6, 3) = \{\mathbb{E}_6, \mathbb{A}_2^3, {}^{(3)}\mathbb{E}_6\}$. *Here the application of the oriflamme construction is not necessary. The automorphism groups are as follows*

T	G_T	$\#G_T^+$	*sgdb*
$\{0\}, \{2\}$	$C_2 \times \mathrm{U}_4(2).2$	$2^6 \cdot 3^4 \cdot 5$	–
$\{1\}$	$D_{12} \wr S_3$	$2^5 \cdot 3^4$	–
$\{0, 2\}$	$3_+^{1+2}.(C_2 \times \mathrm{GL}_2(3))$	$2^3 \cdot 3^4$	533
$\{0, 1\}, \{1, 2\}$	$N.(C_2^2 \times S_4)$	$2^3 \cdot 3^4$	704
$\{0, 1, 2\}$	$N.(C_2^2 \times S_3)$	$2 \cdot 3^4$	10

where $N = O_3(G_{\{1\}}) \cong C_3^3$.

(7) $\mathcal{L}(\mathbb{B}_5 \perp {}^{(3)}\mathbb{B}_1, 3) = \{\mathbb{B}_5 \perp {}^{(3)}\mathbb{B}_1, \mathbb{B}_2 \perp \mathbb{A}_2 \perp {}^{(3)}\mathbb{B}_2; \mathbb{B}_1 \perp {}^{(3)}\mathbb{B}_5\}$. *Here the application of the oriflamme construction is not necessary. The automorphism groups are as follows*

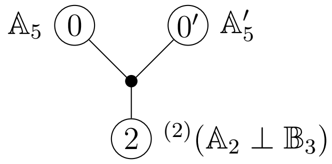
T	G_T	$\#G_T^+$	*sgdb*
$\{0\}, \{2\}$	$C_2 \times C_2 \wr S_5$	$2^6 \cdot 3 \cdot 5$	–
$\{1\}$	$C_2 \wr S_2 \times D_{12} \times C_2 \wr S_2$	$2^5 \cdot 3$	144
$\{0, 2\}$	$C_2^2 \times \mathrm{GL}_2(3)$	$2^3 \cdot 3$	3
$\{0, 1\}, \{1, 2\}$	$C_2^2 \times D_8 \times S_3$	$2^3 \cdot 3$	8
$\{0, 1, 2\}$	$C_2^3 \times S_3$	$2 \cdot 3$	2

For $0 \le i \le 2$ *let* Y_i *be the even sublattice of* $\mathcal{L}(\mathbb{B}_5 \perp {}^{(3)}\mathbb{B}_1, 3)_i$, *see also part (4). Then the admissible one-class chains*

$$\mathcal{L}(Y_0, 3) = \{Y_0, Y_1, Y_2\} \text{ and } \mathcal{L}({}^{(2)}Y_0^{\#,2}, 3) = ({}^{(2)}Y_0^{\#,2}, {}^{(2)}Y_1^{\#,2}, {}^{(2)}Y_2^{\#,2})$$

yield the same groups.

(8) $\mathcal{L}(\mathbb{A}_5, 2) = \{\mathbb{A}_5, {}^{(2)}\mathbb{B}_1 \perp Z, {}^{(2)}(\mathbb{A}_2 \perp \mathbb{B}_3)\}$. *Here* Z *is the even sublattice of* $\mathbb{B}_3 \perp {}^{(3)}\mathbb{B}_1$ *and* $\mathrm{Aut}(Z) = \mathrm{Aut}(\mathbb{B}_3 \perp {}^{(3)}\mathbb{B}_1)$. *After applying the oriflamme construction, the lattice chain becomes*



The automorphism groups are as follows

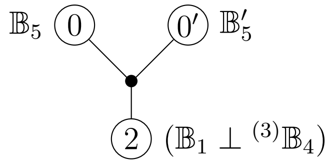
T	G_T	$\#G_T^+$	*sgdb*
$\{0\}, \{0'\}$	$C_2 \times S_6$	$2^3 \cdot 3^2 \cdot 5$	118
$\{2\}$	$D_{12} \times C_2 \wr S_3$	$2^3 \cdot 3^2$	43
$\#T = 2$	$C_2^2 \times S_4$	$2^3 \cdot 3$	12
$\{0, 0', 2\}$	$C_2^2 \times D_8$	2^3	3

The admissible one-class chains

$$\mathcal{L}({}^{(3)}\mathbb{A}_5^{\#,3},2) = \{{}^{(3)}\mathbb{A}_5^{\#,3}, {}^{(6)}\mathbb{B}_1 \perp {}^{(3)}Z^{\#,3}, {}^{(6)}(\mathbb{A}_2^{\#} \perp \mathbb{B}_3)\}$$
$$\mathcal{L}(\mathbb{A}_2 \perp \mathbb{B}_3, 2) = \{\mathbb{A}_2 \perp \mathbb{B}_3, \mathbb{B}_1 \perp {}^{(2)}Z^{\#,2}, {}^{(2)}\mathbb{A}_5^{\#,2}\}$$
$$\mathcal{L}({}^{(3)}(\mathbb{A}_2^{\#} \perp \mathbb{B}_3),2) = \{{}^{(3)}(\mathbb{A}_2^{\#} \perp \mathbb{B}_3), {}^{(3)}\mathbb{B}_1 \perp {}^{(6)}Z^{\#}, {}^{(6)}\mathbb{A}_5^{\#}\}$$

yield the same stabilisers.

