LMS Short Course on Computational Group Theory

Lab session 2

Getting to know GAP— first steps with permutation groups

There are hints on this sheet. We suggest that you first try to solve the exercises without using the hints, however, if you get **stuck** with one, then first read only the first hint and try again. If this does not help, try the second hint and so on. Finally, if nothing helps, **ask** someone.

1. Let G be the group generated by the following two permutations:

   $$(1, 10)(2, 3, 6, 9, 5, 8, 4, 11) \text{ and } (1, 2, 5, 9)(7, 10, 11, 8).$$

   We first want to analyse the action of this group on $M := \{1, 2, \ldots, 11\}$:
   Find, using GAP, the largest k such that G acts k-transitively on M.

   Hint 0: To enter the group, use the `Group` command ($\rightarrow$ `?Group`).

   Hint 1: Compute the orbit of 1 under G and decide, whether or not the action is transitive ($\rightarrow$ `?Orbit` and $\rightarrow$ `?IsTransitive`).

   Hint 2: Compute the stabiliser of 1 in G and apply the same method to it, of course using a different starting point ($\rightarrow$ `?Stabilizer`)

   Hint 3: Repeat.

2. For the k you found in 1: does G act **sharply** k-transitively?

   Hint 1: Look at the last stabiliser you computed.

3. Let's analyse the structure of this group a bit: Compute the group order ($\rightarrow$ `?Size`).

4. Compute the center of this group.

   Hint 1: $\rightarrow$ `?Center`

5. Compute the derived subgroup of this group.

   Hint 1: $\rightarrow$ `?DerivedSubgroup`

6. Check if this group is simple.

   Hint 1: $\rightarrow$ `?IsSimple`

7. Compute the 2-, 3-, 5- and 11-Sylow subgroups of G.

   Hint 1: $\rightarrow$ `?SylowSubgroup`

8. Compute the stabiliser of 1 in G and apply the above methods to it to find out something about its structure.

   Hint 1: $\rightarrow$ `?Stabiliser`

9. Let's study the derived subgroup D of $\mathrm{Stab}_G(1)$: Confirm that it is a simple group of order 360.

10. We suspect that this might be isomorphic to the alternating group A_6 on 6 points. Verify this with GAP and compute an explicit isomorphism.
    Compute the images of the generators of D under this isomorphism and the preimages of the standard generators of A_6.

    Hint 1: $\rightarrow$ `?IsomorphismGroups`

    Hint 2: This gives you a GAP object representing an isomorphism. You can access the generators of a group with `GeneratorsOfGroup`. You can map elements using `ImageElm` and compute preimages with `PreImage`.

11. Find out what the following command does and why it does this (assuming that the above group D is stored in the variable `D`):

    ```
    List(GeneratorsOfGroup(D),x->ImageElm(iso,x));
    ```

    Hint 1: $\to$ `?List`

    Hint 2: $\to$ `?arrow notation`

12. We want to construct this isomorphism in another way. To this end, understand the following sequence of commands:

    ```
    c := ConjugacyClassesMaximalSubgroups(D);
    List(c,Size);
    r := List(c,Representative);
    List(r,Size);
    ```

13. The above computation gave you two subgroups of index 6. Compute the two actions of D on the right cosets of them.

    Hint 1: $\to$ `FactorCosetAction`

14. Derive explicitly an automorphism of D which does not come from conjugation in the symmetric group S_6.