(9) $\mathcal{L}(\mathbb{B}_5, 3) = \{\mathbb{B}_5, \mathbb{B}_2 \perp \mathbb{A}_2 \perp {}^{(3)}\mathbb{B}_1, \mathbb{B}_1 \perp {}^{(3)}\mathbb{B}_4\}$. *After applying the oriflamme construction, the lattice chain becomes*



T	G_T	$\#G_T^+$	*sgdb*
$\{i\}$	$C_2 \wr S_5$	$2^6 \cdot 3 \cdot 5$	11358
$\{2\}$	$C_2 \wr S_4 \times C_2$	$2^5 \cdot 3$	204
$\{0,0'\}$	$(C_2 \times D_8) \times S_3$	$2^3 \cdot 3$	3
$\{i,2\}$	$C_2 \times \mathrm{GL}_2(3)$	$2^3 \cdot 3$	8
$\{0,0',2\}$	$C_2^2 \times S_3$	$2 \cdot 3$	2

where $i \in \{0,0'\}$. *The admissible one-class chain*

$$\mathcal{L}(\mathbb{B}_4 \perp {}^{(3)}\mathbb{B}_1, 3) = \{\mathbb{B}_4 \perp {}^{(3)}\mathbb{B}_1, \mathbb{B}_1 \perp \mathbb{A}_2 \perp {}^{(3)}\mathbb{B}_2, {}^{(3)}\mathbb{B}_5\}$$

yields the same stabilisers.

5.1.2. *Quadratic forms in four variables.* Now assume that $K = E$ and $\dim_K(V) = 4$. By [16, Theorem 7.4.1] there are up to similarity exactly 481 one-class genera of lattices if $K = \mathbb{Q}$ and additionally 607 such genera over 22 other base fields where the largest degree is $[K : \mathbb{Q}] = 5$ ([16, Theorem 7.4.2]). As we are only interested in the case where the rank of $\mathrm{U}_\mathfrak{p}$ is 2, we only need to consider pairs $(L, \mathfrak{p})$ where L is one of these 1088 lattices and $\mathfrak{p}$ a prime ideal such that $V_\mathfrak{p} \cong \mathbb{H}(K_\mathfrak{p}) \perp \mathbb{H}(K_\mathfrak{p})$. In this case the building $\mathcal{B}$ of $\mathrm{U}_\mathfrak{p}$ is of type $A_1 \oplus A_1$ and not connected even after oriflamme construction. We will not list the groups acting chamber transitively on $\mathcal{B}^+$, also because of the numerous cases of one-class lattice chains in this situation.

To list the lattices we need some more notation. We denote by $\mathcal{Q} := \mathcal{Q}_{\alpha,\infty,\mathfrak{p}_1,\ldots,\mathfrak{p}_s}$ a definite quaternion algebra over $K = \mathbb{Q}(\alpha)$ which ramifies exactly at the finite places $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ of K. Given an integral ideal $\mathfrak{a}$ of $\mathbb{Z}_K$ coprime to all $\mathfrak{p}_i$, then $\mathcal{O}_{\alpha,\infty,\mathfrak{p}_1,\ldots,\mathfrak{p}_s;\mathfrak{a}}$ denotes an Eichler order of level $\mathfrak{a}$ in $\mathcal{Q}$.

We omit the subscript α whenever $K = \mathbb{Q}$. Similarly, the subscript $\mathfrak{a}$ is omitted, if $\mathfrak{a} = \mathbb{Z}_K$, i.e. the order is maximal.

Then $\mathcal{O}_{\alpha,\infty,\mathfrak{p}_1,\ldots,\mathfrak{p}_s;\mathfrak{a}}$ with the reduced norm form of $\mathcal{Q}$ yields a quaternary lattice over $\mathbb{Z}_K$. By [24, Corollary 4.6] this lattice is unique in its genus, if and only if all Eichler orders of level $\mathfrak{a}$ in $\mathcal{Q}$ are conjugate.

Hence we identify such orders with their quaternary lattices.

Proposition 5.2. *Let L be a $\mathfrak{p}$-normalised, quaternary lattice over $\mathbb{Z}_K$ such that $\mathcal{L}(L, \mathfrak{p})$ is a fine $\mathfrak{p}$-admissible lattice chain of length 2 and class number one. Then one of the following holds.*

(1) $K = \mathbb{Q}$ *and either*
- $\mathfrak{p} = 2$ *and* $L \cong \mathcal{O}_{\infty,3} \cong \mathbb{A}_2 \perp \mathbb{A}_2$ *or* $\mathcal{O}_{\infty,5}$.
- $\mathfrak{p} \in \{3, 5, 11\}$ *and* $L \cong \mathcal{O}_{\infty,2} \cong \mathbb{D}_4$.
- $\mathfrak{p} = 3$ *and* $L \cong \mathbb{B}_4$.

(2) $K = \mathbb{Q}(\sqrt{5})$ *and either*
  - $\mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{p}) \in \{4, 5, 9, 11, 19, 29, 59\}$ *and* $L \cong \mathcal{O}_{\sqrt{5},\infty}$. *This lattice is called* H_4 *in* [26].
  - $\mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{p}) \in \{5, 11\}$ *and* $L \cong \mathcal{O}_{\sqrt{5},\infty;2\mathbb{Z}_K} \cong \mathbb{D}_4$.
  - $\mathfrak{p} = 2\mathbb{Z}_K$ *and* $L \cong \mathcal{O}_{\sqrt{5},\infty;\mathfrak{a}} \cong \mathcal{L}(\mathcal{O}_{\sqrt{5},\infty}, \mathfrak{a})_2$ *with* $\mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{a}) \in \{5, 11\}$.

(3) $K = \mathbb{Q}(\sqrt{2})$ *and either*
  - $\mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{p}) \in \{2, 7, 23\}$ *and* $L \cong \mathcal{O}_{\sqrt{2},\infty}$.
  - $\mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{p}) = 7$ *and* $L \cong \mathcal{O}_{\sqrt{2},\infty;\sqrt{2}\mathbb{Z}_K} \cong \mathcal{L}(\mathcal{O}_{\sqrt{2},\infty})_2$ *or* L *is isometric to a unimodular lattice of norm* $\sqrt{2}\mathbb{Z}_K$ *in* $(V, \Phi) \cong \langle 1, 1, 1, 1\rangle$. *By* [25, IX:93], *the genus of the latter lattice is uniquely determined and it has class number one by* [16].
  - $\mathfrak{p} = \sqrt{2}\mathbb{Z}_K$ *and* $L \cong \mathcal{O}_{\sqrt{5},\infty;\mathfrak{a}} \cong \mathcal{L}(\mathcal{O}_{\sqrt{5},\infty}, \mathfrak{a})_2$ *with* $\mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{a}) = 7$.

(4) $K = \mathbb{Q}(\sqrt{3})$ *and either*
  - $\mathfrak{p} = \sqrt{3}\mathbb{Z}_K$ *and* $L \cong \mathcal{O}_{\sqrt{3},\infty;\mathfrak{p}_2}$ *or* L *is isometric to a unimodular lattice of norm* $\mathfrak{p}_2$ *in* $(V, \Phi) \cong \langle 1, 1, 1, 1\rangle$. *Again, this lattice is unique up to isometry.*
  - $\mathfrak{p} = \mathfrak{p}_2$ *and* $L \cong \mathcal{O}_{\sqrt{3},\infty;\sqrt{3}\mathbb{Z}_K}$.

(5) $K = \mathbb{Q}(\sqrt{13})$ *and* $\mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{p}) = 3$ *and* $L \cong \mathcal{O}_{\sqrt{13},\infty}$. *This lattice is called* $D_4^{\sim}$ *in* [26].

(6) $K = \mathbb{Q}(\sqrt{17})$ *and* $\mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{p}) = 2$ *and* $L \cong \mathcal{O}_{\sqrt{17},\infty}$. *This lattice is called* $(2A_2)^{\sim}$ *in* [26].

(7) $K = \mathbb{Q}(\theta_9)$ *is the maximal totally real subfield of the cyclotomic field* $\mathbb{Q}(\zeta_9)$ *and* $\mathfrak{p} = 2\mathbb{Z}_K$ *and* $L \cong \mathcal{O}_{\theta,\infty,\mathfrak{p}_3}$.

(8) $K = \mathbb{Q}(\alpha) \cong \mathbb{Q}[X]/(X^3 - X^2 - 3X + 1)$ *is the unique totally real number field of degree* 3 *and discriminant* 148. *Then either* $\mathfrak{p} = \mathfrak{p}_5$ *and* $L \cong \mathcal{O}_{\alpha,\infty;\mathfrak{p}_2}$ *or* $\mathfrak{p} = \mathfrak{p}_2$ *and* $L \cong \mathcal{O}_{\alpha,\infty;\mathfrak{p}_5}$.

(9) $K = \mathbb{Q}(\alpha) \cong \mathbb{Q}[X]/(X^3 - X^2 - 4X + 2)$ *is the unique totally real number field of degree* 3 *and discriminant* 316. *Then* $\mathfrak{p} = \mathfrak{p}_2$ *and* $L \cong \mathcal{O}_{\alpha,\infty;\mathfrak{p}_4}$.

(10) $K = \mathbb{Q}(\alpha) \cong \mathbb{Q}[X]/(X^4 - X^3 - 3X^2 + X + 1)$ *is the unique totally real number field of degree* 4 *and discriminant* 725. *Then* $L \cong \mathcal{O}_{\alpha,\infty}$ *and* $\mathrm{Nr}_{K/\mathbb{Q}}(\mathfrak{p}) \in \{11, 19\}$ *or* $\mathfrak{p}$ *is the ramified prime ideal of norm* 29.

(11) $K = \mathbb{Q}(\alpha) \cong \mathbb{Q}[X]/(X^4 - 4X^2 - X + 1)$ *is the unique totally real number field of degree* 4 *and discriminant* 1957. *Then* $\mathfrak{p} = \mathfrak{p}_3$ *and* $L \cong \mathcal{O}_{\alpha,\infty}$.

(12) $K = \mathbb{Q}(\alpha) \cong \mathbb{Q}[X]/(X^4 - X^3 - 4X^2 + X + 2)$ *is the unique totally real number field of degree* 4 *and discriminant* 2777. *Then* $\mathfrak{p} = \mathfrak{p}_2$ *and* $L \cong \mathcal{O}_{\alpha,\infty}$.

Here $\mathfrak{p}_q$ *denotes a prime ideal of* $\mathbb{Z}_K$ *of norm* q. *Conversely, in all these cases the chain* $\mathcal{L}(L, \mathfrak{p})$ *is* $\mathfrak{p}$*-admissible and has class number one.*

5.2. **Hermitian forms.** In this section we treat the case that $[E : K] = 2$, so E is a totally complex extension of degree 2 of the totally real number field K. The automorphism groups of the hermitian lattices that occur in the tables below are strongly related to maximal finite symplectic matrix groups classified in [14]. We use the notation introduced in this thesis (see also [15]) to name the groups. All hermitian lattices with class number ≤ 2 are classified in [16, Section 8] and listed explicitly for $n \geq 3$ in [16, pp 129-140].

Proposition 5.3. *Let* $\mathcal{L}(L_0, \mathfrak{p})$ *be a fine* $\mathfrak{P}$*-admissible chain of class number one and of length at least* 2. *Then* $K = \mathbb{Q}$, $d := \dim_E(V) \in \{3, 4, 5\}$ *and one the following holds:*

(1) $E = \mathbb{Q}(\sqrt{-3})$, $\mathfrak{p} = 3\mathbb{Z}$ *and* $L_0 \cong \mathbb{B}_5 \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1+\sqrt{-3}}{2}] \cong (\mathbb{A}_2^5)_{\sqrt{-3}}$*:*

$$\mathcal{L}(L_0, 3) = \{L_0, (\mathbb{A}_2^2 \perp {}^{(3)}\mathbb{E}_6^{\#})_{\sqrt{-3}}, (\mathbb{A}_2 \perp \mathbb{E}_8)_{\sqrt{-3}}\}.$$

Here the application of the oriflamme construction is not necessary. The automorphism groups are as follows:

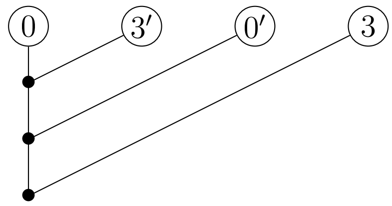
T	G_T	$\#G_T^+$
$\{0\}$	$C_6 \wr S_5$	$2^7 \cdot 3^5 \cdot 5$
$\{1\}$	$C_6 \wr S_2 \times {}_{\sqrt{-3}}[\pm 3_+^{1+2}.\mathrm{SL}_2(3)]_3$	$2^6 \cdot 3^5$
$\{2\}$	$C_6 \times {}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \times C_3]_4$	$2^7 \cdot 3^5 \cdot 5$
$\{0,1\}$	$C_6 \wr S_2 \times {}_{\sqrt{-3}}[\pm 3_+^{1+2}.C_6]_3$	$2^4 \cdot 3^5$
$\{1,2\}$	$C_6 \times {}_{\sqrt{-3}}[\pm(3_+^{1+2}.\mathrm{SL}_2(3) \times C_3)]_4$	$2^4 \cdot 3^5$
$\{0,1\}$	$C_6 \times {}_{\sqrt{-3}}[\pm 3^3 : S_4 \times C_3]_4$	$2^4 \cdot 3^5$
$\{1,2\}$	$C_6 \times {}_{\sqrt{-3}}[\pm(3_+^{1+2}.\mathrm{SL}_2(3) \times C_3)]_4$	$2^4 \cdot 3^5$
$\{0,1,2\}$	$C_6 \times {}_{\sqrt{-3}}[\pm 3_+^{1+2}.C_6 \times C_3]_4$	$2^2 \cdot 3^5$

(2) $E = \mathbb{Q}(\sqrt{-7})$, $\mathfrak{p} = 2\mathbb{Z}$ *and* $L_0 \cong (\mathbb{E}_8)_{\sqrt{-7}}$*:*

$$\mathcal{L}((\mathbb{E}_8)_{\sqrt{-7}}, 2) = \{(\mathbb{E}_8)_{\sqrt{-7}}, (\mathbb{D}_8)_{\sqrt{-7}}, (\mathbb{D}_4 \perp \mathbb{D}_4)_{\sqrt{-7}}, ({}^{(2)}\mathbb{D}_8)_{\sqrt{-7}}\}.$$

After applying the variant of the oriflamme construction described in Remark 4.8, the lattice chain becomes

$(\mathbb{E}_8)_{\sqrt{-7}} \cong$



The automorphism groups are as follows:

T	G_T	$\#G_T^+$	*sgdb*
$\#T = 1$	$2.\mathrm{Alt}_7$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	–
$\{0,0'\}, \{3,3'\}$	$\mathrm{SL}_2(3) \times C_3 : 2$	$2^4 \cdot 3^2$	124
$\{0,3\}, \{0,3'\}, \{0',3\}, \{0',3'\}$	$\mathrm{SL}_2(7)$	$2^4 \cdot 3 \cdot 7$	114
$\#T = 3$	$2.S_4$	$2^4 \cdot 3$	28
$\{0,0',3,3'\}$	Q_{16}	2^4	9

(3) $E = \mathbb{Q}(\sqrt{-3})$, $\mathfrak{p} = 2\mathbb{Z}$ *and* $L_0 \cong (\mathbb{E}_8)_{\sqrt{-3}}$*:*

$$\mathcal{L}((\mathbb{E}_8)_{\sqrt{-3}}, 2) = \{(\mathbb{E}_8)_{\sqrt{-3}}, (\mathbb{D}_4 \perp \mathbb{D}_4)_{\sqrt{-3}}, ({}^{(2)}\mathbb{E}_8)_{\sqrt{-3}}\}.$$

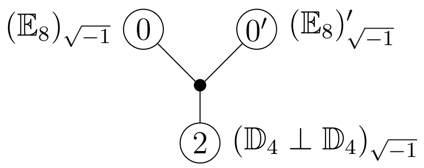
Here the application of the oriflamme construction is not necessary. The automorphism groups are as follows:

T	G_T	$\#G_T^+$
$\{0\}, \{2\}$	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \times C_3]_4$	$2^7 \cdot 3^4 \cdot 5$
$\{1\}$	${}_{\sqrt{-3}}[\mathrm{SL}_2(3) \times C_3]_2^2$	$2^7 \cdot 3^3$
$\{0,2\}$	$2_-^{1+4}.\mathrm{Alt}_5 \times C_3$	$2^7 \cdot 3 \cdot 5$
$\{0,1\}, \{1,2\}$	$\mathrm{SL}_2(3) \wr C_2 \times C_3$	$2^7 \cdot 3^2$
$\{0,1,2\}$	$(Q_8 \wr S_2) : C_3 \times C_3$	$2^7 \cdot 3$

(4) $E = \mathbb{Q}(\sqrt{-1})$, $\mathfrak{p} = 2\mathbb{Z}$ *and* $L_0 \cong (\mathbb{E}_8)_{\sqrt{-1}}$*:*

$$\mathcal{L}((\mathbb{E}_8)_{\sqrt{-1}}, 2) = \{(\mathbb{E}_8)_{\sqrt{-1}}, (\mathbb{D}_8)_{\sqrt{-1}}, (\mathbb{D}_4 \perp \mathbb{D}_4)_{\sqrt{-1}}\}.$$

Here the application of the oriflamme construction is not necessary. After applying the oriflamme construction, one obtains the following lattices

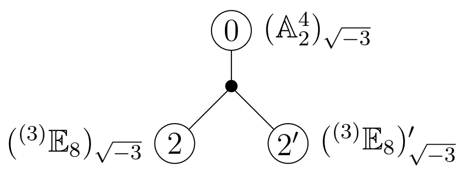


T	G_T	$\#G_T^+$
$\{0\}, \{0'\}$	${}_i[(2_+^{1+4}\mathrm{Y}C_4).S_6]_4$	$2^9 \cdot 3^2 \cdot 5$
$\{2\}$	${}_i[(D_8\mathrm{Y}C_4).S_3]_2^2$	$2^9 \cdot 3^2$
$\{0,2\}, \{0',2\}$		$2^9 \cdot 3$
$\{0,0'\}$		$2^9 \cdot 3$
$\{0,2,0'\}$		2^9

(5) $E = \mathbb{Q}(\sqrt{-3})$, $\mathfrak{p} = 3\mathbb{Z}$ *and* $L_0 = \mathbb{B}_4 \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1+\sqrt{3}}{1}] \cong (\mathbb{E}_8)_{\sqrt{-3}}$*: Here the application of the oriflamme construction is not necessary.*

$$\mathcal{L}(L_0, 3) = \{(\mathbb{A}_2^4)_{\sqrt{-3}}, (\mathbb{A}_2 \perp {}^{(3)}\mathbb{E}_6^{\#})_{\sqrt{-3}}, ({}^{(3)}\mathbb{E}_8)_{\sqrt{-3}}\}.$$

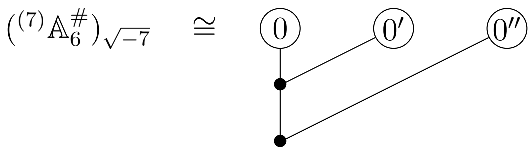
After applying the oriflamme construction, the chain becomes:



The automorphism groups are as follows:

T	G_T	$\#G_T^+$	*sgdb*
$\{0\}$	$C_6 \wr S_4$	$2^6 \cdot 3^4$	–
$\{2\}, \{2'\}$	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \times C_3]_4$	$2^7 \cdot 3^4 \cdot 5$	–
$\{0,2\}, \{0,2'\}$	$(\pm C_3^4).S_4$	$2^4 \cdot 3^4$	3085
$\{2,2'\}$	$(C_6 \times 3_+^{1+2}).S_3$	$2^4 \cdot 3^4$	2895
$\{0,2,2'\}$	$(C_6 \times C_3 \wr C_3).2$	$2^2 \cdot 3^4$	68

(6) $E = \mathbb{Q}(\sqrt{-7})$, $\mathfrak{p} = 2\mathbb{Z}$ *and* $L_0 = ({}^{(7)}\mathbb{A}_6^{\#})_{\sqrt{-7}}$*. After applying the variant of the oriflamme construction described in Remark 4.8, the chain becomes:*



The automorphism groups are as follows:

T	G_T	$\#G_T^+$
$\#T = 1$	$\pm C_7 : 3$	$3 \cdot 7$
$\#T = 2$	C_6	3
$\{0, 0', 0''\}$	C_2	1

5.3. **Quaternionic hermitian forms.** In this section we treat the case that $[E : K] = 4$, so E is a totally definite quaternion algebra over the totally real number field K. All quaternionic hermitian lattices with class number ≤ 2 are classified in [16, Section 9] and listed explicitly for $n \geq 2$ in [16, pp 147-150].

Proposition 5.4. *Suppose E is a definite quaternion algebra and let $\mathcal{L}(L_0, \mathfrak{p})$ be a fine $\mathfrak{P}$-admissible chain of length at least 2 and of class number one. Then $K = \mathbb{Q}$, $d := \dim_E(V) = 2$ and one of the following holds:*

(1) $E \cong \mathcal{Q}_{\infty,2}$, *the rational quaternion algebra ramified at* 2 *and* ∞, $\mathfrak{p} = 3\mathbb{Z}$ *and* $L_0 \cong (\mathbb{E}_8)_{\infty,2}$ *is the unique* $\mathcal{M}$*-structure of the* $\mathbb{E}_8$*-lattice whose automorphism group is called* ${}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$ *in* [24]*. The oriflamme construction is not necessary and the automorphism groups are*

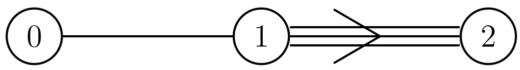
T	G_T	$\#G_T$	*sgdb*
$\{0\}, \{2\}$	${}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$	$2^7 \cdot 3 \cdot 5$	–
$\{1\}$	$Q_8 : \mathrm{SL}_2(3)$	$2^6 \cdot 3$	1022
$\{0,1\}, \{1,2\}$	$C_2 \times \mathrm{SL}_2(3)$	$2^4 \cdot 3$	32
$\{0,2\}$	$C_3 : SD_{16}$	$2^4 \cdot 3$	16
$\#T = 3$	$C_2 \times C_6$	$2^2 \cdot 3$	9

(2) $E \cong \mathcal{Q}_{\infty,3}$ *and* $\mathfrak{p} = 2\mathbb{Z}$ *and* $L_0 \cong (\mathbb{E}_8)_{\infty,3}$ *is the unique* $\mathcal{M}$*-structure of the* $\mathbb{E}_8$*-lattice whose automorphism group is called* ${}_{\infty,3}[\mathrm{SL}_2(9)]_2$ *in* [24]*. The oriflamme construction is not necessary and the automorphism groups are*

T	G_T	$\#G_T$	*sgdb*
$\{0\}, \{2\}$	${}_{\infty,3}[\mathrm{SL}_2(9)]_2$	$2^4 \cdot 3^2 \cdot 5$	409
$\{1\}$	$\mathrm{SL}_2(3).S_3$	$2^4 \cdot 3^2$	124
$\#T = 2$	$C_2.S_4$	$2^4 \cdot 3$	28
$\{0,1,2\}$	Q_{16}	2^4	9

Note that the above quaternion algebras only have one conjugacy class of maximal orders and for any such order $\mathcal{M}$*, the above* $\mathcal{M}$*-lattice* L_0 *is uniquely determined up to isometry.*

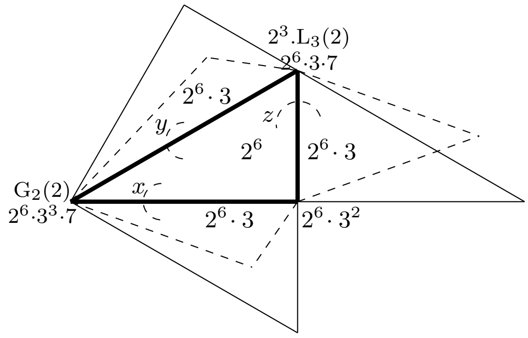
5.4. **The exceptional groups.** The exceptional groups have been dealt with in [16, Chapter 10], where it is shown that only the group G_2 admits one-class genera defined by a coherent family of parahoric subgroups. In all cases the number field is the field of rational numbers. The one-class genera of lattice chains correspond to the coherent families of parahoric subgroups $(P_q)_{q \text{ prime}}$ where for one prime p the parahoric subgroup P_p is the Iwahori subgroup, a stabiliser of a chamber in the corresponding p-adic building. Hence [16, Theorem 10.3.1] shows directly that there is a unique S-arithmetic group of type G_2 with a discrete and chamber transitive action. It is given by the $\mathbb{Z}$-form $\mathbf{G}_2$ where each parahoric subgroup P_q is hyperspecial. This integral model of G_2 is described in [8] (see also [5] for more one-class genera of G_2). Here $\mathbf{G}_2(\mathbb{Z}) \cong \mathrm{G}_2(2)$ and the S-arithmetic group is $\mathbf{G}_2(\mathbb{Z}[\frac{1}{2}])$ (so $S = \{(2)\}$). The extended Dynkin diagram of G_2 is as follows.



The stabilisers G_T of the simplices $T \subseteq \{0,1,2\}$ in the corresponding building of $\mathrm{G}_2(\mathbb{Q}_2)$ are given in [16, Section 10.3]:

T	G_T	$\#G_T$	*sgdb*
$\{0\}$	$\mathrm{G}_2(2)$	$2^6 \cdot 3^3 \cdot 7$	–
$\{2\}$	$2^3.\mathrm{GL}_3(2)$	$2^6 \cdot 3 \cdot 7$	814
$\{1\}$	$2_+^{1+4}.((C_3 \times C_3).2)$	$2^6 \cdot 3^2$	8282
$\{1,2\}$	$2_+^{1+4}.S_3$	$2^6 \cdot 3$	1494
$\{0,2\}$	$((C_4 \times C_4).2).S_3$	$2^6 \cdot 3$	956
$\{0,1\}$	$2_+^{1+4}.S_3$	$2^6 \cdot 3$	988
$\{0,1,2\}$	$\mathrm{Syl}_2(\mathrm{G}_2(2))$	2^6	134

One may visualise the chamber transitive action of $\mathbf{G}_2(\mathbb{Z}[\frac{1}{2}])$ on the Bruhat-Tits building of $\mathrm{G}_2(\mathbb{Q}_2)$ by indicating the three generators x, y, z of $\mathbf{G}_2(\mathbb{Z}[\frac{1}{2}])$ of order 3 mapping the standard chamber to one of the (three times) two neighbours.



Using a suitable embedding $\mathrm{G}_2 \hookrightarrow \mathrm{O}_7$ we find matrices for the three generators

$$x := \begin{pmatrix} 0 & 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}, \; y := \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; z := \frac{1}{2}\begin{pmatrix} 2 & 2 & 0 & -1 & 0 & -2 & -1 \\ 1 & 0 & 2 & 0 & -1 & 0 & -1 \\ 2 & 2 & 2 & -3 & -2 & -2 & -1 \\ 2 & 2 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 4 & -2 & -2 & 0 & -2 \\ 2 & -2 & 0 & 1 & 2 & 0 & -1 \\ 0 & 2 & 0 & -1 & 0 & -2 & 1 \end{pmatrix}.$$

To obtain a presentation in these generators, one only needs to compute the relations between the pairs of generators that hold in the finite group generated by the two matrices (in the stabiliser of a vertex).

6. Chamber transitive actions on p-adic buildings.

In this section we tabulate the chamber transitive actions on the p-adic buildings obtained from the one-class genera of lattice chains given in the previous section.

We use the names and the local Dynkin diagrams as given in [31]. The name for $\mathrm{U}_\mathfrak{p}$ usually does not give the precise type of the p-adic group. For instance the lattices $\mathbb{E}_6$ and $\mathbb{D}_6$ define two non isomorphic non-split forms of the algebraic group O_6 over $\mathbb{Q}_2$ which we both denote by $\mathrm{O}_6^-(\mathbb{Q}_2)$. To distinguish these groups, we also give the Tits index as in [30] and [31, Section 4.4]. Note that the isomorphism $\mathrm{O}_6^- \cong \mathrm{U}_4$ is given by the action of O_6 on the even part of the Clifford algebra. So we find the one-class genera of lattice chains also in a hermitian geometry, for $\mathcal{L}(\mathbb{E}_6, 2)$ (from 5.1 (3)) we get the same stabilisers as for $\mathcal{L}((\mathbb{E}_8)_{\sqrt{-3}}, 2)$ (from 5.3 (2)) in the projective group. Such coincidences are indicated by listing the lattices L_0 and the corresponding references (ref) in Table 1. The last column of Table 1 refers to a construction of the respective chamber transitive action in the literature. For a more detailed description of the different unitary groups U_p associated to the various types of local Dynkin diagrams we refer the reader to [31, Section 4.4].

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TABLE 1. Buildings with chamber transitive discrete actions

p	r	L_0	ref	U_p	index	root system	local Dynkin diagram	Lit
2	4	$\mathbb{E}_8$	5.1 (1)	$\mathrm{O}_8^+(\mathbb{Q}_2)$	${}^1D_{4,4}^{(1)}$	$\tilde{D}_4$	$\begin{array}{ccccc} 4 & \diagdown & & \diagup & 0 \\ & & 2 & & \\ 4' & \diagup & & \diagdown & 0' \end{array}$	[11] [9]
2	3	$\mathbb{E}_7$	5.1 (2)	$\mathrm{O}_7(\mathbb{Q}_2)$	$B_{3,3}$	$\tilde{B}_3$	$\begin{array}{cccc} & & \diagup & 0 \\ 3 \Leftarrow 2 & & & \\ & & \diagdown & 0' \end{array}$	[11] [9]
2	3	$\mathbb{A}_6$ $(\mathbb{E}_8)_{\sqrt{-7}}$	5.1 (3) 5.3 (2)	$\mathrm{O}_6^+(\mathbb{Q}_2) \cong$ $\mathrm{SL}_4(\mathbb{Q}_2)$	${}^1A_3$	$\tilde{A}_3$	$\begin{array}{cccc} & 0 & — & 3 \\ \diagup & & \diagup & \\ 3' & — & 0' & \end{array}$	[11] [10]
2	2	$\mathbb{E}_6$ $(\mathbb{E}_8)_{\sqrt{-3}}$	5.1 (4) 5.3 (3)	$\mathrm{O}_6^-(\mathbb{Q}_2) \cong$ $\mathrm{U}_4(\mathbb{Q}_2(\sqrt{-3}))$	${}^2A_{3,2}^{(1)}$	$\tilde{C}_2$	$0 \Rightarrow 2 \Leftarrow 0'$	[11] [22]
2	2	$\mathbb{D}_6$ $(\mathbb{E}_8)_{\sqrt{-1}}$	5.1 (5) 5.3 (4)	$\mathrm{O}_6^-(\mathbb{Q}_2) \cong$ $\mathrm{U}_4(\mathbb{Q}_2(\sqrt{-1}))$	${}^2D_{3,2}^{(1)}$	$C\text{-}B_2$	$0 \Leftarrow 2 \Rightarrow 0'$	[11] [9]
3	2	$\mathbb{E}_6$ $\mathbb{B}_5 \perp {}^{(3)}\mathbb{B}_1$ $(\mathbb{E}_8)_{\sqrt{-3}}$	5.1 (6) 5.1 (7) 5.3 (5)	$\mathrm{O}_6^-(\mathbb{Q}_3) =$ $\mathrm{O}_6^-(\mathbb{Q}_3) \cong$ $\mathrm{U}_4(\mathbb{Q}_3(\sqrt{-3}))$	${}^2D_{3,2}^{(1)}$	$C\text{-}B_2$	$0 \Leftarrow 2 \Rightarrow 0'$	[11] [12]
2	2	$\mathbb{A}_5$ $(\mathbb{E}_8)_{\infty,3}$	5.1 (8) 5.4 (1)	$\mathrm{O}_5(\mathbb{Q}_2) \cong$ $\mathrm{Sp}_4(\mathbb{Q}_2)$	$C_{2,2}^{(1)}$	$\tilde{C}_2$	$0 \Rightarrow 2 \Leftarrow 0'$	[11] [22]
3	2	$\mathbb{B}_5$ $(\mathbb{E}_8)_{\infty,2}$	5.1 (9) 5.4 (2)	$\mathrm{O}_5(\mathbb{Q}_3) \cong$ $\mathrm{Sp}_4(\mathbb{Q}_3)$	$C_{2,2}^{(1)}$	$\tilde{C}_2$	$0 \Rightarrow 2 \Leftarrow 0'$	[11] [12]
2	2	$({}^{(7)}\mathbb{A}_6^{\#})_{\sqrt{-7}}$	5.3 (6)	$\mathrm{SL}_3(\mathbb{Q}_2)$	${}^1A_2$	$\tilde{A}_2$	$\begin{array}{ccccc} & & 0 & & \\ & \diagup & & \diagdown & \\ 0'' & & — & & 0' \end{array}$	[11] [19] [23]
3	2	$(\mathbb{A}_2^5)_{\sqrt{-3}}$	5.3 (1)	$\mathrm{U}_5(\mathbb{Q}_3(\sqrt{-3}))$	${}^2A_{4,2}^1$	$C\text{-}BC_2$	$0 \Leftarrow 2 \Leftarrow 0'$	new
2	2	$\mathbf{G}_2(\mathbb{Z}[\frac{1}{2}])$	5.4	$\mathrm{G}_2(\mathbb{Q}_2)$	$G_{2,2}$	$\tilde{G}_2$	$0 — 1 \Rrightarrow 2$	[11] [9]

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